§7.3 System of Linear (algebraic) Equations
Eigen Values, Eigen Vectors

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Consider a system of $m$ linear equations, in $n$ (unknown) variables:

\begin{align*}
 a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
 a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\
 a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\
 & \quad \vdots \\
 a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m
\end{align*}  \hspace{1cm} (1)

where $a_{ij}$, $b_j$ are real or complex numbers.
Continued

Write

\[
A = \begin{pmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  a_{31} & a_{32} & \cdots & a_{3n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix}
  b_1 \\
  b_2 \\
  \vdots \\
  b_m
\end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n
\end{pmatrix}
\]

Then, \(A\) is called the coefficient matrix of the system (1). We also write \(A = (a_{ij})\).

In matrix form, the system (1) is written as

\[Ax = \mathbf{b}\] (2)
The Homogeneous Equation

- If $b = 0$, then the system (2) would be called a homogeneous system. So,

\[ Ax = 0 \]  \hspace{1cm} (3)

is a homogeneous system of linear equation.

- Then, $x = 0$ is a solution of the homogeneous system (3), to be called the trivial solution.
A system and the homogeneous system

- Suppose $x^{(0)}$ is a solution of the system (2): $Ax = b$.
- Then, any solution of (2): $Ax = b$ is of the form

$$x = x^{(0)} + \xi$$

(4)

where $\xi$ is a solution of the corresponding homogeneous system $Ax = 0$. 
Augmented Matrix

- Corresponding to system (1), define the augmented matrix

\[
A|b = \begin{pmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} & | & b_1 \\
    a_{21} & a_{22} & \cdots & a_{2n} & | & b_2 \\
    a_{31} & a_{32} & \cdots & a_{3n} & | & b_3 \\
    \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
    a_{m1} & a_{m2} & \cdots & a_{mn} & | & b_m
\end{pmatrix}
\] (5)

- In deed, the system (1) and the augmented matrix (5) has the same information/data. The Up-shot: the row operations performed on system (1), can be performed on the augmented matrix (5), in stead.
Solving the system (1)

- There are three possibilities:
  - The system (1), may not have any solution.
  - The system (1), may have infinitely many solution.
  - The system (1), may have a unique solution. For this possibility, we need at least \( n \) equations.

- To solve system (1), we can use TI-84 (\texttt{ref}, \texttt{rref}). Consult any TI-84 site for instructions.
$n = m$: System of $n$ equations and $n$ unknown

The textbook focuses on the case when $m = n$: the number of equations is same as number of unknown $x_1, \ldots, x_n$. In this section we assume $n = m$

- When $n = m$, then the coefficient matrix $A$ of (1) is a square matrix of size $n \times n$.
- Recall, a square matrix $A$ is invertible $\iff |A| \neq 0$.
- If $|A| \neq 0$, then the unique solution of system (2)

$$Ax = b \quad \text{is} \quad x = A^{-1}b \quad (6)$$
Linear Indpendence

- A set $x_1, x_2, \ldots, x_k$ of vectors (in $\mathbb{R}^n$) is said to be linearly dependent over $\mathbb{R}$ if there are scalars $c_1, \ldots, c_k$ in $\mathbb{R}$, not all zero such that $c_1 x_1 + c_2 x_2 + \cdots + c_k x_k = 0$.

- Likewise, a set $x_1, x_2, \ldots, x_k$ of vectors (in $\mathbb{C}^n$) is said to be linearly dependent over $\mathbb{C}$ if there are scalars $c_1, \ldots, c_k$ in $\mathbb{C}$, not all zero such that $c_1 x_1 + c_2 x_2 + \cdots + c_k x_k = 0$.

- A set $x_1, x_2, \ldots, x_k$ of vectors is said to be linearly independent over $\mathbb{R}$ or $\mathbb{C}$, if they are not linearly dependent. That means, if

$$c_1 x_1 + c_2 x_2 + \cdots + c_k x_k = 0 \implies c_1 = c_2 = \cdots = c_k = 0.$$
Given a set $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ (in $\mathbb{R}^n$ or $\mathbb{C}^n$) of vectors, we can form an $n \times k$ matrix $\mathbf{X} := (\mathbf{x}_1 \mathbf{x}_2 \cdots \mathbf{x}_k)$.

Then, $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ is linearly independent, if $\mathbf{Xc} = 0 \implies c = 0$. In other words, $\mathbf{Xc} = 0$ has no non-trivial solution.

For $n$ such vectors, $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ (in $\mathbb{R}^n$ or $\mathbb{C}^n$), they are linearly independent, if the determinant $|\mathbf{X}| \neq 0$. 
Suppose $A$ is a square matrix of size $n \times n$.

- A scalar $\lambda \in \mathbb{C}$ is said to be an Eigenvalue of $A$, if $|A - \lambda I| = 0$.

- The following are equivalent:
  - $\lambda \in \mathbb{C}$ is an Eigenvalue of $A$
  - $|A - \lambda I| = 0$
  - The system $(A - \lambda I)x = 0$ has nontrivial solutions.
  - There are non-zero vectors $x$ such that $Ax = \lambda x$.

- Accordingly, a vector $x \neq 0$ is said to be an eigenvector, for an eigenvalue $\lambda$ of $A$, if $Ax = \lambda x$. 
Eigenvalues are also called characteristic roots of $A$. (The German word "eigen" means "particular" or "peculier".)

The equation $|A - \lambda I| = 0$, is a polynomial equation in $\lambda$, of degree $n$, to be called the characteristic equation of $A$.

Counting multiplicity of roots, the characteristic equation $|A - \lambda I| = 0$, has $n$ complex roots.

Matlab can be used to compute eigenvalues and eigenvectors. Consult instructions in my site. The commands $\text{eig}(A)$, $[V,D]=\text{eig}(A)$ will be useful. However, Matlab does not work too well in this case. Eventually, we will use TI-84 to handle all these. Although, TI-84 does not have any direct command to do all these.
Sometimes, there is no choice but to use analytic methods. This will be the case, when we have to deal with complex eigenvalues.

Main thrust of this section is to compute eigenvalues and eigenvectors.
Sample I: Ex 17

Find the eigenvalues and the corresponding eigenvector of

\[ A = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \]

Use Matlab \texttt{eig}(V, D)

- Analytically: The characteristic equation:

\[
|A - \lambda I| = \begin{vmatrix} 3 - \lambda & -2 \\ 4 & -1 - \lambda \end{vmatrix} = 0
\]

\[(3 - \lambda)(-1 - \lambda) + 8 = 0 \iff \lambda^2 - 2\lambda + 5 = 0 \]

Eigenvalues are \( \lambda = 1 \pm 2i \)
To compute an eigenvector \( \lambda = 1 + 2i \), we solve 
\[(A - \lambda I)x = 0,\]
which is
\[
\begin{pmatrix}
3 - (1 + 2i) & -2 \\
4 & -1 - (1 + 2i)
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} = \begin{pmatrix}
0 \\
0
\end{pmatrix}
\]
\[
\begin{pmatrix}
2 - 2i & -2 \\
4 & -2 - 2i
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} = \begin{pmatrix}
0 \\
0
\end{pmatrix}
\]
\[
\begin{cases}
(2 - 2i)x_1 - 2x_2 = 0 \\
4x_1 - (2 + 2i)x_2 = 0
\end{cases} \implies \begin{cases}
(1 - i)x_1 - x_2 = 0 \\
2x_1 - (1 + i)x_2 = 0
\end{cases}
\]
Subtracting $1 + i$-times the first equation from the second, we get

\[
\begin{align*}
(1 - i)x_1 - x_2 &= 0 \\
0 &= 0 \\
\Rightarrow &\quad \begin{cases} 
(1 - i)x_1 - x_2 = 0 \\
x_2 = (1 - i)x_1
\end{cases}
\end{align*}
\]

Taking $x_1 = 1$, an eigenvector for $\lambda = 1 + 2i$, is

\[
x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 - i \end{pmatrix}
\]

(7)
Eigenvectors for $\lambda = 1 - 2i$

- An eigenvectors for $\lambda = 1 - 2i$ can be computed, as in the case of its conjugate $1 + 2i$.
- **Alternately**, An eigenvectors for $\lambda = 1 - 2i$ is the conjugate of (7):

\[
\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1+i \end{pmatrix}
\]
Sample II: Ex 20

Find the eigenvalues and the corresponding eigenvector of

\[ A = \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} \].

Use Matlab `eig[V, D]`

The characteristic equation:

\[ |A - \lambda I| = \begin{vmatrix} 1 - \lambda & \sqrt{3} \\ \sqrt{3} & -1 - \lambda \end{vmatrix} = 0 \]

\((1 - \lambda)(-1 - \lambda) - 3 = 0 \Longleftrightarrow \lambda^2 - 4 = 0\)

Eigenvalues are \(\lambda = 2, -2\)
Eigenvectors for $\lambda = 2$

For $\lambda = 2$, solve $(A - \lambda I)x = 0$, which is

\[
\begin{pmatrix}
1 - 2 & \sqrt{3} \\
\sqrt{3} & -1 - 2
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} =
\begin{pmatrix}
0 \\
0
\end{pmatrix}
\]

\[
\begin{pmatrix}
-1 & \sqrt{3} \\
\sqrt{3} & -3
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} =
\begin{pmatrix}
0 \\
0
\end{pmatrix}
\]

\[
\begin{cases}
-x_1 + \sqrt{3}x_2 = 0 \\
\sqrt{3}x_1 - 3x_2 = 0
\end{cases} \implies \begin{cases}
x_1 = \sqrt{3}x_2 \\
0 = 0
\end{cases}
\]

The 2nd-line is obtained by adding $\sqrt{3}$-times the first equation to the second.
Taking $x_2 = 1$, an eigenvector for $\lambda = 2$, is

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$$  \hspace{1cm} (8)$$

- Since $\lambda = 2$ has multiplicity one, we expect only one linearly independent eigenvector for $\lambda = 2$. 
Eigenvectors for $\lambda = -2$

For $\lambda = -2$, solve $(A - \lambda I)x = 0$, which is

$$
\begin{pmatrix}
1 + 2 & \sqrt{3} \\
\sqrt{3} & -1 + 2
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} =
\begin{pmatrix}
0 \\
0
\end{pmatrix}
$$

$$
\begin{pmatrix}
3 & \sqrt{3} \\
\sqrt{3} & 1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} =
\begin{pmatrix}
0 \\
0
\end{pmatrix}
$$

\[
\begin{cases}
3x_1 + \sqrt{3}x_2 = 0 \\
\sqrt{3}x_1 + x_2 = 0
\end{cases}
\implies
\begin{cases}
0 = 0 \\
x_2 = -\sqrt{3}x_1
\end{cases}
\]

*The 1st-line is obtained by subtracting $\sqrt{3}$-times the second equation to the first.*
Taking $x_1 = 1$, an eigenvector for $\lambda = -2$, is

\[
x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}
\]  

(9)

Since $\lambda = -2$ has multiplicity one, we expect only one linearly independent eigenvector for $\lambda = -2$. 
Sample III: Ex 23

Find the eigenvalues and the corresponding eigenvector of

\[
A = \begin{pmatrix}
3 & 2 & 2 \\
1 & 4 & 1 \\
-2 & -4 & -1 \\
\end{pmatrix}.
\]

Use Matlab \textit{eig} \([V, D]\).

\[\text{Analytically: The characteristic equation:}\]

\[
\left| A - \lambda I \right| = \begin{vmatrix}
3 - \lambda & 2 & 2 \\
1 & 4 - \lambda & 1 \\
-2 & -4 & -1 - \lambda \\
\end{vmatrix} = 0
\]
(3−\lambda)\begin{vmatrix}4−\lambda & 1 & -2 \\ -4 & -1−\lambda & 1 \\ 1 & -2 & -1−\lambda\end{vmatrix}+2\begin{vmatrix}1 & 4−\lambda \\ -2 & -4\end{vmatrix}=0

=\lambda^3−6\lambda^2−11\lambda+6=0 \implies \\
=\lambda^2(\lambda−1)+5\lambda(\lambda−1)−6(\lambda−1)=−(\lambda−1)(\lambda^2−5\lambda+6)=0 \implies \\
=−(\lambda−1)(\lambda−2)(\lambda−3)=0 \implies \lambda=1, 2, 3

are the eigenvalues of A.
Eigenvectors for $\lambda = 1$

For $\lambda = 1$, solve $(A - \lambda I)x = 0$, which is

$$
\begin{pmatrix}
3 & -1 & 2 \\
1 & 4 & -1 \\
-2 & -4 & -1
\end{pmatrix}
\begin{pmatrix}
2 & 2 & 2 \\
1 & 3 & 1 \\
-2 & -4 & -2
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
$$

$$
\begin{align*}
2x_1 + 2x_2 + 2x_3 &= 0 \\
x_1 + 3x_2 + x_3 &= 0 \\
-2x_1 - 4x_2 - 2x_3 &= 0
\end{align*}
\Rightarrow
\begin{align*}
x_1 + x_2 + x_3 &= 0 \\
x_1 + 3x_2 + x_3 &= 0 \\
x_1 + 2x_2 + x_3 &= 0
\end{align*}
$$
Subtracting first equation from second and third:

\[
\begin{align*}
    \begin{cases}
        x_1 + x_2 + x_3 &= 0 \\
        2x_2 &= 0 \\
        x_2 &= 0
    \end{cases} 
\end{align*}
\Rightarrow
\begin{align*}
    \begin{cases}
        x_1 &= -x_3 \\
        x_2 &= 0 \\
        x_2 &= 0
    \end{cases}
\end{align*}
\]

Taking \( x_3 = 1 \), an eigenvector for \( \lambda = 1 \), is

\[
x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}
\]

(11)
Since $\lambda = 1$ has multiplicity one, we expect only one linearly independent eigenvector for $\lambda = 1$.

It would be much simpler, if we use TI-84 (rref) to solve (10).
Eigenvectors for $\lambda = 2$

For $\lambda = 2$, solve $(A - \lambda I)x = 0$, which is

\[
\begin{pmatrix}
3 - 2 & 2 & 2 \\
1 & 4 - 2 & 1 \\
-2 & -4 & -1 - 2
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 2 & 2 \\
1 & 2 & 1 \\
-2 & -4 & -3
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\]  

(12)

Use rref in TI-84:

\[
\begin{pmatrix}
1 & 2 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\]  

⇒
Continued

\[ \begin{cases} 
  x_1 + 2x_2 = 0 \\
  x_3 = 0 \\
  0 = 0 
\end{cases} \quad \implies \quad \begin{cases} 
  x_1 = -2x_2 \\
  x_3 = 0 \\
  0 = 0 
\end{cases} \]

Taking \( x_1 = 1 \), an eigenvector for \( \lambda = 2 \), is

\[ x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \]
Eigenvectors for $\lambda = 3$

For $\lambda = 3$, solve $(A - \lambda I)x = 0$, which is

$$
\begin{pmatrix}
3 - 3 & 2 & 2 \\
1 & 4 - 3 & 1 \\
-2 & -4 & -1 - 3
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
$$

$$
\begin{pmatrix}
0 & 2 & 2 \\
1 & 1 & 1 \\
-2 & -4 & -4
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
$$

(13)

Use rref in TI-84:

$$
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 1 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\Rightarrow
$$
Continued

\[
\begin{align*}
\begin{cases}
    x_1 &= 0 \\
    x_2 + x_3 &= 0 \\
    0 &= 0
\end{cases}
\end{align*}
\implies \begin{align*}
\begin{cases}
    x_1 &= 0 \\
    x_2 &= -x_3 \\
    0 &= 0
\end{cases}
\end{align*}
\]

Taking \( x_3 = 1 \), an eigenvector for \( \lambda = 3 \), is

\[
x = \begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix} = \begin{pmatrix}
0 \\
-1 \\
1
\end{pmatrix}
\]
Read Example 4-5 (They are helpful).

Homework: §7.3 See the Homework Site!