Chapter 2: First Order ODE
§2.6 Equilibrium Solutions

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First Order ODE

- We recall the general form of the First Order ODEs:

$$\frac{dy}{dt} = f(t, y) \tag{1}$$

where $f(t, y)$ is a function of both the independent variable t and the (unknown) dependent variable y.

- The general solution $y = \varphi(t)$ of (1), is a family of solutions. The graphs of these solutions are called the integral curves.

Goals

- Over last half a century, interest to solve ODEs analytically exclusively, in undergraduate classes, has partly diminished. Emphasis has partly shifted to numerical methods and geometric methods. Neither gives exact solutions. Numerical methods give approximations to the exact solution (which is close enough, giving error estimates). Geometric methods teach the general shape of the graphs of the solutions (integral curves).
- This is a paradigm shift in teaching of ODE in undergraduate classes.

Goals: Continued

- As a result, study of ODE geometrically, using direction fields or direction of the tangents found a special place. The Equilibrium Solutions provide further structure to the geometric shapes of the graphs of the solutions.
- I took easy examples, and tried to provide both geometric approach, using intuition, as well analytic solutions.

Definition

Definition. The y-axis(line $t = 0$) is called the Phase Line.

- Often (as in §2.1), the initial value $y(0) = y_0$ determines the integral curve. The intersection $(0, \varphi(0))$ of an integral curve $y = \varphi(t)$ of (1) may be called the Phase line intercept.
- We **emphasize** that the Phase Line (y-axis) is a choice (not imposed). In this course, the phase line represent the initial $t = t_0$-value. If the initial condition is $y(t_0) = y_0$, we translate the y-axis to $t = t_0$, and the new phase line is again $t = 0$.

Continued

- The rest of this section we would mainly be interested on the right side of the Phase line. That means, we assume $t \geq 0$.

Definition.

Definition. It $f(y_0, t_0) = 0$, then we say that (t_0, y_0) is a critical points of the ODE (1).

- If (t_0, y_0) is a critical point of the ODE (1), then $t = t_0$ is a critical point of the integral curve $y = \varphi(t)$ passing through (t_0, y_0) (*As defined in Calculus-I*).

Definition

Definition: A constant solution $y = \varphi(t) = y_0$ (for all $t \geq 0$) of (1) would be called an Equilibrium Solution.

- If for some y_0, $f(y_0, t) = 0$ for all t, then the constant function $y = y_0$ would be an equilibrium solution of (1).
- The converse is also true. That is, if the constant function $y = y_0$ is a solution of the ODE (1), then

$$f(y_0, t) = \frac{dy}{dt} = 0 \qquad \text{for all } t \geq 0$$

Stabilization and Equilibrium

- **Definition.** Recall, a solution $y = \varphi(t)$ of the ODE (1) is said to stabilize (at infinity), if

$$\lim_{t \to \infty} \varphi(t) = y_0 \qquad \text{is finite.}$$

In this case, we say that $y = \varphi(t)$ stabilizes to $y = y_0$.

- If the solution $y = \varphi(t)$ fails to stabilize, we say that it does not stabilize.
- In this section, we classify the Equilibrium solutions as stable or unstable equilibrium.

Stable Equilibrium

Definition. (Fix a Phase Line.) An Equilibrium solution $y = y(t) = y_0$, of the ODE (1), is said to be a Stable Equilibrium, if there is an open interval $(y_0 - \epsilon, y_0 - \epsilon)$, such that for any integral curve $y = \varphi(t)$ of (1),

$$\text{if} \quad y_0 - \epsilon < \varphi(0) < y_0 - \epsilon, \quad \text{we have} \quad \lim_{t\to\infty} \varphi(t) = y_0$$

So, the line $y = y_0$ is a horizontal asymptote to the integral curve $y = \varphi(t)$

Continued

- The Definition above means, all the integral curves $y = \varphi(t)$, near the Equilibrium Solution $y = y_0$, stabilizes to the Equilibrium Solution $y = y_0$, itself.
- That means, there is a horizontal strip $(y_0 - \epsilon, y_0 + \epsilon)$, such that any integral curve $y = \varphi(t)$ with its Phase Line intercept within this strip, would stabilize to the horizontal line $y = y_0$.
- Note that our definition of Equilibrium solutions and of its stability, depends on the choice of the Phase Line.

Continued

- An equilibrium solution $y = y_0$ is said to be a unstable Equilibrium, if it is not a stable Equilibrium.
- The Direction fields sometimes provide some visual insight in this topic.
- For an ODE $\frac{dy}{dt} = f(y, t)$, we follow the following steps:
  - List all the Equilibrium solutions, by solving $f(y, t) = 0$
  - Then, classify them in to Stable and Unstable Equilibrium.

Example 1

Consider the ODE

$$\frac{dy}{dt} = (y-1)t \quad \text{with} \quad -\infty < y(0) = y_0 < \infty$$

- Determine the Equilibrium Solutions.
- Classify them as Stable or unstable Equilibrium.

Solution:

To compute Equilibrium Solutions, we solve

$$\frac{dy}{dt} = f(y,t) = (y-1)t = 0. \quad \text{Therefore,}$$

$$\left\{ \; y = 1 \quad \forall \; t \geq 0 \qquad \text{is the only Equilibrium Solution.} \right.$$

The Equilibriums solution $y = 1$ divides the ty-plane into two horizontal strips. Check the signs of $\frac{dy}{dt} = f(y,t) = (y-1)t$ in these strips.

Horizontal Strips	$sign(f(y,t))$	Behavior of $y = \varphi(t)$
$1 < y < \infty$	$+ \quad (\forall\ t > 0)$	strictly increasing
$-\infty < y < 1$	$- \quad (\forall\ t > 0)$	strictly decreasing

In the third column, $y = \varphi(t)$ denotes a solution to the ODE.

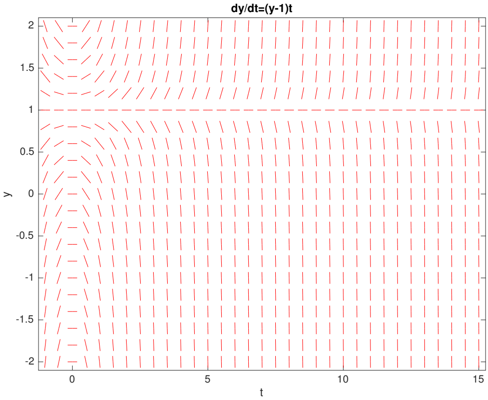
Continued

Intuitively, from the above table one can see that, for any integral curve $y = \varphi(t)$:

Horizontal Strips	The Limit
$1 < \varphi(0) < \infty$	$\lim_{\to\infty} \varphi(t) \geq 1$
$-\infty < \varphi(0) < 1$	$\lim_{\to\infty} \varphi(t) \leq 1$

So, in either case, $\lim_{\to\infty} \varphi(t) \neq 1$.
Intuitively, we conclude that $y = 1$ is NOT a Stable Equilibrium.

The Direction fields also indicate the same conclusion.



Analytic Solution

To conclude the same fully analytically, we solve the ODE. It is a separable equation and we have

$$\int \frac{1}{(y-1)}dy = \int tdt + c \Longrightarrow \ln|y-1| = \frac{t^2}{2} + c$$

$$\text{So,} \quad |y-1| = Ce^{\frac{t^2}{2}} \Longrightarrow |y-1| = |\varphi(0)-1|e^{\frac{t^2}{2}}$$

for any integral curve $y = \varphi(t)$, we have

$$\begin{cases} y = 1 + (\varphi(0)-1)e^{\frac{t^2}{2}} & \textit{if } \varphi(0) > 1 \\ y = 1 - (1-\varphi(0))e^{\frac{t^2}{2}} & \textit{if } \varphi(0) < 1 \end{cases}$$

So, for any integral curve $y = \varphi(t)$, we have

- if $\varphi(0) > 1$ then

$$\lim_{t\to\infty} \varphi(t) = 1 + \lim_{t\to\infty} \left((\varphi(0) - 1)e^{\frac{t^2}{2}} \right) = \infty$$

- if $\varphi(0) < 1$ then

$$\lim_{t\to\infty} \varphi(t) = 1 - \lim_{t\to\infty} \left((1 - \varphi(0))e^{\frac{t^2}{2}} \right) = -\infty$$

So, $y = 1$ is DEFINITELY not a stable equilibrium solution.

Example 2

Consider the ODE

$$\frac{dy}{dt} = -(y-1)t \quad \text{with} \quad -\infty < y(0) = y_0 < \infty$$

- Determine the Equilibrium Solutions.
- Classify them as Stable or unstable Equilibrium.

Solution:

To compute Equilibrium Solutions, we solve

$$\frac{dy}{dt} = f(y,t) = -(y-1)t = 0. \quad \text{Therefore,}$$

$$\left\{ \; y = 1 \quad \forall \; t \geq 0 \qquad \text{is the only Equilibrium Solution.} \right.$$

The Equilibriums solution $y = 1$ divides the ty-plane into two horizontal strips. Check the signs of $\frac{dy}{dt} = f(y, t) = -(y - 1)t$ in these strips.

Horizontal Strips	$sign(f(y, t))$	Behavior of $y = \varphi(t)$
$1 < y < \infty$	$- \quad (\forall\ t > 0)$	strictly decreasing
$-\infty < y < 1$	$+ \quad (\forall\ t > 0)$	strictly increasing

In the third column, $y = \varphi(t)$ denotes a solution to the ODE.

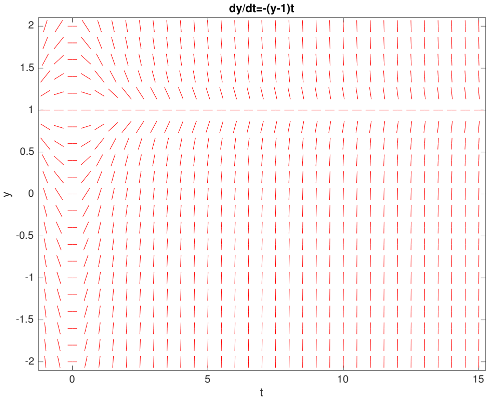
Continued

Intuitively, from the above table one can see that, for any integral curve $y = \varphi(t)$:

- If $\varphi(0) > 1$, then it seems that $y = \varphi(t)$ is steadily decreasing to the horizontal line $y = 1$.
  Therefore, it seems $\lim_{t\to\infty} \varphi(t) = 1$
- If $\varphi(0) < 1$, then it seems that $y = \varphi(t)$ is steadily increasing to the horizontal line $y = 1$.
  Therefore, it seems $\lim_{t\to\infty} \varphi(t) = 1$.

Therefore, intuitively, we conclude that $y = 1$ is a Stable Equilibrium.

The Direction fields also indicate the same conclusion.



Analytic Solution

To conclude the same fully analytically, we solve the ODE. It is a separable equation and we have

$$\int \frac{1}{(y-1)}dy = -\int tdt + c \Longrightarrow \ln|y-1| = -\frac{t^2}{2} + c$$

$$\text{So,} \quad |y-1| = Ce^{-\frac{t^2}{2}} \Longrightarrow |y-1| = |\varphi(0)-1|e^{-\frac{t^2}{2}}$$

for any integral curve $y = \varphi(t)$, we have

$$\begin{cases} y = 1 + (\varphi(0)-1)e^{-\frac{t^2}{2}} & \text{if } \varphi(0) > 1 \\ y = 1 - (1-\varphi(0))e^{-\frac{t^2}{2}} & \text{if } \varphi(0) < 1 \end{cases}$$

So, for any integral curve $y = \varphi(t)$, we have

- if $\varphi(0) > 1$ then

$$\lim_{t\to\infty} \varphi(t) = 1 + \lim_{t\to\infty} \left((\varphi(0) - 1)e^{-\frac{t^2}{2}} \right) = 1$$

- if $\varphi(0) < 1$ then

$$\lim_{t\to\infty} \varphi(t) = 1 - \lim_{t\to\infty} \left((1 - \varphi(0))e^{-\frac{t^2}{2}} \right) = 1$$

So, $y = 1$ is DEFINITELY a stable equilibrium solution.

Example 3

Consider the ODE

$$\frac{dy}{dt} = (y-1)\cos t \quad \text{with} \quad -\infty < y(0) = y_0 < \infty$$

- Determine the Equilibrium Solutions.
- Classify them as Stable or unstable Equilibrium.

Solution:

To compute Equilibrium Solutions, we solve

$$\frac{dy}{dt} = f(y,t) = (y-1)\cos t = 0. \quad \text{Therefore,}$$

$$\begin{cases} y = 1 & \forall\ t \geq 0 \\ \forall\ y & t = n\pi + \frac{\pi}{2} \qquad n = 0, \pm 1, \pm 2, \cdots \end{cases}$$

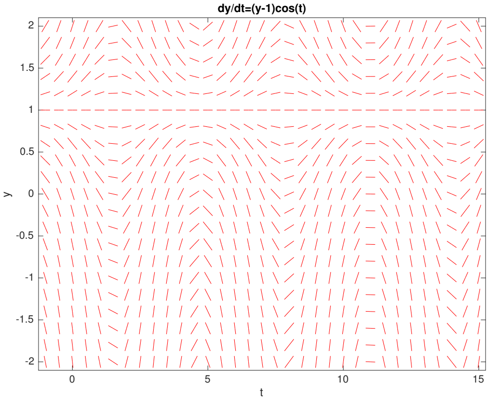
- So, $y = 1$ is the only Equilibriums solution.
- While the second line indicate that $\left(n\pi + \frac{\pi}{2}, y\right)$ is a Critical Point of the ODE, for all y. (*Studying critical points is not the focus of this section.*)

The Equilibriums solution $y = 1$ divides the ty-plane into two horizontal strips. Check the signs of
$\frac{dy}{dt} = f(y,t) = (y-1)\cos t$ in these strips.

Horizontal Strips	$sign(f(y,t))$	Behavior of $y = \varphi(t)$
$1 < y < \infty$	Keeps Switching	may be periodic
$-\infty < y < 1$	Keeps Switching	may be periodic

In the third column, $y = \varphi(t)$ denotes a solution to the ODE. Unless one has strong intuition, one cannot decide, if $y = 1$ is a Stable of Unstable Equilibrium, without solving the ODE.

The Direction fields also indicate the same.



Analytic Solution

To conclude the same fully analytically, we solve the ODE. It is a separable equation and we have

$$\int \frac{1}{(y-1)} dy = \int \cos t dt + c \Longrightarrow \ln|y-1| = \sin t + c$$

$$\text{So,} \quad |y-1| = Ce^{\sin t} \Longrightarrow |y-1| = |\varphi(0)-1|e^{\sin t}$$

for any integral curve $y = \varphi(t)$, we have

$$\begin{cases} y = 1 + (\varphi(0)-1)e^{\sin t} & if \;\; \varphi(0) > 1 \\ y = 1 - (1-\varphi(0))e^{\sin t} & if \;\; \varphi(0) < 1 \end{cases}$$

Conclution

As $-1 \leq \sin t \leq 1$, we have the following, for the integral curve $y = \varphi(t)$:

- If $\varphi(0) > 1$, the integral curve $y = \varphi(t) = 1 + (\varphi(0) - 1)e^{\sin t}$ oscillates between $1 + (\varphi(0) - 1)e^{-1}$ and $1 + (\varphi(0) - 1)e$.
- If $\varphi(0) < 1$, the integral curve $y = \varphi(t) = 1 - (1 - \varphi(0))e^{\sin t}$ oscillates between $1 - (1 - \varphi(0))e$ and $1 - (1 - \varphi(0))e^{-1}$.

So, $y = \varphi(t)$ does not approach to $y = 1$.
So, $y = 1$ is not a stable Equilibrium Solution.

Example 4

Consider the ODE

$$\frac{dy}{dt} = (y-1)(-2+\cos t) \quad \text{with} \quad -\infty < y(0) = y_0 < \infty$$

- Determine the Equilibrium Solutions.
- Classify them as Stable or unstable Equilibrium.

Solution:

To compute Equilibrium Solutions, we solve

$$\frac{dy}{dt} = f(y,t) = (y-1)(-2+\cos t) = 0. \quad \text{Therefore,}$$

$$\{ \; y = 1 \quad \forall \; t \geq 0$$

- So, $y = 1$ is the only Equilibriums solution.

The Equilibriums solution $y = 1$ divides the *ty*-plane into two horizontal strips. Check the signs of $\frac{dy}{dt} = f(y,t) = (y-1)(-2+\cos t)$ in these strips.

Horizontal Strips	$sign(f(y,t))$	Behavior of $y = \varphi(t)$
$1 < y < \infty$	negative	decreasing
$-\infty < y < 1$	positive	increasing

In the third column, $y = \varphi(t)$ denotes a solution to the ODE.

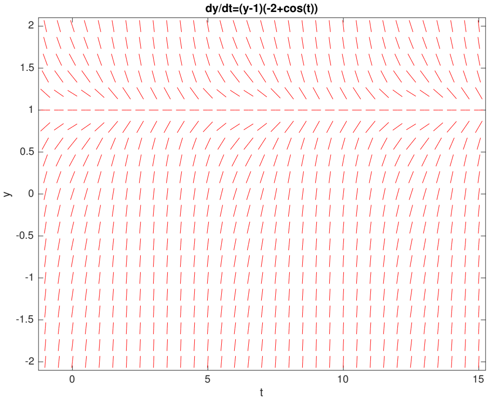
Continued

Intuitively, from the above table one can see that, for any integral curve $y = \varphi(t)$:

- If $\varphi(0) > 1$, then it seems that $y = \varphi(t)$ is steadily decreasing to the horizontal line $y = 1$.
  Therefore, it seems $\lim_{t\to\infty} \varphi(t) = 1$
- If $\varphi(0) < 1$, then it seems that $y = \varphi(t)$ is steadily increasing to the horizontal line $y = 1$.
  Therefore, it seems $\lim_{t\to\infty} \varphi(t) = 1$.

Therefore, intuitively, we conclude that $y = 1$ is a Stable Equilibrium.

The Direction fields also indicate the same.



Analytic Solution

To conclude the same fully analytically, we solve the ODE. It is a separable equation and we have

$$\int \frac{1}{(y-1)}dy = \int (-2+\cos)t dt + c \Longrightarrow \ln|y-1| = -2t + \sin t + c$$

$$\text{So,} \quad |y-1| = Ce^{-2t+\sin t} \Longrightarrow |y-1| = |\varphi(0)-1|e^{-2t+\sin t}$$

for any integral curve $y=\varphi(t)$, we have

$$\begin{cases} y = 1 + (\varphi(0)-1)e^{-2t+\sin t} & \text{if } \varphi(0) > 1 \\ y = 1 - (1-\varphi(0))e^{-2t+\sin t} & \text{if } \varphi(0) < 1 \end{cases}$$

Conclution

For an integral curve $y = \varphi(t)$, we compute the limit $\lim_{t\to\infty} \varphi(t)$:

- If $\varphi(0) > 1$, we have

$$\lim_{t\to\infty} \varphi(t) = \lim_{t\to\infty} \left(1 + (\varphi(0) - 1)e^{-2t+\sin t}\right) = 1$$

- If $\varphi(0) < 1$, we have

$$\lim_{t\to\infty} \varphi(t) = \lim_{t\to\infty} \left(1 - (1 - \varphi(0))e^{-2t+\sin t}\right) = 1$$

So, $y = 1$ is a stable Equilibrium Solution.

Example 5

Consider the ODE

$$\frac{dy}{dt} = y(y-1)t \quad \text{with} \quad -\infty < y_0 < \infty$$

- Determine the Equilibrium Solutions.
- Classify them as Stable or unstable Equilibrium.

Solution:

To compute Equilibrium Solutions, we solve

$$\frac{dy}{dt} = f(y,t) = y(y-1)t = 0. \quad \text{Therefore,}$$

$$\begin{cases} y = 1 & \forall\ t \geq 0 \\ y = 0 & \forall\ t \geq 0 \\ \forall\ y & t = 0 \end{cases}$$

So, the Equilibrium solutions are $y = 0$ and $y = 1$

The Equilibriums divide the *ty*-plane into horizontal strips. Check the signs of $\frac{dy}{dt} = f(y,t) = y(y-1)t = 0$ in these strips.

Horizontal Strips	$sign(f(y,t))$	Behavior of $y = \varphi(t)$
$1 < y < \infty$	$+$	increasing
$0 < y < 1$	$-$	decreasing
$-\infty < y < 0$	$+$	increasing

In the third column, $y = \varphi(t)$ denotes a solution to the ODE.

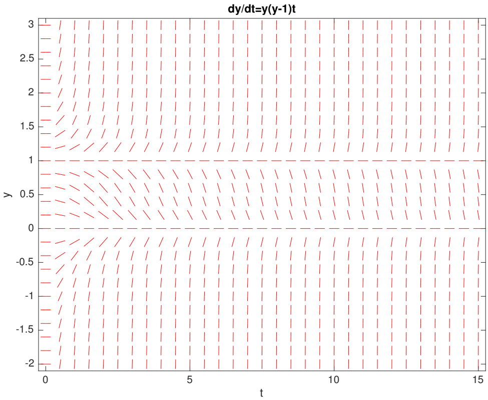
Continued

Intuitively, from the above table one can see that, for any integral curve $y = \varphi(t)$:

Horizontal Strips	The Limit
$1 < \varphi(0) < \infty$	$\lim_{\to\infty} \varphi(t) = \infty$
$0 < \varphi(0) < 1$	$\lim_{\to\infty} \varphi(t) = 0$
$-\infty < \varphi(0) < 0$	$\lim_{\to\infty} \varphi(t) = 0$

Intuitively, we conclude that $y = 0$ is a Stable Equilibrium solution and $y = 1$ is an Unstable Equilibrium solution.

The Direction fields also indicate the same, intuitively.



Analytic Solution

To conclude the same fully analytically, we solve the ODE. It is a separable equation and we have

$$\int \frac{1}{y(y-1)}dy = \int tdt + c$$

Use method of partial fractions, we get

$$-\int \frac{1}{y}dy + \int \frac{1}{(y-1)}dy = \frac{t^2}{2} + c + c \Longrightarrow$$

$$\ln\left(\frac{|y-1|}{|y|}\right) = \frac{t^2}{2} + c \Longrightarrow \left|\frac{y-1}{y}\right| = Ce^{\frac{t^2}{2}} \Longrightarrow$$

Continued

For an integral curve $y = \varphi(t)$, (with $\varphi(0) \neq 0$ of $\varphi(0) \neq 1$), we have

$$\left|\frac{\varphi(0) - 1}{\varphi(0)}\right| = C \Longrightarrow$$

$$\left|\frac{y - 1}{y}\right| = \left|\frac{\varphi(0) - 1}{\varphi(0)}\right| e^{\frac{t^2}{2}}$$

Continued

For an integral curve $y = \varphi(t)$, as above, we have:

- If $1 < \varphi(0)$, then

$$\left|\frac{y-1}{y}\right| = \frac{\varphi(0)-1}{\varphi(0)} e^{\frac{t^2}{2}} \Longrightarrow \frac{y-1}{y} = \frac{\varphi(0)-1}{\varphi(0)} e^{\frac{t^2}{2}}$$

(Last step is justified, because for $y = \varphi(t)$ and $y-1$ to have opposite sign, $\varphi(t_0) = 1$, for some t_0. That would be contradiction, because at $t = t_0$, $LHS = 0$ and $RHS \neq 0$.)

- So,

$$y = \varphi(t) = \frac{1}{1 - \left(\frac{\varphi(0)-1}{\varphi(0)} e^{\frac{t^2}{2}}\right)} \quad \text{which is undefined,}$$

for $t = t_0 > 0$, given by

$$\frac{\varphi(0) - 1}{\varphi(0)} e^{\frac{t_0^2}{2}} = 1. \quad \text{Also, note} \quad 0 < \frac{\varphi(0) - 1}{\varphi(0)} < 1$$

- In deed, we have the left limit $\lim_{t \to t_0^-} \varphi(t) = \infty$
- So, $y = 1$ is NOT a stable Equilibrium solution.

Continued

- If $0 < \varphi(0) < 1$, then

$$\left|\frac{y-1}{y}\right| = -\frac{\varphi(0)-1}{\varphi(0)} e^{\frac{t^2}{2}} \implies -\frac{y-1}{y} = -\frac{\varphi(0)-1}{\varphi(0)} e^{\frac{t^2}{2}}$$

(*Last step is justified, as above.*) So,

$$y = \varphi(t) = \frac{1}{1 + \left(\frac{1-\varphi(0)}{\varphi(0)} e^{\frac{t^2}{2}}\right)}$$

(*Note, the denominator is always negative.*)

Continued

Therefore,

$$\lim_{t\to\infty} \varphi(t) = \lim_{t\to\infty} \left(\frac{1}{1 + \left(\frac{1-\varphi(0)}{\varphi(0)} e^{\frac{t^2}{2}} \right)} \right) = 0$$

Continued

- If $\varphi(0) < 0$, then

$$\left|\frac{y-1}{y}\right| = \frac{\varphi(0)-1}{\varphi(0)} e^{\frac{t^2}{2}} \Longrightarrow \frac{y-1}{y} = \frac{\varphi(0)-1}{\varphi(0)} e^{\frac{t^2}{2}}$$

(*Last step is justified, as above.*) So,

$$y = \varphi(t) = \frac{1}{1 - \left(\frac{\varphi(0)-1}{\varphi(0)} e^{\frac{t^2}{2}}\right)}$$

Continued

Note $\varphi(0) - 1 < \varphi(0) < 0$. So, $\frac{\varphi(0)-1}{\varphi(0)} > 1$.
Therefore, the denominator of $\varphi(t)$, above
$1 - \left(\frac{\varphi(0)-1}{\varphi(0)} e^{\frac{t^2}{2}}\right) < 0$ for all t.

$$\text{Therefore,} \quad \lim_{t\to\infty} \varphi(t) = \lim_{t\to\infty} \left(\frac{1}{1 - \left(\frac{\varphi(0)-1}{\varphi(0)} e^{\frac{t^2}{2}}\right)} \right) = 0$$

Finally, we conclude that $y = 0$ is a Stable Equilibrium, as we intuitively concluded before.

Autonomous Equations

While discussing Stability of integral curves of (1), some authors and instructors (Textbooks and Online Lecture Notes) considered Autonomous Equations.

- An ODE of the form

$$\frac{dy}{dt} = f(y) \tag{2}$$

  is said to be an Autonomous ODE. That means, when $f(t, y) = f(y)$ can be written as a function of y only, (*while, a* $y = y(t)$ *is a function of* t)

- Simplest among them is

$$\frac{dy}{dt} = ry \tag{3}$$

This ODE is called the Exponential Growth Model, which we discussed before.

Why Autonomous?

Preference for Autonomous Equations, in this context may be the following:

- Autonomous Equations are separable, hence solved

$$\text{as follows}: \quad \int \frac{1}{f(y)} dy = \int dt + c$$

  This will give a solution, at least in implicit form. Give initial value $y(0) = y_0$, c is also determined.

Continued

- Secondly, along a horizontal line $y = y_0$, $\frac{dy}{dt}_{|y=y_0} = f(y_0)$ remains constant. Hence, direction tangents remain parallel, along this line.
- Third, for such an ODE (2), the critical points occur only on the Equilibrium solutions $y = y_0$, where $f(y_0) = 0$. There is no other point (t, y), where direction tangent is horizontal. All the Examples given above, have critical points outside the Equilibrium solutions.
- We would discuss a few Autonomous Equations, in this context of Stability and Equilibrium.

Example A1

Consider the ODE

$$\frac{dy}{dt} = (y+1)y(y-1) \quad \text{with} \quad -\infty < y_0 < \infty$$

- Determine the Equilibrium Solutions.
- Classify them as Stable or unstable Equilibrium.

Solution:

To compute Equilibrium Solutions, we solve

$$\frac{dy}{dt} = f(y,t) = (y+1)y(y-1) = 0. \quad \text{Therefore,}$$

$$\begin{cases} y = -1 & \forall\ t \geq 0 \\ y = 0 & \forall\ t \geq 0 \\ y = 1 & \forall\ t \geq 0 \end{cases} \qquad \text{are the Equilibrium Solutions.}$$

The Equilibriums divide the *ty*-plane into horizontal strips. Check the signs of $\frac{dy}{dt} = f(y,t) = (y+1)y(y-1) = 0$ in these strips.

Horizontal Strips	$sign(f(y,t))$	Behavior of $y = \varphi(t)$
$1 < y < \infty$	$+$	increasing
$0 < y < 1$	$-$	decreasing
$-1 < y < 0$	$+$	increasing
$-\infty < y < -1$	$-$	decreasing

In the third column, $y = \varphi(t)$ denotes a solution to the ODE.

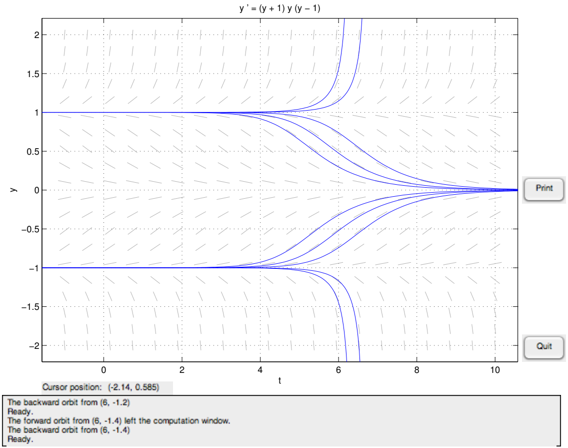
Continued

Intuitively, from the above table one can see that, for any integral curve $y = \varphi(t)$:

Horizontal Strips	The Limit
$1 < \varphi(0) < \infty$	$\lim_{\to\infty} \varphi(t) = \infty$
$0 < \varphi(0) < 1$	$\lim_{\to\infty} \varphi(t) = 0$
$-1 < \varphi(0) < 0$	$\lim_{\to\infty} \varphi(t) = 0$
$-\infty < \varphi(0) < -1$	$\lim_{\to\infty} \varphi(t) = -\infty$

Intuitively, we conclude that $y = 0$ is a Stable Equilibrium and $y = -1$, $y = 1$ are Unstable.

The Direction fields also indicate the same conclusion, intuitively.



Analytic Solution

To conclude the same fully analytically, we solve the ODE. It is a separable equation and we have

$$\int \frac{1}{(y+1)y(y-1)}dy = \int dt + c$$

Use method of partial fractions, we get

$$\int \frac{1}{2(y+1)}dy - \frac{1}{y}dy + \int \frac{1}{2(y-1)}dy = t + c \Longrightarrow$$

$$\ln\left(\frac{\sqrt{|y^2-1|}}{|y|}\right) = t + c \Longrightarrow \sqrt{\left|1 - \frac{1}{y^2}\right|} = Ce^t \Longrightarrow$$

$$\left|1 - \frac{1}{y^2}\right| = Ce^{2t}$$

For an integral curve $y = \varphi(t)$, we have

$$\left|1 - \frac{1}{\varphi(0)^2}\right| = C \Longrightarrow$$

$$\left|1 - \frac{1}{y^2}\right| = \left|1 - \frac{1}{\varphi(0)^2}\right| e^{2t}$$

Stability Questions

First we examine whether $y = 0$ is Stable or not?
In the Horizontal Strip $-1 < y < 1$, the solutions $y = \varphi(t)$ is:

$$-\left(1 - \frac{1}{y^2}\right) = -\left(1 - \frac{1}{\varphi(0)^2}\right) e^{2t}$$

So, whenever $-1 < \varphi(0) < 1$, $\varphi(0) \neq 0$, we have

$$-\left(1 - \frac{1}{\varphi(t)^2}\right) = -\left(1 - \frac{1}{\varphi(0)^2}\right) e^{2t}$$

$$\text{So,}\quad \left(\frac{1}{\varphi(t)^2}-1\right)=\left(\frac{1}{\varphi(0)^2}-1\right)e^{2t}\Longrightarrow$$

$$\frac{1}{\varphi(t)^2}=1+\left(\frac{1}{\varphi(0)^2}-1\right)e^{2t}\Longrightarrow$$

$$\varphi(t)=\pm\sqrt{\frac{1}{1+\left(\frac{1}{\varphi(0)^2}-1\right)e^{2t}}}$$

The $+$ or $-$ sign depends on, whether $\varphi(0)$ is positive or negative.

Taking limits, we have

$$\lim_{t\to\infty} \varphi(t) = \pm \lim_{t\to\infty} \sqrt{\frac{1}{1 + \left(\frac{1}{\varphi(0)^2} - 1\right) e^{2t}}} = 0$$

Therefore, such an integral curve $y = \varphi(t)$ stabilizes to $y = 0$. Hence, $y = 0$ is a stable Equilibrium.

Continued

Now examine, whether the Equilibrium Solution $y = -1$ is Stable or not.

- We already saw, for an integral curve $y = \varphi(t)$, with $-1 < \varphi(0) < 0$ (above the line $y = -1$), the limit $\lim_{t\to\infty} \varphi(t) = 0$.
- So, $y = \varphi(t)$ stabilizes to $y = 0$, not to $y = 1$.
- So, $y = 1$ is not a stable Equilibrium.
- *Note, to check failure of stability, it was no more necessary to check what happens, if* $\varphi(0) > 1$.

Continued

Likewise examine, whether the Equilibrium Solution $y = 1$ is Stable or not.

- We already saw, for an integral curve $y = \varphi(t)$, with $0 < \varphi(0) < 1$ (below the line $y = 1$), the limit $\lim_{t \to \infty} \varphi(t) = 0$.
- So, $y = \varphi(t)$ stabilizes to $y = 0$, not to $y = -1$.
- So, $y = -1$ is not a stable Equilibrium.
- *Note, to check failure of stability, it was no more necessary to check what happens, if $\varphi(0) < -1$.*

Example A2

Example Consider the autonomous ODE

$$\frac{dy}{dt} = e^{y(y-1)} - 1 \quad \text{with} \quad -\infty < y_0 < \infty$$

- Determine the equilibrium (critical) points.
- Sketch the integral curves, as needed, and classify the solutions as asymptotically stable or unstable.
- *We would avoid Analytic Solutions!*

Solution

- We use Matlab. We make a few comments:
- The equilibrium solutions are given by
  $\frac{dy}{dt} = e^{y(y-1)} - 1 = 0$
- So, the quilibrium solutions are $y = 0$ and $y = 1$.

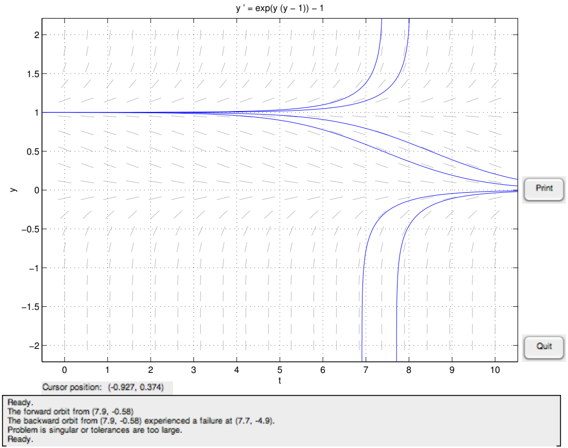
Continued

- Sign of $f(y) = e^{y(y-1)} - 1$ is given by

$$sign(f(y)) = \begin{cases} + & \text{if } -\infty < y < 0 \\ - & \text{if } 0 < y < 1 \\ + & \text{if } 1 < y < \infty \end{cases}$$

- Mentally think of the direction fields (or sketch on a paper), it becomes clear that $y = 0$ is asymptotically stable and $y = 1$ is asymptotically unstable.

The Direction Field: Look at how if behaves on the right side. This shows stable at $y = 0$, not stable at $y = 1$:



Example A3

Example Consider the autonomous of ODE

$$\frac{dy}{dt} = y - \sqrt{y} \quad \text{with} \quad 0 \leq y_0 < \infty$$

- Determine the equilibrium (critical) points.
- Sketch the integral curves, as needed, and classify the solutions as asymptotically stable or unstable.
- *Avoid Analytic Solution*

Solution

- We use Matlab. We make a few comments:
- The equilibrium solutions are given by $\frac{dy}{dt} = y - \sqrt{y} = 0$
- So, the equilibrium solutions are $y = 0$ and $y = 1$.

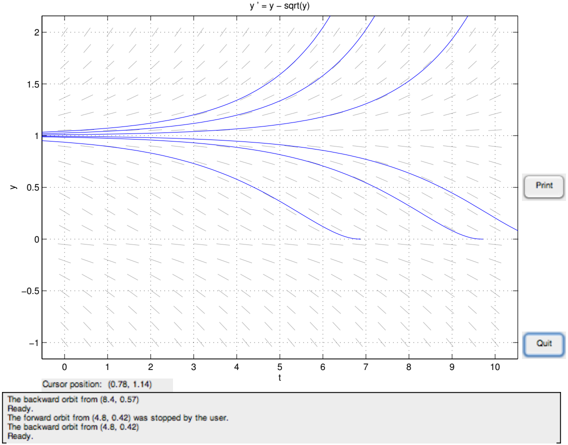
Continued

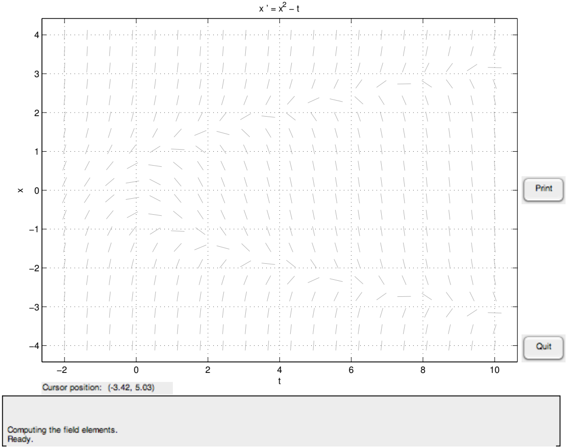
- Sign of $f(y) = y - \sqrt{y}$ is given by

$$sign(f(y)) = \begin{cases} - & \text{if } 0 < y < 1 \\ + & \text{if } 1 < y < \infty \end{cases}$$

- Mentally think of the direction fields (or sketch on a paper), it becomes clear that $y = 0$ is asymptotically stable and $y = 1$ are asymptotically unstable.

The Direction Field:Look at how if behaves on the right side. Stable at $y = 0$, not stable at $y = 1$:




x ' = x2 – t
x
t
Cursor position: (-3.42, 5.03)
Print
Quit
Computing the field elements.
Ready.