Chapter 2: First Order ODE §2.7 Exact Equations

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First Order ODE

- Recall the general form of the First Order DEs (FODE):

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

(In this section use x as the independent variable; not t.)

- This can also be written as

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{2}$$

with variety of choices of $M(x, y), N(x, y)$.

Exact FODE

Sometimes, we may be lucky:

- Suppose there is a function $\psi(x,y)$ such that

$$\frac{\partial\psi}{\partial x}(x,y) = M(x,y) \quad \text{and} \quad \frac{\partial\psi}{\partial y}(x,y) = N(x,y). \qquad (3)$$

- In that case, the equation (2) would reduce to

$$\frac{d\psi}{dx} = \frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{dy}{dx} = M + N\frac{dy}{dx} = 0 \implies \psi(x,y) = c$$

where c is an arbitrary constant.

Exact FODE

- Therefore, any solution $y = \varphi(x)$ of (2) would satisfy the equation $\psi(x, \varphi(x)) = c$.
- If we can find such a function $\psi(x, y)$, DE (2) would be called an Exact ODE.
- Question: When or how can we tell that a ODE of the type (2) is exact? The Answer: The following Theorem

Exactness Theorem

Theorem 2.7.1

- Assume $M(x,y), N(x,y), \frac{\partial M}{\partial y}, \frac{\partial N}{\partial x}$ are continuous on

$$\text{the rectangle} \quad \mathcal{R} := \left\{ (x,y) : \begin{array}{c} a < x < b \\ c < y < d \end{array} \right\}$$

- Then the ODE (2) is EXACT in $\mathcal{R}$ $\Longleftrightarrow$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{at each} \quad (x,y) \in \mathcal{R} : . \qquad (4)$$

Continued

- That means there is a function $\psi(x, y)$ satisfying equation

$$(3) : \frac{\partial \psi}{\partial x} = M, \frac{\partial \psi}{\partial y} = N \iff M, N \text{ satisfy equation } (4).$$

- Proof. We skip the proof. However, same steps in the proof is followed to solve such equations.
- For solving problems, it is important to follow the steps, as in the following samples.
- **Remark.** Note that Equation (3) is a Partial Differential Equation (PDE), which is cover in a latter course.

Example 1

Consider the ODE

$$(7x + 4y) + (2x - 13y)\frac{dy}{dx} = 0 \qquad (5)$$

- Determine, if it is exact.
- If yes, solve it.

Continued: Solution

- Here $M = (7x + 4y)$ and $N = 2x - 13y$.
- So, $\frac{\partial M}{\partial y} = 4$ and $\frac{\partial N}{\partial x} = 2$
- So, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. So, equation (5) is not exact.
- In the next Sample, I change equation (5) slightly, to make it exact.

Example 2

Consider the DE

$$(2 + 4yx) + (2x^2 - 2y)\frac{dy}{dx} = 0 \tag{6}$$

- Determine, if it is exact.
- If yes, solve it.

Example 2

- Here $M = 2 + 4yx$ and $N = 2x^2 - 2y$.
- So, $\frac{\partial M}{\partial y} = 4x$ and $\frac{\partial N}{\partial x} = 4x$
- So, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 4x$. So, equation (6) is exact.

Continued: Step 1

- We set

$$\frac{\partial \psi}{\partial x} = M = 2 + 4yx, \qquad \frac{\partial \psi}{\partial y} = N = 2x^2 - 2y \qquad (7)$$

- Integrating the first one, with respect to x:

$$\psi(x, y) = \int (2 + 4yx)dx + h(y)$$

where $h(y)$ is function of y, to be determined. So,

$$\psi(x, y) = 2x + 2x^2y + h(y) \qquad (8)$$

Continued: Step 2

- Differentiating equation (8), then combining with the second part of the equation (7)

$$\frac{\partial \psi}{\partial y} = 2x^2 + \frac{dh(y)}{dy} = 2x^2 - 2y \qquad (9)$$

- (*we $h(y)$ to be independent of x*) and we have:

$$\frac{dh(y)}{dy} = -2y \quad \Longrightarrow \quad h(y) = -y^2$$

(We do not need to have a constant).

- Combining with equation (8), we have

$$\psi(x,y) = 2x + 2x^2 y + h(y) \quad \Longrightarrow \quad \psi(x,y) = 2x + 2x^2 y - y^2$$

The Answer

- So, the general solution to the DE (6) is

$$\psi(x, y) = 2x + 2x^2y - y^2 = c$$

where c is an arbitrary constant.
- For each value of c, we get a solution of (6).

Example 3

Consider the ODE

$$\frac{dy}{dx} = \frac{-e^x + by}{bx - cy} \tag{10}$$

- Determine, under what condition, the ODE is exact?
- Then, solve it.

Solution

- Rewrite the ODE (10): $-(-e^x + by) + (bx - cy)\frac{dy}{dx} = 0$
- Here $M = -(-e^x + by)$ and $N = bx - cy$.
- So, $\frac{\partial M}{\partial y} = -b$ and $\frac{\partial N}{\partial x} = b$
- So, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ if $-b = b$. So, equation (10) if $b = 0$.
- With $b = 0$, we rewrite DE (10) as:

$$e^x - cy\frac{dy}{dx} = 0, \quad \text{With} \quad M = e^x, \ N = -cy. \qquad (11)$$

Continued

- We set

$$\frac{\partial\psi}{\partial x} = M = e^x, \qquad \frac{\partial\psi}{\partial y} = N = -cy \tag{12}$$

- Integrating the first one, with respect to x:

$$\psi(x,y) = \int e^x dx + h(y)$$

where $h(y)$ is function of y, to be determined. So,

$$\psi(x,y) = e^x + h(y) \tag{13}$$

Continued

- Differentiate (13), with respect to y and use (12)

$$\frac{\partial \psi}{\partial y} = \frac{dh(y)}{dy} = -cy \tag{14}$$

- (*we expect* $h(y)$ *to be independent of* x) and we have:

$$\frac{dh(y)}{dy} = -cy \quad \Longrightarrow \quad h(y) = -\frac{cy^2}{2}$$

(We do not need to have a constant).

Continued

- From (13), we have

$$\psi(x, y) = e^x + h(y) \quad \Longrightarrow \quad \psi(x, y) = e^x - \frac{cy^2}{2}$$

- So, the general solution to the DE (10) is

$$\psi(x, y) = e^x - \frac{cy^2}{2} = c$$

where c is an arbitrary constant.

Example 4

Consider the ODE

$$(y \cos x + 6x) + (\sin x + e^{y})\frac{dy}{dx} = 0 \qquad (15)$$

- Determine, if it is exact.
- If yes, solve it.

Solution: Step 1

- Here $M = y\cos x + 6x$ and $N = \sin x + e^y$.
- So, $\frac{\partial M}{\partial y} = \cos x$ and $\frac{\partial N}{\partial x} = \cos x$
- So, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ So, the ODE (15) is exact.

Step 2

- We set

$$\frac{\partial \psi}{\partial x} = M = y\cos x + 6x, \qquad \frac{\partial \psi}{\partial y} = N = \sin x + e^y \quad (16)$$

- Integrating the first one, with respect to x:

$$\psi(x,y) = \int (y\cos x + 6x)\, dx + h(y)$$

where $h(y)$ is function of y, to be determined. So,

$$\psi(x,y) = y\sin x + 3x^2 + h(y) \qquad (17)$$

Step 3

- Differentiate (17), with respect to y and use (12)

$$\frac{\partial \psi}{\partial y} = \sin x + \frac{dh}{dx} = \sin x + e^y \qquad (18)$$

- (*we* $h(y)$ *to be independent of* x) and we have:

$$\frac{dh(y)}{dy} = e^y \quad \Longrightarrow \quad h(y) = e^y$$

(We do not need to have a constant).

Answer

- From (17), we have

$$\psi(x,y) = y\sin x + 3x^2 + h(y) = y\sin x + 3x^2 + e^y$$

- So, the general solution to the DE (15) is

$$\psi(x,y) = y\sin x + 3x^2 + e^y = c$$

where c is an arbitrary constant.