Chapter 4: Higher Order ODE
§4.1 General Overview of Theory

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Objective

- Having discussed 1^{st} and 2^{nd}-Order ODE in previous chapters, by Higher Order ODE, we would mean ODE of order three and above.
- In this short exposition of Higher Order ODE, we discuss that the Theory of Higher Order ODE is strikingly similar to that of second order ODE, as discussed in Chapter 3.
- We would give an overview of the same, so that you get the flavor, and not go deep into solving problems.

ODE of Order n

- For an integer $n \geq 1$, an ODE of order n is given by

$$\frac{d^n y}{dt^n} = f\left(t, y, \frac{dy}{dt}, \ldots, \frac{d^{n-1}y}{dt^{n-1}}\right) \tag{1}$$

Sometime, we use the notation: $y^{(r)} := \frac{d^r y}{dt^r}$. So, the above can be written as This is also written as

$$y^{(n)} = f\left(t, y, y', y^{(2)}, \ldots, y^{(n-1)}\right)$$

Linear ODE of Order n

An ODE of order n, in either one of following two forms

$$\begin{cases} \frac{d^n y}{dt^n} + p_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_1(t)\frac{dy}{dt} + p_0(t)y = g(t) \\ P_n(t)\frac{d^n y}{dt^n} + P_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + P_1(t)\frac{dy}{dt} + P_0(t)y = g(t) \end{cases} \quad (2)$$

would be called a Linear ODE of order n. We usually assume that $p_i(t)$, $P_i(t)$, $g(t)$ are continuous on an open interval I.

Linear Operators

In the context of (2), define linear operators:

$$\left\{ \begin{array}{l} \mathcal{L} := \frac{d^n}{dt^n} + p_{n-1}(t)\frac{d^{n-1}}{dt^{n-1}} + \cdots + p_1(t)\frac{d}{dt} + p_0(t) \\ \\ \mathcal{L} := P_n(t)\frac{d^n}{dt^n} + P_{n-1}(t)\frac{d^{n-1}}{dt^{n-1}} + \cdots + P_1(t)\frac{d}{dt} + P_0(t) \end{array} \right. \tag{3}$$

Such operators act on all n-times differentiable functions $y = y(t)$. Further, the Linear ODE (2) can be written as

$$\mathcal{L}(y) = g(t) \tag{4}$$

An Initial Value Problem

Definition. Let $\mathcal{L}$ be a differential operator, as in (3). An Initial Value Problem (IVP), of order n is as follows:

$$\begin{cases} \mathcal{L}(y) = g(t) & \text{as in (1)} \\ y(t_0) = y_0, y'(t_0) = y_1, \ldots, y^{(n-1)}(t_0) = y_{n-1} & t_0 \in I \end{cases} \tag{5}$$

The Existence and Uniqueness Theorem

Theorem 4.1.1. Consider the Initial value Problem (5). As before, assume $p_i(t), g(t)$ or $P(t), g(t)$ are continuous on the interval I. Then,

- The IVP (5) has a solution $y = \varphi(t)$.
- The domain of $y = \varphi(t)$ is I,
- The solution $y = \varphi(t)$ is unique, on I.

Homogeneous Linear ODE

Consider the Linear ODE (2). If $g(t) = 0$, in (2), then (2), would be called Homogenous. So, a homogenous ODE can be written as

$$\mathcal{L}(y) = 0 \tag{6}$$

where as in (3)

$$\begin{cases} \mathcal{L} := \frac{d^n}{dt^n} + p_{n-1}(t)\frac{d^{n-1}}{dt^{n-1}} + \cdots + p_1(t)\frac{d}{dt} + p_0(t) \\ \mathcal{L} := P_n(t)\frac{d^n}{dt^n} + P_{n-1}(t)\frac{d^{n-1}}{dt^{n-1}} + \cdots + P_1(t)\frac{d}{dt} + P_0(t) \end{cases} \tag{7}$$

Linearity Lemmas

Lemma 4.1.2 Let $\mathcal{L}$ be a differential operator, as in (7). Then, for any two n-times differentiable functions $y = \varphi_1(t)$ and $y = \varphi_2(t)$, and real numbers c_1, c_2, we have

$$\mathcal{L}(c_1\varphi_1 + c_2\varphi_2) = c_1\mathcal{L}(\varphi_1) + c_2\mathcal{L}(\varphi_2)$$

Linear Combination of Solutions

Lemma 4.1.3 Let $y = y_1(t), y = y_2(t), \cdots, y = y_k(t)$ be solutions of the Homogeneous ODE (6), and $c_1, \ldots, c_k$ be real numbers. Then, the linear combination

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_k y_k \quad \text{is a solutions of (6).}$$

Proof. Follows from Lemma 4.1.1 ∎

Further Goals

We know,

- The Linear Homogeneous ODE (6) has a trivial solution $y = 0$.
- By Lemma 4.1.3, any constant linear combination of solutions of (6) is also a solution of (6) .

Continued

Recall, n is the Order of the Linear Homogeneous ODE (6) .

- Question: Fix n solutions $y = y_1(t), y = y_2(t), \cdots, y = y_n(t)$ of the Linear Homogenous ODE (6). Suppose, $y = \varphi(t)$ is any other solution of (6). Question is, whether or when we write write φ as a constant linear combinations of $y = y_1(t), y = y_2(t), \cdots, y = y_n(t)$?

  As in the case of Order two ODE, we answer this question subsequently.

Definition: The Fundamental Set

Definition: Fix n solutions $y = y_1(t), y = y_2(t), \cdots, y = y_n(t)$ of the Linear Homogenous ODE (6). We say that they form a Fundamental Set of solutions of (6), if any solution $y = \varphi(t)$ of (6) is a constant linear combination That means, if

$$y = \varphi(t) = \sum_{i=1}^{n} c_i y_i(t) \quad \text{for some } c_1, \ldots, c_n \in \mathbb{R}, \ \forall \ t \in I,$$

Wronskian

Definition. Let $y = y_1(t), y = y_2(t), \cdots, y = y_n(t)$ be $n-1$-times differentiable functions on an open interval $I : \alpha < t < \beta$. The Wronskian $W(t)$, of these functions is defined to be the determinant function:

$$W(t) = \begin{vmatrix} y_1(t) & y_2(t) & y_3(t) & \cdots & y_n(t) \\ y_1'(t) & y_2'(t) & y_3'(t) & \cdots & y_n'(t) \\ y_1^{(2)}(t) & y_2^{(2)}(t) & y_3^{(2)}(t) & \cdots & y_n^{(2)}(t) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & y_3^{(n-1)}(t) & \cdots & y_n^{(n-1)}(t) \end{vmatrix} \quad t \in I \tag{8}$$

Continued

Sometimes, to indicate its dependence on $y = y_1(t), y = y_2(t), \cdots, y = y_n(t)$, the Wronskian $W(t)$ is denoted by

$$W(y_1, y_2, \cdots, y_n)(t) := W(t)$$

The (Wronskian) Theorem 4.1.4

Theorem 4.1.4 We consider the former, of the two, forms of the Linear Homogeneous ODE

$$\mathcal{L}(y) = \frac{d^n y}{dt^n} + p_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_1(t)\frac{dy}{dt} + p_0(t)y = 0 \quad (9)$$

Assume $p_i(t)$ are continuous on and open interval I. Fix n solutions $y = y_1(t), y = y_2(t), \cdots, y = y_n(t)$ of (9).
Let $W(t)$ denote the Wronskian of $y = y_1(t)$, $y = y_2(t)$, $\cdots$, $y = y_n(t)$.

Theorem 4.1.4: Continued

Then, the following three conditions are equivalent:

(1) $W(t) \neq 0$ for all $t \in I$.

(2) $W(t_0) \neq 0$ for some $t_0 \in I$.

(3) $y = y_1(t), y = y_2(t), \cdots, y = y_n(t)$ form a Fundamental set of Solutions of (9).

Rest of This Chapter

Goal of this chapter remains to provide a flavor of the theory of Higher Order ODEs.

- The next section gives a overview of Homogeneous Linear ODE (6), with constant coefficients.
- The last section comments on the Methods to solve Nonhomogenous Linear ODE with constant coefficients. Again, these methods are strikingly similar, to that of 2^{nd}-Order Linear ODEs, namely, the Method of Variation of Parameter and the Method of Undetermined Coefficients.

Definition

Definition A Homogeneous Linear ODE (6) is said to have constant coefficient, if $p_i(t), P_i(t)$ are constant functions. So, a linear Homogeneous ODE, of order n, with constant coefficients looks like

$$\mathcal{L}(y) = a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = 0 \quad (10)$$

with $a_0, a_1, \cdots, a_n \in \mathbb{R}$ and $a_n \neq 0$.

Definition

Definition A nonHomogeneous Linear ODE (2) is said to have constant coefficient, if $p_i(t), P_i(t)$ are constant functions. So, a linear Homogeneous ODE, of order n, with constant coefficients looks like

$$\mathcal{L}(y) = a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = g(t) \quad (11)$$

with $a_0, a_1, \cdots, a_n \in \mathbb{R}$, $a_n \neq 0$ and $g(t) \neq 0$.