Chapter 6: The Laplace Transform §6.4 Systems with Discontinuous Functions

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Discontinuous Force Functions

- As in §6.2, in this section we solve systems of nonhomogeneous linear second order with constant coefficients, using Laplace Transform and Inverse Laplace Transforms.
- However, the nonhomogeneous part $g(t)$ would be discontinuous, mostly with an eye for the Derac δ function and $d_\tau(t-c)$.
- As in §6.2, usully we would use the charts of Laplace Transform and Inverse Laplace Transforms.

Example 1

Solve the IVP

$$y'' + 9y = f(t) = \begin{cases} 1 & \pi \leq t < 3\pi \\ 0 & 3\pi \leq t < \infty \end{cases} \qquad y(0) = 0, y'(0) = 1.$$

- We write $f(t) = u_\pi(t) - u_{3\pi}(t)$.
- So, we have $y'' + 9y = u_\pi(t) - u_{3\pi}(t)$
- We write $Y(s) = \mathcal{L}\{y\}$. So,

$$\mathcal{L}\{y''\} + 9\mathcal{L}\{y\} = \mathcal{L}\{u_\pi\} - \mathcal{L}\{u_{3\pi}\} \implies$$

$$(s^2Y - sy(0) - y'(0)) + 9Y = \frac{e^{-\pi s}}{s} - \frac{e^{-3\pi s}}{s} \Longrightarrow$$

$$(s^2 + 9)Y = 1 + \frac{e^{-\pi s}}{s} - \frac{e^{-3\pi s}}{s} \Longrightarrow$$

$$Y = \frac{1}{s^2+1} + \frac{e^{-\pi s}}{s(s^2+1)} - \frac{e^{-3\pi s}}{s(s^2+1)}$$

$$= \mathcal{L}\{\sin t\} + \left(e^{-\pi s} - e^{-3\pi s}\right)\left(\frac{a}{s} + \frac{bs+c}{s^2+1}\right)$$

$$= \mathcal{L}\{\sin t\} + \left(e^{-\pi s} - e^{-3\pi s}\right)\left(\frac{1}{s} - \frac{s}{s^2+1}\right)$$

Continued

$$Y = \mathcal{L}\{\sin t\} + \frac{e^{-\pi s}}{s} - \frac{e^{-3\pi s}}{s} + \frac{se^{-\pi s}}{s^2+1} - \frac{se^{-3\pi s}}{s^2+1}$$

$$= \mathcal{L}\{\sin t\} + \mathcal{L}\{u_\pi\} - \mathcal{L}\{u_{3\pi}\} + e^{-\pi s}\mathcal{L}\{\cos t\} - e^{-3\pi s}\mathcal{L}\{\cos t\}$$

$$= \mathcal{L}\{\sin t\} + \mathcal{L}\{u_\pi\} - \mathcal{L}\{u_{3\pi}\} + \mathcal{L}\{u_\pi \cos(t-\pi)\} - \mathcal{L}\{u_{3\pi} \cos(t-3\pi)\}$$

$$= \mathcal{L}\{\sin t\} + \mathcal{L}\{u_\pi\} - \mathcal{L}\{u_{3\pi}\} - \mathcal{L}\{u_\pi \cos(t)\} + \mathcal{L}\{u_{3\pi} \cos(t)\}$$

$$\text{So } Y = \mathcal{L}\{y\} = \mathcal{L}\left\{\sin t + u_\pi - u_{3\pi} - u_\pi \cos(t) + u_{3\pi}\cos(t)\right\}$$

$$\text{So} \quad y = \sin t + u_\pi - u_{3\pi} - u_\pi \cos(t) + u_{3\pi}\cos(t)$$

Example 2

Solve the IVP

$$y'' + y' - 6y = u_2(t) \quad y(0) = 0, y'(0) = 1.$$

- We write $Y(s) = \mathcal{L}\{y\}$. There are two steps:
  - Compute $Y(s)$, by application of Laplace transform $\mathcal{L}$.
  - Compute $y = \mathcal{L}^{-1}\{Y(s)\}$ by application of Inverse Laplace transform $\mathcal{L}$.

$$\mathcal{L}\{y''\} + \mathcal{L}\{y'\} - 6\mathcal{L}\{y\} = \mathcal{L}\{u_2(t)\}$$

$$(s^2Y(s) - sy(0) - y'(0)) + (sY(s) - y(0)) - 6Y(s) = \frac{1}{s}e^{-2s}$$

$$(s^2Y(s) - 1) + sY(s) - 6Y(s) = \frac{1}{s}e^{-2s}$$

$$Y(s) = \frac{1}{s(s^2+s-6)}e^{-2s} + \frac{1}{s^2+s-6}$$

$$= \frac{1}{s(s^2+3s+2)}e^{-2s} + \frac{1}{s-2} - \frac{1}{s+3}$$

$$= \frac{1}{s(s^2+3s+2)}e^{-2s} + \mathcal{L}\{e^{2t}\} - \mathcal{L}\{e^{-3t}\}$$

Continued

$$\text{So,}\quad Y = \frac{1}{s(s+3)(s-2))}e^{-2s} + \mathcal{L}\{e^{2t} - e^{-3t}\}$$

$$= e^{-2s}\left(\frac{a}{s} + \frac{b}{s-2} + \frac{c}{s+3}\right) + \mathcal{L}\{e^{2t} - e^{-3t}\}$$

$$= e^{-2s}\left(-\frac{1}{6}\frac{1}{s} + \frac{1}{10}\frac{1}{s-2} + \frac{1}{15}\frac{1}{s+3}\right) + \mathcal{L}\{e^{2t} - e^{-3t}\}$$

$$= e^{-2s}\left(-\frac{1}{6}\mathcal{L}\{1\} + \frac{1}{10}\mathcal{L}\{e^{2t}\} + \frac{1}{15}\mathcal{L}\{e^{-3t}\}\right) + \mathcal{L}\{e^{2t} - e^{-3t}\}$$

$$= e^{-2s}\left(\mathcal{L}\left\{-\frac{1}{6} + \frac{1}{10}e^{2t} + \frac{1}{15}e^{-3t}\right\}\right) + \mathcal{L}\{e^{2t} - e^{-3t}\}$$

Continued

$$\text{So,} \quad Y = e^{-2s}\left(\mathcal{L}\left\{\frac{-5+3e^{2t}+2e^{-3t}}{30}\right\}\right) + \mathcal{L}\{e^{2t}-e^{-3t}\}$$

$$= \left(\mathcal{L}\left\{u_2(t)\frac{-5+3e^{2(t-2)}+2e^{-3(t-2)}}{30}\right\}\right) + \mathcal{L}\{e^{2t}-e^{-3t}\} \Longrightarrow$$

$$\mathcal{L}\{y\} = Y = \mathcal{L}\left\{u_2(t)\frac{-5+3e^{2(t-2)}+2e^{-3(t-2)}}{30} + e^{2t} - e^{-3t}\right\}$$

$$\text{So,} \quad y = u_2(t)\frac{-5+3e^{2(t-2)}+2e^{-3(t-2)}}{30} + e^{2t} - e^{-3t}$$

Example 3

Solve the IVP

$$y'' + 9y = g(t) = \begin{cases} t & 0 \leq t < 3 \\ 3 & 3 \leq t \end{cases} \qquad y(0) = 0, y'(0) = 1.$$

- We write $Y(s) = \mathcal{L}\{y\}$. There are two steps:
  - Compute $Y(s)$, by application of Laplace transform $\mathcal{L}$.
  - Compute $y = \mathcal{L}^{-1}\{Y(s)\}$ by application of Inverse Laplace transform $\mathcal{L}$.

- Write $g(t) = (1 - u_3(t))t + 3u_3(t) = t - u_3(t)(t-3)$
- From the Charts So,

$$\mathcal{L}\{g(t)\} = \frac{1}{s^2} - e^{-3s}\frac{1}{s^2}$$

- Apply $\mathcal{L}$ the System:

$$\mathcal{L}\{y"\} + 9\mathcal{L}\{y\} = \mathcal{L}\{g(t)\} = \frac{1}{2s^2} - \frac{e^{-6s}}{2s^2}$$

$$s^2Y(s) - sy(0) - y'(0) + 9Y(s) = \frac{1}{s^2} - e^{-3s}\frac{1}{s^2}$$

- $$s^2Y(s) - 1 + 9Y(s) = \frac{1}{s^2} - e^{-3s}\frac{1}{s^2}$$

$$Y(s) = \frac{1}{s^2+9} + \frac{1}{s^2(s^2+9)} - e^{-3s}\frac{1}{s^2(s^2+9)}$$

$$= \frac{1}{s^2+9} + \frac{1}{9}\frac{1}{s^2} - \frac{1}{9}\frac{1}{s^2+9} - e^{-3s}\frac{1}{s^2(s^2+9)}$$

$$= \frac{1}{9}\frac{1}{s^2} + \frac{8}{9}\frac{1}{s^2+9} - e^{-3s}\frac{1}{s^2(s^2+9)}$$

$$= \frac{1}{9}\mathcal{L}\{t\} + \frac{8}{27}\mathcal{L}\{\sin 3t\} - e^{-3s}\frac{1}{s^2(s^2+9)}$$

Continued

So,

$$Y = \mathcal{L}\left\{\frac{3t + 8\sin 3t}{27}\right\} - e^{-3s}\left(\frac{1}{9}\frac{1}{s^2} - \frac{1}{9}\frac{1}{s^2+9}\right)$$

$$= \mathcal{L}\left\{\frac{3t + 8\sin 3t}{27}\right\} - e^{-3s}\left(\frac{1}{9}\mathcal{L}\{t\} - \frac{1}{27}\mathcal{L}\{\sin 3t\}\right)$$

$$= \mathcal{L}\left\{\frac{3t + 8\sin 3t}{27}\right\} - e^{-3s}\mathcal{L}\left\{\frac{3t - \sin 3t}{27}\right\}$$

$$= \mathcal{L}\left\{\frac{3t + 8\sin 3t}{27}\right\} - \mathcal{L}\left\{u_3(t)\frac{3(t-3) - \sin 3(t-3)}{27}\right\}$$

Continued

So, $\mathcal{L}\{y\} = Y$

$$= \mathcal{L}\left\{\frac{3t + 8\sin 3t}{27} - u_3(t)\frac{3(t-3) - \sin 3(t-3)}{27}\right\}$$

So,

$$y = \frac{3t + 8\sin 3t}{27} - u_3(t)\frac{3(t-3) - \sin 3(t-3)}{27}$$