Chapter 6: The Laplace Transform

§6.2 Solutions of Initial Value Problems

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The Goal of this section is to use Laplace Transform to solve Initial value problems, in second order linear ODE (as in Chapter 3).

This way, the methods may become more algebraic.

Two theorem that follows would be instrumental for this method.
Theorem 6.2.1: Suppose $f$ is a continuous function on an interval $0 \leq t \leq \alpha$.

- Assume $f'$ is and is piecewise continuous on the interval $0 \leq t \leq \alpha$.
- Assume there are constants $\kappa, \lambda, \beta$, with $\kappa > 0$, $\beta > 0$, such that

$$|f(t)| \leq \kappa e^{\lambda t} \quad \text{for all} \quad t \geq \beta$$

(*In words, $f$ has (at most) exponential growth.*)

Then,

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0) \quad (1)$$
Corollary 6.2.2: Suppose $f$ is a continuous function on an interval $0 \leq t \leq \alpha$. Assume $f'$, $f^{(2)}$, $\ldots$, $f^{(n-1)}$ are continuous, and $f^{(n)}$ is piecewise continuous on the interval $0 \leq t \leq \alpha$. Assume there are constants $\kappa$, $\lambda$, $\beta$, with $\kappa > 0$, $\beta > 0$,

\[ \exists \quad |f^{(i)}(t)| \leq \kappa e^{\lambda t} \quad \text{for all} \quad i = 0, 1, \ldots, n \quad \text{and} \quad t \geq \beta \]

Then, $\mathcal{L}\{f^{(n)}(t)\} =$

\[ s^n \mathcal{L}\{f(t)\} - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0) \quad (2) \]
The cases $n = 1$ (1) and $n = 2$ will be used more frequently:

- **$n = 1$ case:**

  Then,  
  \[ \mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0) \]  
  (3)

- **$n = 2$ case:**

  \[ \mathcal{L}\{f^{(2)}(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0) \]  
  (4)
1-1 Correspondence

Laplace Transformation can be to solve IVP, due to the following:

- Suppose \( f, g \) are two continuous functions on an interval. Then,
\[
\mathcal{L}\{f(t)\} = \mathcal{L}\{g(t)\} \implies f = g
\]  

- If \( \mathcal{L}\{f(t)\} = F(s) \), we write \( \mathcal{L}^{-1}\{F(s)\} = f \), to be called the inverse Laplace transform of \( g \). Further, inverse Laplace transform is linear, in the sense, for \( \alpha, \beta \in \mathbb{R} \),
\[
\mathcal{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha \mathcal{L}^{-1}\{F(s)\} + \beta \mathcal{L}^{-1}\{G(s)\}
\]
Solving: Charts of Laplace Transform and Inverse Laplace Transforms are available, in the internet and any standard Textbook. Download one and use for this section.

To solve initial value problems, when $y(0) = y_0$, $y'(0) = y'_0$ are given, we compute the Laplace transform $\mathcal{L}(\{\varphi(t)\})$ of the solution $y = \varphi(t)$ and use the chart to compare.
Example 1

Compute the Inverse Laplace Transform of \( F(s) = \frac{3s-6}{s^2-4s+13} \).

- We have
  \[
  F(s) = 3 \frac{s - 2}{(s - 2)^2 + 3^2}
  \]

- By Formula chart:
  \[
  F(s) = 3 \frac{s - 2}{(s - 2)^2 + 3^2} = 3 \mathcal{L}\{e^{2t} \cos 3t\} = \mathcal{L}\{3e^{2t} \cos 3t\}
  \]

- So,
  \[
  \mathcal{L}^{-1}\{F(s)\} = 3e^{2t} \cos 3t
  \]
Example 2

Compute the Inverse Laplace Transform of $F(s) = \frac{-1-2s}{s^2+4s+13}$.

We have

$$F(s) = \frac{-1-2s}{s^2+4s+13} = \frac{-1-2s}{(s+2)^2+3^2}$$

$$= \frac{3}{(s+2)^2+3^2} - 2\frac{s+2}{(s+2)^2+3^2}$$

By Formula Chart:

$$F(s) = \mathcal{L}\{e^{-2t} \sin 3t\} - 2\mathcal{L}\{e^{-2t} \cos 3t\}$$

$$= \mathcal{L}\{e^{-2t} \sin 3t - 2e^{-2t} \cos 3t\}$$
So,

\[ \mathcal{L}^{-1}\{F(s)\} = e^{-2t} \sin 3t - 2e^{-2t} \cos 3t \]
Example 3

Compute the Inverse Laplace Transform of

\[ F(s) = \frac{5s^3 - 7s^2 - 4s}{(s^2 + 2s + 5)(s^2 - 2s + 2)}. \]

► The method of partial fractions is used frequently, in this section. Review all these examples.
Solution

Use method of partial fractions:

\[ F(s) = \frac{5s^3 - 7s^2 - 4s}{(s^2 + 2s + 5)(s^2 - 2s + 2)} = \frac{as + b}{s^2 + 2s + 5} + \frac{cs + d}{s^2 - 2s + 2} = \frac{(as + b)(s^2 - 2s + 2) + (cs + d)(s^2 + 2s + 5)}{(s^2 + 2s + 5)(s^2 - 2s + 2)} = \frac{s^3(a + c) + s^2(-2a + b + 2c + d) + s(2a - 2b + 5c + 2d) + (2b + c + d)}{(s^2 + 2s + 5)(s^2 - 2s + 2)} \]
\[ \begin{align*}
  a + c &= 5 \\
  -2a + b + 2c + d &= -7 \\
  2a - 2b + 5c + 2d &= -4 \\
  2b + 5d &= 0
\end{align*} \]

In matrix form:

\[
\begin{pmatrix}
1 & 0 & 1 & 0 \\
-2 & 1 & 2 & 1 \\
2 & -2 & 5 & 2 \\
0 & 2 & 0 & 5
\end{pmatrix}
\begin{pmatrix}
a \\
b \\
c \\
d
\end{pmatrix}
= 
\begin{pmatrix}
5 \\
-7 \\
-4 \\
0
\end{pmatrix}
\]
Use TI84 (rref):

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\begin{pmatrix}
a \\
b \\
c \\
d \\
\end{pmatrix} =
\begin{pmatrix}
5 \\
5 \\
0 \\
-2 \\
\end{pmatrix}
\]

So,

\[a = 5, \quad b = 5, \quad c = 0, \quad d = -2\]
So,

\[ F(s) = \frac{5s + 5}{s^2 + 2s + 5} - \frac{2}{s^2 - 2s + 2} = 5 \frac{(s + 1)}{(s + 1)^2 + 4} - 2 \frac{1}{(s - 1)^2 + 1} \]

By Formula 10, 9:

\[ F(s) = 5L\{e^{-t} \cos 2t\} - 2L\{e^t \sin t\} \]

\[ F(s) = L\{5e^{-t} \cos 2t - 2e^t \sin t\} \]

So,

\[ L^{-1}\{F(s)\} = 5e^{-t} \cos 2t - 2e^t \sin t \]
Example 4

Solve the IVP

\[ y'' - y' - 6y = 0; \quad y(0) = 1, \; y'(0) = 0 \]

- Let \( y = \varphi(t) \) be the solution, and write
  \[ Y(s) = \mathcal{L}\{y\} = \mathcal{L}\{\varphi\}. \]
- Apply Laplace transform to the equation:
  \[ \mathcal{L}\{y'' - y' - 6y\} = \mathcal{L}\{0\} \implies \mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 6\mathcal{L}\{y\} = 0 \]
By (3), (4):

\[
[s^2 Y(s) - sy(0) - y'(0)] - [sY(s) - y(0)] - 6Y(s) = 0
\]

\[
[s^2 Y(s) - s] - [sY(s) - 1] - 6Y(s) = 0
\]

\[
Y(s) = \frac{s - 1}{s^2 - s - 6} = \frac{s - 1}{(s + 2)(s - 3)} = \frac{a}{s + 2} + \frac{b}{s - 3} \Rightarrow
\]

\[
Y(s) = \frac{3}{5(s + 2)} + \frac{2}{5(s - 3)}
\]

Use the Chart \(Y(s) = \frac{3}{5} \mathcal{L}(e^{-2t})(s) + \frac{2}{5} \mathcal{L}(e^{3t})(s) \quad s > 3\)

\[
y = \mathcal{L}^{-1}\{Y(s)\} = \frac{3}{5}e^{-2t} + \frac{2}{5}e^{3t}
\]
Example 5

Solve the IVP

\[ y'' - 6y' + 9y = 0; \quad y(0) = 1, y'(0) = 1 \]

- Let \( y = \varphi(t) \) be the solution. Write \( Y(s) = \mathcal{L}\{y\} \).
- Apply Laplace transform to the equation:

\[
\mathcal{L}\{y'' - 6y' + 9y\} = \mathcal{L}\{0\} \quad \Rightarrow \quad \mathcal{L}\{y''\} - 6\mathcal{L}\{y'\} + 9\mathcal{L}\{y\} = 0
\]
By (3), (4):

\[ [s^2 Y(s) - sy(0) - y'(0)] - 6[sY(s) - y(0)] + 9Y(s) = 0 \]

\[ [s^2 Y(s) - s - 1] - 6[sY(s) - 1] + 9Y(s) = 0 \]

\[ Y(s) = \frac{s - 5}{s^2 - 6s + 9} = \frac{s - 5}{(s - 3)^2} = \frac{1}{s - 3} - 2\frac{1}{(s - 3)^2} \]

Use the Chart  \( Y(s) = \mathcal{L}\{e^{3t}\} - \mathcal{L}\{te^{3t}\} = \mathcal{L}\{e^{3t} - te^{3t}\} \)

So,

\[ y = \mathcal{L}^{-1}\{Y(s)\} = e^{3t} - te^{3t} \]
Example 6

Solve the IVP

\[ y^{(4)} - 9y = 0; \quad y(0) = 1, y'(0) = 0, y''(0) = 3, y'''(0) = 0 \]

- Let \( y = \varphi(t) \) be the solution. Write \( Y(s) = \mathcal{L}\{y\} \).
- Apply Laplace transform to the equation:

\[
\mathcal{L}\{y^{(4)} - 9y\} = \mathcal{L}\{0\} \implies \mathcal{L}\{y^{(4)}\} - 9\mathcal{L}\{y\} = 0
\]
By the theorem

\[ s^4 Y(s) - s^3 y(0) - s^2 y'(0) - sy''(0) - y^{(3)}(0) - 9 Y(s) = 0 \]

\[ [s^4 Y(s) - s^3 - 3s] - 9 Y(s) = 0 \]

\[ Y(s) = \frac{s^3 + 3s}{s^4 - 9} = \frac{s}{s^2 - 3} = \frac{a}{s - \sqrt{3}} + \frac{b}{s + \sqrt{3}} \]

So,

\[ Y(s) = \frac{1}{2(s - \sqrt{3})} + \frac{1}{2(s + \sqrt{3})} = \frac{1}{2} \mathcal{L}\{e^{\sqrt{3}t}\} + \frac{1}{2} \mathcal{L}\{e^{-\sqrt{3}t}\} \]

\[ = \mathcal{L}\{\frac{e^{\sqrt{3}t} + e^{-\sqrt{3}t}}{2}\} \]

So,

\[ y = \mathcal{L}^{-1}\{Y(s)\} = \frac{e^{\sqrt{3}t} + e^{-\sqrt{3}t}}{2} \]
Example 7

Solve the IVP

\[ y'' + 9y = \cos 2t; \quad y(0) = 1, \; y'(0) = 0 \]

Let \( y = \varphi(t) \) be the solution. Write \( Y(s) = \mathcal{L}\{y\} \).

Apply Laplace transform to the equation:

\[
\mathcal{L}\{y'' + 9y\} = \mathcal{L}\{\cos 2t\} \implies \mathcal{L}\{y''\} + 9\mathcal{L}\{y\} = \frac{s}{s^2 + 4}
\]
By the theorem

\[ s^2 Y(s) - sy(0) - y'(0) + 9Y(s) = \frac{s}{s^2 + 4} \implies \]

\[ s^2 Y(s) - s + 9Y(s) = \frac{s}{s^2 + 4} \]

\[ Y(s) = \frac{s^3 + 5s}{(s^2 + 4)(s^2 + 9)} \]
Use partial fraction: Write

\[
\frac{s^3 + 5s}{(s^2 + 4)(s^2 + 9)} = \frac{as + b}{s^2 + 4} + \frac{cs + d}{s^2 + 9}
\]

\[
\begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
9 & 0 & 4 & 0 \\
0 & 9 & 0 & 4
\end{pmatrix}
\begin{pmatrix}
a \\
b \\
c \\
d
\end{pmatrix}
= 
\begin{pmatrix}
1 \\
0 \\
5 \\
0
\end{pmatrix}
\]

\[
\begin{cases}
a + c = 1 \\
b + d = 0 \\
9a + 4c = 5 \\
9b + 4d = 0
\end{cases}
\]
Use TI84 (rref):

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
a \\
b \\
c \\
d
\end{pmatrix}
= 
\begin{pmatrix}
.2 \\
0 \\
.8 \\
0
\end{pmatrix}
\]
So, \( a = .2, b = 0, c = .8, d = 0 \) and

\[
Y(s) = .2 \frac{s}{s^2 + 4} + .8 \frac{s}{s^2 + 9} = .2\mathcal{L}\{\cos 2t\} + .8\mathcal{L}\{\cos 3t\}
\]

\[
Y(s) = \mathcal{L}\{.2 \cos 2t + .8 \cos 3t\}
\]

So, the solution

\[
y = \mathcal{L}^{-1}(Y(s)) = .2 \cos 2t + .8 \cos 3t
\]
Example 8

Consider the IVP:

\[ y'' + 13y = \begin{cases} 
  t & \text{if } 0 \leq t < 1 \\
  1 & \text{if } 1 \leq t \leq \infty 
\end{cases} \quad y(0) = 0, y'(0) = 0 \]

Let \( y = \varphi(t) \) be the solution. Compute \( Y(s) = \mathcal{L}\{y\} \).

- Also, write \( g(t) = \begin{cases} 
  t & \text{if } 0 \leq t < 1 \\
  1 & \text{if } 1 \leq t \leq \infty 
\end{cases} \)

- Apply Laplace transform to the equation:

\[
\mathcal{L}\{y'' + 13y\} = \mathcal{L}\{g(t)\} \implies \mathcal{L}\{y''\} + 13\mathcal{L}\{y\} = \mathcal{L}\{g(t)\}
\]
Compute $\mathcal{L}\{g(t)\}$ by direct computation (We did a similar problem in §6.1):

$$\mathcal{L}\{g(t)\}(s) = \int_{0}^{\infty} e^{-st} g(t) \, dt = \int_{0}^{1} e^{-st} \, dt + \int_{1}^{\infty} e^{-st} \, dt$$

$$= \frac{1}{-s} \int_{0}^{1} t e^{-st} \, dt + \left[ \frac{e^{-st}}{-s} \right]_{t=1}^{\infty}$$

$$= \frac{1}{-s} \left[ t e^{-st} \right]_{t=0}^{1} - \int_{0}^{1} e^{-st} \, dt + \frac{e^{-s}}{s}$$

$$= \frac{1}{-s} \left[ e^{-s} + \left[ \frac{e^{-st}}{s} \right]_{t=0}^{1} \right] + \frac{e^{-s}}{s}$$

$$= \frac{1}{-s} \left[ e^{-s} + e^{-s} - \frac{1}{s} \right] + \frac{e^{-s}}{s} = \frac{1 - e^{-s}}{s^2}$$
We have

\[ \mathcal{L}\{y''\} + 13\mathcal{L}\{y\} = \mathcal{L}\{g(t)\} \Rightarrow \]

\[ s^2 Y(s) - sy(0) - y'(0) + 13Y(s) = \frac{1 - e^{-s}}{s^2} \]

So,

\[ Y(s) = \frac{1 - e^{-s}}{s^2(s^2 + 13)} \]