Chapter 6: The Laplace Transform §6.2 Solutions of Initial Value Problems

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Goals

- The Goal of this section is to use Laplace Transform to solve Initial value problems, in second order linear ODE (as in Chapter 3).
- This way, the methods may become more algebraic.
- Two theorem that follows would be instrumental for this method.

Theorem 6.2.1

Theorem 6.2.1: Suppose f is a continuous function on an interval $0 \leq t \leq \alpha$.

- Assume f' is and is piecewise continuous on the interval $0 \leq t \leq \alpha$.
- Assume there are constants κ, λ, β, with $\kappa > 0$, $\beta > 0$, such that

$$|f(t)| \leq \kappa e^{\lambda t} \qquad \text{for all} \quad t \geq \beta$$

(*In words, f has (at most) exponential growth.*)

$$\text{Then,} \quad \mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0) \qquad (1)$$

Corollary 6.2.2

Corollary 6.2.2: Suppose f is a continuous function on an interval $0 \leq t \leq \alpha$. Assume $f', f^{(2)}, \ldots, f^{(n-1)}$ are continuous, and $f^{(n)}$ is piecewise continuous on the interval $0 \leq t \leq \alpha$. Assume there are constants κ, λ, β, with $\kappa > 0$, $\beta > 0$,

$$\ni \quad |f^{(i)}(t)| \leq \kappa e^{\lambda t} \qquad \text{for all} \quad i = 0, 1, \ldots, n \;\; \text{and} \;\; t \geq \beta$$

Then, $\mathcal{L}\{f^{(n)}(t)\} =$

$$s^n \mathcal{L}\{f(t)\} - s^{n-1} f(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0) \qquad (2)$$

- The cases $n = 1$ (1) and $n = 2$ will be used more frequently:
  - $n = 1$ case:

$$\text{Then,} \quad \mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0) \tag{3}$$

  - $n = 2$ case:

$$\mathcal{L}\{f^{(2)}(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0) \tag{4}$$

1-1 Correspondence

Laplace Transformation can be to solve IVP, due to the following:

- Suppose f, g are two continuous functions on an interval.

$$\text{Then,} \quad \mathcal{L}\{f(t)\} = \mathcal{L}\{g(t)\} \quad \Longrightarrow \quad f = g \qquad (5)$$

- If $\mathcal{L}\{f(t)\} = F(s)$, we write $\mathcal{L}^{-1}\{F(s)\} = f$, to be called the inverse Laplace transform of g. Further, inverse Laplace transform is linear, in the sense, for $\alpha, \beta \in \mathbb{R}$,

$$\mathcal{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha\mathcal{L}^{-1}\{F(s)\} + \beta\mathcal{L}^{-1}\{G(s)\}$$

- Solving: Charts of Laplace Transform and Inverse Laplace Transforms are available, in the internet and any standard Textbook. Download one and use for this section.
- To solve initial value problems, when $y(0) = y_0$, $y'(0) = y_0'$ are given, we compute the Laplace transform $\mathcal{L}(\{\varphi(t)\}$ of the solution $y = \varphi(t)$ and use the chart to compare.

Example 1

Compute the Inverse Laplace Transform of $F(s) = \frac{3s-6}{s^2-4s+13}$.

- We have
$$F(s) = 3\frac{s-2}{(s-2)^2+3^2}$$
- By Formula chart:
$$F(s) = 3\frac{s-2}{(s-2)^2+3^2} = 3\mathcal{L}\{e^{2t}\cos 3t\} = \mathcal{L}\{3e^{2t}\cos 3t\}$$
- So,
$$\mathcal{L}^{-1}\{F(s)\} = 3e^{2t}\cos 3t$$

Example 2

Compute the Inverse Laplace Transform of $F(s) = \frac{-1-2s}{s^2+4s+13}$.

- We have

$$F(s) = \frac{-1-2s}{s^2+4s+13} = \frac{-1-2s}{(s+2)^2+3^2}$$

$$= \frac{3}{(s+2)^2+3^2} - 2\frac{s+2}{(s+2)^2+3^2}$$

- By Formula Chart:

$$F(s) = \mathcal{L}\{e^{-2t}\sin 3t\} - 2\mathcal{L}\{e^{-2t}\cos 3t\}$$

$$= \mathcal{L}\{e^{-2t}\sin 3t - 2e^{-2t}\cos 3t\}$$

- So,
$$\mathcal{L}^{-1}\{F(s)\} = e^{-2t}\sin 3t - 2e^{-2t}\cos 3t$$

Example 3

Compute the Inverse Laplace Transform of

$$F(s) = \frac{5s^3 - 7s^2 - 4s}{(s^2 + 2s + 5)(s^2 - 2s + 2)}.$$

- The method of partial fractions is used frequently, in this section. Review all these examples.

Solution

Use method of partial fractions:

$$F(s) = \frac{5s^3 - 7s^2 - 4s}{(s^2+2s+5)(s^2-2s+2)}.$$

$$= \frac{as+b}{s^2+2s+5} + \frac{cs+d}{s^2-2s+2}$$

$$= \frac{(as+b)(s^2-2s+2)+(cs+d)(s^2+2s+5)}{(s^2+2s+5)(s^2-2s+2)} =$$

$$\frac{s^3(a+c)+s^2(-2a+b+2c+d)+s(2a-2b+5c+2d)+(2b+}{(s^2+2s+5)(s^2-2s+2)}$$

- $$\begin{cases} a + c = 5 \\ -2a + b + 2c + d = -7 \\ 2a - 2b + 5c + 2d = -4 \\ 2b + 5d = 0 \end{cases}$$
- In matrix form:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ -2 & 1 & 2 & 1 \\ 2 & -2 & 5 & 2 \\ 0 & 2 & 0 & 5 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 5 \\ -7 \\ -4 \\ 0 \end{pmatrix}$$

- Use TI84 (rref):

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \\ 0 \\ -2 \end{pmatrix}$$

- So,

$$a = 5, \quad b = 5 \quad c = 0 \quad d = -2$$

- So,

$$F(s) = \frac{5s+5}{s^2+2s+5} - \frac{2}{s^2-2s+2}$$
$$= 5\frac{(s+1)}{(s+1)^2+4} - 2\frac{1}{(s-1)^2+1}$$

- By Formula 10, 9:

$$F(s) = 5\mathcal{L}\{e^{-t}\cos 2t\} - 2\mathcal{L}\{e^{t}\sin t\}$$

$$F(s) = \mathcal{L}\{5e^{-t}\cos 2t - 2e^{t}\sin t\}$$

- So,

$$\mathcal{L}^{-1}\{F(s)\} = 5e^{-t}\cos 2t - 2e^{t}\sin t$$

Example 4

Solve the IVP

$$y'' - y' - 6y = 0; \quad y(0) = 1, y'(0) = 0$$

- Let $y = \varphi(t)$ be the solution, and write $Y(s) = \mathcal{L}\{y\} = \mathcal{L}\{\varphi\}$.
- Apply Laplace transform to the equation:

$$\mathcal{L}\{y'' - y' - 6y\} = \mathcal{L}\{0\} \Longrightarrow \mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 6\mathcal{L}\{y\} = 0$$

By (3), (4):

$$[s^2Y(s) - sy(0) - y'(0)] - [sY(s) - y(0)] - 6Y(s) = 0$$

$$[s^2Y(s) - s] - [sY(s) - 1] - 6Y(s) = 0$$

$$Y(s) = \frac{s-1}{s^2 - s - 6} = \frac{s-1}{(s+2)(s-3)} = \frac{a}{s+2} + \frac{b}{s-3} \Longrightarrow$$

$$Y(s) = \frac{3}{5(s+2)} + \frac{2}{5(s-3)}$$

$$\text{Use the Chart} \quad Y(s) = \frac{3}{5}\mathcal{L}(e^{-2t})(s) + \frac{2}{5}\mathcal{L}(e^{3t})(s) \quad s > 3$$

$$y = \mathcal{L}^{-1}\{Y(s)\} = \frac{3}{5}e^{-2t} + \frac{2}{5}e^{3t}$$

Example 5

Solve the IVP

$$y'' - 6y' + 9y = 0; \quad y(0) = 1, y'(0) = 1$$

- Let $y = \varphi(t)$ be the solution. Write $Y(s) = \mathcal{L}\{y\}$.
- Apply Laplace transform to the equation:

$$\mathcal{L}\{y'' - 6y' + 9y\} = \mathcal{L}\{0\} \Longrightarrow \mathcal{L}\{y''\} - 6\mathcal{L}\{y'\} + 9\mathcal{L}\{y\} = 0$$

By (3), (4):

$$[s^2Y(s) - sy(0) - y'(0)] - 6[sY(s) - y(0)] + 9Y(s) = 0$$

$$[s^2Y(s) - s - 1] - 6[sY(s) - 1] + 9Y(s) = 0$$

$$Y(s) = \frac{s-5}{s^2-6s+9} = \frac{s-5}{(s-3)^2} = \frac{1}{s-3} - 2\frac{1}{(s-3)^2}$$

Use the Chart $\quad Y(s) = \mathcal{L}\{e^{3t}\} - \mathcal{L}\{te^{3t}\} = \mathcal{L}\{e^{3t} - te^{3t}\}$

So,

$$y = \mathcal{L}^{-1}\{Y(s)\} = e^{3t} - te^{3t}$$

Example 6

Solve the IVP

$$y^{(4)} - 9y = 0; \quad y(0) = 1, y'(0) = 0, y''(0) = 3, y^{(3)}(0) = 0$$

- Let $y = \varphi(t)$ be the solution. Write $Y(s) = \mathcal{L}\{y\}$.
- Apply Laplace transform to the equation:

$$\mathcal{L}\{y^{(4)} - 9y\} = \mathcal{L}\{0\} \Longrightarrow \mathcal{L}\{y^{(4)}\} - 9\mathcal{L}\{y\} = 0$$

- By the theorem

$$[s^4Y(s)-s^3y(0)-s^2y'(0)-sy''(0)-y^{(3)}(0)]-9Y(s)=0$$

$$[s^4Y(s)-s^3-3s]-9Y(s)=0$$

$$Y(s)=\frac{s^3+3s}{s^4-9}=\frac{s}{s^2-3}=\frac{a}{s-\sqrt{3}}+\frac{b}{s+\sqrt{3}}$$

- So,

$$Y(s)=\frac{1}{2(s-\sqrt{3})}+\frac{1}{2(s+\sqrt{3})}=\frac{1}{2}\mathcal{L}\{e^{\sqrt{3}t}\}+\frac{1}{2}\mathcal{L}\{e^{-\sqrt{3}t}\}$$

$$=\mathcal{L}\{\frac{e^{\sqrt{3}t}+e^{-\sqrt{3}t}}{2}\}$$

- So,

$$y=\mathcal{L}^{-1}\{Y(s)\}=\frac{e^{\sqrt{3}t}+e^{-\sqrt{3}t}}{2}$$

Example 7

Solve the IVP

$$y'' + 9y = \cos 2t; \quad y(0) = 1, y'(0) = 0$$

- Let $y = \varphi(t)$ be the solution. Write $Y(s) = \mathcal{L}\{y\}$.
- Apply Laplace transform to the equation:

$$\mathcal{L}\{y'' + 9y\} = \mathcal{L}\{\cos 2t\} \Longrightarrow \mathcal{L}\{y''\} + 9\mathcal{L}\{y\} = \frac{s}{s^2 + 4}$$

- By the theorem

$$s^2Y(s) - sy(0) - y'(0) + 9Y(s) = \frac{s}{s^2+4} \Longrightarrow$$

$$s^2Y(s) - s + 9Y(s) = \frac{s}{s^2+4}$$

$$Y(s) = \frac{s^3+5s}{(s^2+4)(s^2+9)}$$

- Use partial fraction: Write

$$\frac{s^3+5s}{(s^2+4)(s^2+9)} = \frac{as+b}{s^2+4} + \frac{cs+d}{s^2+9}$$

$$\begin{cases} a+c=1 \\ b+d=0 \\ 9a+4c=5 \\ 9b+4d=0 \end{cases}$$

- In matrix form:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 9 & 0 & 4 & 0 \\ 0 & 9 & 0 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 5 \\ 0 \end{pmatrix}$$

- Use TI84 (rref):

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} .2 \\ 0 \\ .8 \\ 0 \end{pmatrix}$$

- So, $a = .2, b = 0, c = .8, d = 0$ and

$$Y(s) = .2\frac{s}{s^2+4} + .8\frac{s}{s^2+9} = .2\mathcal{L}\{\cos 2t\} + .8\mathcal{L}\{\cos 3t\}$$

$$Y(s) = \mathcal{L}\{.2\cos 2t + .8\cos 3t\}$$

- So, the solution

$$y = \mathcal{L}^{-1}(Y(s)) = .2\cos 2t + .8\cos 3t$$

Example 8

Consider the IVP:

$$y" + 13y = \begin{cases} t & \textit{if } 0 \leq t < 1 \\ 1 & \textit{if } 1 \leq t \leq \infty \end{cases} \qquad y(0) = 0, y'(0) = 0$$

Let $y = \varphi(t)$ be the solution. Compute $Y(s) = \mathcal{L}\{y\}$.

- Also, write $g(t) = \begin{cases} t & \textit{if } 0 \leq t < 1 \\ 1 & \textit{if } 1 \leq t \leq \infty \end{cases}$
- Apply Laplace transform to the equation:

$$\mathcal{L}\{y" + 13y\} = \mathcal{L}\{g(t)\} \Longrightarrow \mathcal{L}\{y"\} + 13\mathcal{L}\{y\} = \mathcal{L}\{g(t)\}$$

- Compute $\mathcal{L}\{g(t)\}$ by direct computation (*We did a similar problem in §6.1*):

$$\mathcal{L}\{g(t)\}(s) = \int_0^\infty e^{-st} g(t)dt = \int_0^1 e^{-st} tdt + \int_1^\infty e^{-st} dt$$

$$= \frac{1}{-s}\int_0^1 tde^{-st} + \left[\frac{e^{-st}}{-s}\right]_{t=1}^{\infty}$$

$$= \frac{1}{-s}\left[\left[te^{-st}\right]_{t=0}^{1} - \int_0^1 e^{-st} dt\right] + \frac{e^{-s}}{s}$$

$$= \frac{1}{-s}\left[e^{-s} + \left[\frac{e^{-st}}{s}\right]_{t=0}^{1}\right] + \frac{e^{-s}}{s}$$

$$= \frac{1}{-s}\left[e^{-s} + \left[\frac{e^{-s}}{s} - \frac{1}{s}\right]\right] + \frac{e^{-s}}{s} = \frac{1-e^{-s}}{s^2}$$

- We have

$$\mathcal{L}\{y''\} + 13\mathcal{L}\{y\} = \mathcal{L}\{g(t)\} \Longrightarrow$$

$$s^2Y(s) - sy(0) - y'(0) + 13Y(s) = \frac{1-e^{-s}}{s^2}$$

- So,

$$Y(s) = \frac{1-e^{-s}}{s^2(s^2+13)}$$