Chapter 2: First Order DE
§2.1 Linear DE: Integrating Factors

Satya Mandal, KU

January 18, 2018

First Order DE

- This chapter deals with first order DE. That means only first order derivative $\frac{dy}{dt}$ appears in the equation. We write them as
$$\frac{dy}{dt} = f(t, y) \tag{1}$$
where $f(t, y)$ is a function of both the independent variable t and the (unknown) dependent variable y.
- If $f(t, y) = \frac{G(t,y)}{P(t,y)}$ is a fraction, (1) can be written as
$$P(t, y)\frac{dy}{dt} = G(t, y) \tag{2}$$

Linear ODE of 1^{st}-order

- 1^{st}-order DEs (1) are called linear, if $f(t, y)$ is "linear" in y, in the following sense:

$$\frac{dy}{dt} + p(t)y = g(t) \tag{3}$$

where $p(t), g(t)$ are functions of t.

- In analogy to equation (2), this can also be written as

$$P(t)\frac{dy}{dt} + Q(t)y = G(t) \tag{4}$$

where $P(t), Q(t), G(t)$ are functions of t.

Method of Integrating Factor

A method to solve linear equations of the form (3):

$$\frac{dy}{dt} + p(t)y = g(t) \tag{5}$$

- $$\text{Let} \quad \mu(t) = \exp\left(\int p(t)dt\right). \quad \text{Then} \quad \frac{d\mu}{dt} = p(t)\mu(t).$$
- Multiply the equation (5) by $\mu(t)$, we get

$$\mu(t)\frac{dy}{dt} + \mu(t)p(t)y = \mu(t)g(t) \quad OR$$

$$\mu(t)y' + \mu'(t)y = \mu(t)g(t)$$

Continued

- So,
  $\frac{d}{dt}(\mu(t)y) = \mu(t)g(t) \Longrightarrow \mu(t)y = \int \mu(t)g(t)dt + c$,
  where c is an arbitrary constant. So, with

  $$\mu(t) = \exp\left(\int p(t)dt\right)$$

  a general solution of (5) is

  $$y = \frac{1}{\mu(t)}\left[\int \mu(t)g(t)dt + c\right] \qquad (6)$$

  This solution is valid on the domain $\mu(t) \neq 0$

Convenient form for Numerical Solutions

- In particular, a solution of (5) is

$$y = \frac{1}{\mu(t)}\left[\int_{t_0}^{t} \mu(s)g(s)ds + c\right] \qquad (7)$$

  where t_0 is a suitable number (often zero). This solution (7) is useful for numerical solution.
- $\mu(t)$ is called an integrating factor (IF).

Example I

Solve the initial value problem $\begin{cases} y' - \frac{y}{1+t} & = (1+t)e^t \\ y(0) & = 0 \end{cases}$

- Integrating factor (IF):

$$\mu(t) = \exp\left(\int \frac{-1}{1+t} dt\right) = \exp\left(-\ln(1+t)\right) = \frac{1}{1+t}$$

- First Method: use solution (6):

$$y = \frac{1}{\mu(t)}\left[\int \mu(t)g(t)dt + c\right]$$

$$= (1+t)\left[\int \frac{1}{1+t}(1+t)e^t dt + c\right]$$

Continued: Sample I

- $$y = (1+t)\left[\int e^t dt + c\right] = (1+t)(e^t + c)$$
- Using the initial value: $0 = y(0) = (1+c)$ or $c = -1$.
- The Final Solution:

$$y = (1+t)(e^t - 1)$$

Continued: Sample I

- Direct Method (without using solution 6):
  - Multiply the equation by the IF $\mu(t) = \frac{1}{1+t}$:

$$\frac{1}{1+t}y' - \frac{1}{1+t}\frac{y}{1+t} = \frac{1}{1+t}(1+t)e^t$$

$$\frac{1}{1+t}y' - \frac{y}{(1+t)^2} = e^t$$

  - We expect the LHS to be $\frac{d}{dt}(\mu(t)y)$: In deed:

$$LHS = \frac{d}{dt}\left(\frac{y}{1+t}\right) = e^t$$

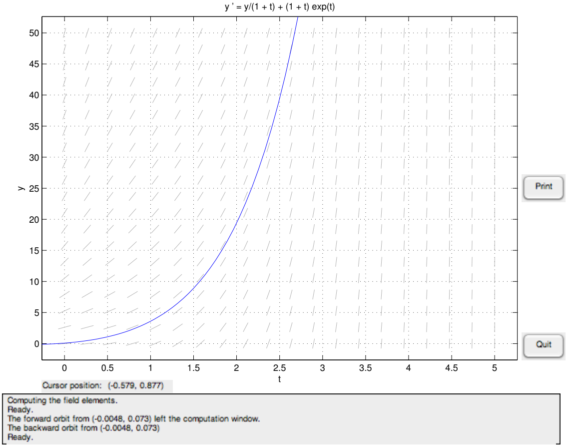
Continued: Sample I

- Integrating both sides:

$$\frac{y}{1+t} = \int e^t dt + c \quad \Longrightarrow \frac{y}{1+t} = e^t + c$$

- As before, we use initial value condition $y(0) = 0$ and get $c = -1$.
- So, again, the final solution

$$y = (1+t)(e^t - 1)$$


y ' = y/(1 + t) + (1 + t) exp(t)
50
45
40
35
30
25
20
15
10
5
0
y
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
t
Print
Quit
Cursor position: (-0.579, 0.877)
Computing the field elements.
Ready.
The forward orbit from (-0.0048, 0.073) left the computation window.
The backward orbit from (-0.0048, 0.073)
Ready.

Example II

Solve the initial value problem

$$\begin{cases} (1+t^2)^3 y' + 4t(1+t^2)^2 y = 1 \\ y(0) = 0 \end{cases}$$

- First Step; Reduce the problem to the form (3). To do this divide the equation by $(1+t^2)^3$. We get:

$$y' + \frac{4t}{t^2+1} y = \frac{1}{(t^2+1)^3}$$

- The integrating factor: $\mu(t) = \exp\left(\int \frac{4t}{t^2+1} dt\right) =$

$$= \exp\left(2\int \frac{2t}{t^2+1} dt\right) = \exp\left(2\ln(t^2+1)\right) = (t^2+1)^2$$

Sample II: Use solution (5)

- By solution (5): $y = \frac{1}{\mu(t)}\left[\int \mu(t)g(t)dt + c\right]$

$$= \frac{1}{(t^2+1)^2}\left[\int (t^2+1)^2 \frac{1}{(t^2+1)^3} dt + c\right]$$

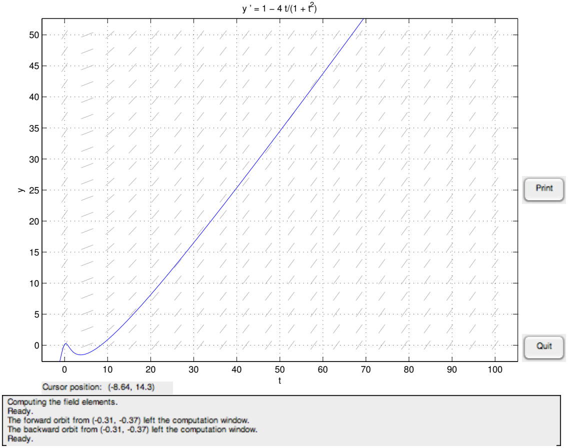
- $$y = \frac{1}{(t^2+1)^2}\left[\int \frac{1}{(t^2+1)} dt + c\right]$$

- $$y = \frac{1}{(t^2+1)^2}\left[\tan^{-1} t + c\right]$$

Continued

- Using initial value condition $y(0) = 0$, we have $0 = \tan^{-1} 0 + c = c$.
- So, the final solution

$$y = \frac{1}{(t^2+1)^2}\left[\tan^{-1} t\right]$$


y ' = 1 – 4 t/(1 + t²)
y
t
Print
Quit
Cursor position: (-8.64, 14.3)
Computing the field elements.
Ready.
The forward orbit from (-0.31, -0.37) left the computation window.
The backward orbit from (-0.31, -0.37) left the computation window.
Ready.

Example III

Consider the initial value problem: $\begin{cases} y' - y = 1 + 3\sin t \\ y(0) = y_0 \end{cases}$

Find the value of y_0 so that $\lim_{t\to\infty} y(t)$ is finite.

- Integrating factor (IF): $\mu(t) = \exp\left(\int -dt\right) = e^{-t}$.
- By solution (5): $y = \frac{1}{\mu(t)}\left[\int \mu(t)g(t)dt + c\right]$

$$= e^t \left[\int e^{-t}(1 + 3\sin t)dt + c\right]$$

$$= e^t \left[-e^{-t} + 3\int e^{-t}\sin t dt + c\right]$$

Continued

- Recall $\int e^{-t} \sin t dt = -\frac{e^{-t}(\sin t + \cos t)}{2}$
- So, the solution

$$y = e^t \left[-e^{-t} - 3\frac{e^{-t}(\sin t + \cos t)}{2} + c \right]$$

$$y = -1 - \frac{3(\sin t + \cos t)}{2} + ce^t$$

- Use the initial value condition $y(0) = y_0$:

$$y_0 = -1 - \frac{3}{2} + c \qquad or \qquad c = y_0 + \frac{5}{2}$$

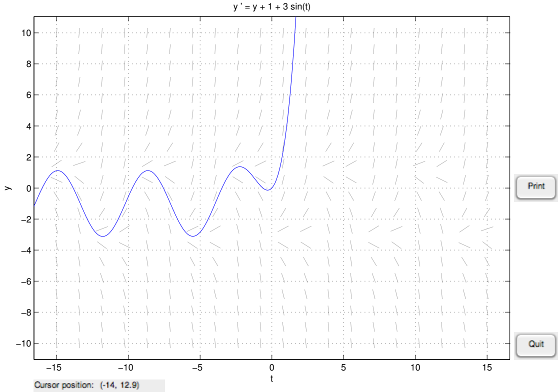
Continued

- So, the solution of the initial value problem:

$$y = -1 - \frac{3(\sin t + \cos t)}{2} + \left(y_0 + \frac{5}{2}\right) e^t$$

- Now, unless $\left(y_0 + \frac{5}{2}\right) \neq 0$, $\lim_{t\to\infty} y = \pm\infty$.
- So, for the $\lim_{t\to\infty} y$ to remain finite,

$$y_0 + \frac{5}{2} = 0 \qquad \text{OR} \qquad y_0 = -\frac{5}{2}.$$



Computing the field elements.
Ready.
The forward orbit from (-0.032, -0.029) left the computation window.
The backward orbit from (-0.032, -0.029)
Ready.