Chapter 2: First Order ODE
§2.3 miscellaneous ODEs

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First Order ODE

- Recall the general form of the First Order ODEs (FODE):

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

- There is a large classes of Examples of ODEs in the literature (Textbook) that easily reduce to the types we learned in the previous sections (Linear and Separable). We would discuss some of them in this section.

Homogeneous ODE: Definition

Definition: For the purpose of this course, an ODE (1) would be called a homogeneous ODE, if we can write $f(x, y)$ as a function $g(v)$ of $v = \frac{y}{x}$. So,

$$\frac{dy}{dx} = f(x, y) = g(v) \qquad \text{where} \qquad v = \frac{y}{x}$$

Such an equation changes to a Separable Equations, by substitution $v = \frac{y}{x}$.

Example 1

Solve the ODE

$$\frac{dy}{dx} = \frac{x+y}{x-y} \qquad \text{Assume} \qquad x > 0.$$

Solution: Divide numerator and denominator by x:

$$\frac{dy}{dx} = \frac{x+y}{x-y} = \frac{1+\frac{y}{x}}{1-\frac{y}{x}} = \frac{1+v}{1-v} \qquad \text{where} \qquad v = \frac{y}{x}$$

So, this is a Homogeneous ODE

Example 1: Solution

- Substitute $y = xv$. We have $\frac{dy}{dx} = v + x\frac{dv}{dx}$. So, the ODE reduces to

$$v + x\frac{dv}{dx} = \frac{1+v}{1-v} \Longrightarrow x\frac{dv}{dx} = \frac{1+v}{1-v} - v = \frac{1+v^2}{1-v} \Longrightarrow$$

$$\int \frac{1-v}{1+v^2}dv = \int \frac{dx}{x} + c$$

$$\int \frac{1}{1+v^2}dv - \int \frac{v}{1+v^2}dv = \ln|x| + c \Longrightarrow$$

Continued

- $$\tan^{-1} v - \frac{1}{2}\ln(1+v^2) = \ln x + c \Longrightarrow$$
  $$\tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2}\ln\left(1+\left(\frac{y}{x}\right)^2\right) = \ln x + c$$

Example 2

Solve the ODE

$$\frac{dy}{dx} = \frac{x^2 - xy}{y^2 - xy} \qquad \text{Assume} \qquad x > 0.$$

Solution: Divide numerator and denominator by x^2:

$$\frac{dy}{dx} = \frac{x^2 - xy}{y^2 - xy} = \frac{1 - \frac{y}{x}}{\left(\frac{y}{x}\right)^2 - \frac{y}{x}} = \frac{1 - v}{v^2 - v} \quad \text{where} \quad v = \frac{y}{x}$$

So, this is a Homogeneous ODE

Example 2: Solution

- Substitute $y = xv$. We have $\frac{dy}{dx} = v + x\frac{dv}{dx}$. So, the ODE reduces to

$$v + x\frac{dv}{dx} = \frac{1-v}{v^2-v} \Longrightarrow$$

$$x\frac{dv}{dx} = \frac{1-v}{v^2-v} - v = \frac{1-v+v^2-v^3}{v^2-v} = \frac{(1-v)(1+v^2)}{v^2-v} \Longrightarrow$$

$$x\frac{dv}{dx} = -\frac{(1+v^2)}{v} \Longrightarrow \quad \int \frac{v}{1+v^2}dv = -\int \frac{dx}{x} + c \Longrightarrow$$

$$\frac{1}{2}\ln(1+v^2) = -\ln x + c \Longrightarrow \quad \frac{1}{2}\ln\left(1+\left(\frac{y}{x}\right)^2\right) = -\ln x + c$$

Example 3

Solve the ODE

$$x\frac{dy}{dx} = y(\ln y - \ln x + 1) \qquad \text{Assume} \qquad x > 0, y > 0.$$

Solution: Divide both sides by x:

$$\frac{dy}{dx} = \frac{y}{x}\left(\ln\left(\frac{y}{x}\right) - 1\right) = v\,(\ln v - 1) \qquad \text{where} \qquad v = \frac{y}{x}$$

So, this is a Homogeneous ODE

Example 3: Solution

- Substitute $y = xv$. We have $\frac{dy}{dx} = v + x\frac{dv}{dx}$. So, the ODE reduces to

$$v+x\frac{dv}{dx} = v(\ln v - 1) \Longrightarrow x\frac{dv}{dx} = v(\ln v - 1) - v = v\ln v$$

$$\int \frac{dv}{v\ln v} = \int \frac{dx}{x} + c \Longrightarrow \ln|\ln v| = \ln x + c \Longrightarrow$$

$$\ln\left|\ln\left(\frac{y}{x}\right)\right| = \ln x + c$$

Bernoulli's Equation: Defintion

Definition: The ODE of the form

$$\frac{dy}{dt} + p(t)y = g(t)y^n \quad \text{is said to be a Bernoulli's Equation} \tag{2}$$

where $p(t)$ and $g(t)$ are functions of t.

- Such an ODE reduces to a Linear Equation (as in § 2.1), by a substitution $z = y^{-(n-1)}$.
- There are other ODEs that easily reduce to this form, by some other substitution.

Example 1

Solve the ODE (Bernoulli's Equation)

$$\frac{dy}{dx} + \frac{y}{x+1} = \frac{x^3}{y^2} \qquad \text{Assume} \quad x > 0$$

Solution: Multiply the ODE by y^2. The ODE reduces to

$$y^2\frac{dy}{dx} + \frac{y^3}{x+1} = x^3$$

Substitute

$$z = y^3 \qquad \text{Then,} \quad \frac{dz}{dx} = 3y^2\frac{dy}{dx}$$

Solution

- So, the ODE reduces to

$$\frac{1}{3}\frac{dy}{dx} + \frac{z}{x+1} = x^3 \Longrightarrow \frac{dz}{dx} + 3\frac{z}{x+1} = 3x^3$$

which is a Linear ODE in z. The Integrating factor

$$\mu(x) = \exp(\int p(x)dx) = \exp\left(\int 3\frac{dx}{x+1}\right) = (x+1)^3$$

Solution

- Multiplying the ODE, by $\mu(x) = (x+1)^3$, we have

$$\frac{d}{dx}\left((x+1)^3 z\right) = 3x^3(x+1)^3 \Longrightarrow$$

$$(x+1)^3 z = \int 3x^3(x+1)^3 dx + c \Longrightarrow$$

$$(x+1)^3 z = \int \left(3x^3(x^3+3x^2+3x+1)\right) dx + c$$

$$= \frac{3x^7}{7} + \frac{9x^6}{6} + \frac{9x^5}{5} + \frac{3x^4}{4} + c$$

Solution

- So,

$$y^3 = z = \frac{1}{(x+1)^3}\left(\frac{3x^7}{7} + \frac{9x^6}{6} + \frac{9x^5}{5} + \frac{3x^4}{4} + c\right)$$

  in the solution of the ODE, in implicit form.
- In this case, in the explicit form, the solution is

$$y = \frac{1}{(x+1)}\left(\frac{3x^7}{7} + \frac{9x^6}{6} + \frac{9x^5}{5} + \frac{3x^4}{4} + c\right)^{1/3}$$

Example 2

Solve the ODE (Bernoulli's Equation)

$$\frac{dy}{dt} - 2yt = t^3y^2$$

Solution: Dividing the ODE by y^2. The ODE reduces to

$$y^{-2}\frac{dy}{dt} - \frac{2t}{y} = t^3$$

Substitute

$$z = \frac{1}{y} \qquad \text{Then,} \quad \frac{dz}{dt} = -\frac{1}{y^2}\frac{dy}{dt}$$

Solution

- So, the ODE reduces to

$$-\frac{dz}{dt} - 2tz = t^3 \Longrightarrow \frac{dz}{dt} + 2tz = -t^3$$

which is a linear ODE. The integrating factor:

$$\mu(t) = \exp\left(\int 2tdt\right) = e^{t^2}$$

- Multiplying the ODE by $\mu(t)$, we hav

$$e^{t^2}\frac{dz}{dt} + 2te^{t^2}z = -t^3e^{t^2} \Longrightarrow \frac{d}{dt}\left(e^{t^2}z\right) = -t^3e^{t^2} \Longrightarrow$$

Solution

$$e^{t^2}z = -\int t^3 e^{t^2} + c$$

This integral is computed by Integration by Parts *(you can leave it in this form, for partial credit)*.

$$e^{t^2}z = -\frac{1}{2}\int t^2 d\left(e^{t^2}\right) + c = -\frac{1}{2}\left(t^2 e^{t^2} - \int 2te^{t^2}dt\right) + c \Longrightarrow$$

$$e^{t^2}z = -\frac{1}{2}\left(t^2 e^{t^2} - e^{t^2}\right) + c \Longrightarrow$$

$$z = -\frac{e^{-t^2}}{2}\left(t^2 e^{t^2} - e^{t^2}\right) + ce^{-t^2} = -\frac{1}{2}\left(t^2 - 1\right) + ce^{-t^2}$$

Solution

So, the solution in the implicit form:

$$z = \frac{1}{y} = -\frac{e^{-t^2}}{2}\left(t^2 e^{t^2} - e^{t^2}\right) + ce^{-t^2} = -\frac{1}{2}\left(t^2 - 1\right) + ce^{-t^2}$$

Example 3

Solve the ODE (Bernoulli's Equation)

$$\frac{dy}{dt} + y \sin t = y^2 \sin t$$

Solution: Dividing the ODE by y^2. The ODE reduces to

$$y^{-2}\frac{dy}{dt} + \frac{\sin t}{y} = \sin t$$

Substitute

$$z = \frac{1}{y} \qquad \text{Then,} \quad \frac{dz}{dt} = -\frac{1}{y^2}\frac{dy}{dt}$$

Solution

- So, the ODE reduced to

$$-\frac{dz}{dt} + (\sin t)z = \sin t \Longrightarrow \frac{dz}{dt} - (\sin t)z = -\sin t$$

which is a linear equation in z. The integrating factor:

$$\mu(t) = \exp\left(\int -\sin t dt\right) = e^{\cos t}$$

- Multiplying the ODE by $\mu(t) = e^{\cos t}$, we have

$$\frac{d}{dt}\left(ze^{\cos t}\right) = -e^{\cos t}\sin t$$

Solution

So,

$$ze^{\cos t} = \int -e^{\cos t} \sin t + c = e^{\cos t} + c \Longrightarrow$$

$$\frac{1}{y} = z = e^{-\cos t}\left(e^{\cos t} + c\right) = 1 + ce^{-\cos t}$$

is the solution in implicit form. In explicit form, the solution is

$$y = \frac{1}{1 + ce^{-\cos t}}$$