Chapter I: Introduction
§1.1 Modeling and Direction Fields

Satya Mandal, KU

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Introduction

- The objective, of this introductory section, is to Introduce the idea of modeling natural or social system in terms of Differential Equations (DE).
- We study of equations involving derivatives of functions.
- We learn a variety of methods to solve DEs.
- For modeling, main thing to remember, is that the derivative $\frac{dy}{dt}$ is the rate at which y changes with t. The notation t was chosen to indicate that the independent variable, sometimes represent time.

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- A DE that describes a physical or social process, closely enough, is called a Mathematical Model of the same.

When Gopher took a honeypot from his lunch box, Pooh just couldn't stand it any longer. "Please Gopher," he pleaded, "could you spare a small smackeral of honey?" — Pooh Story

Force, velocity, acceleration

- Suppose an object is dropped from a point (the point of ejection). We measure position of the object, by the distance s (in meters) from the point of ejection.
- The velocity $v = v(t)$, at time t, is the rate of change in s at time t. So, $v = \frac{ds}{dt}$.
- Again, the acceleration $a(t)$, at time t, is the rate of change in velocity. So, $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$

Newton's second law of motion

- Newton's second law of motion states: Force F needed to be applied on an object of mass m, to create an acceleration a is given by $F = ma$
- So, $F = m\frac{dv}{dt}$, would be a model for this physical system.
- Now, what are the forces acting on such an object? Gravitational pull $g = 9.8 meters/second^2$ and drag δ are among them. It is known (modeled) that $\delta = \gamma v$ a constant times the velocity v. The drag, acts opposite to the direction of the motion. So, $F = mg - \gamma v$.

Newton's second law of motion

- Putting these two together, the model of this "falling" is

$$m\frac{dv}{dt} = mg - \gamma v \quad (1)$$

By solving this DE, the velocity function $v = v(t)$ is obtained.

- In fact, modeling is approximating the physical system, as closely as we can. Formula (or the model) for the drag could be more complex, depending on the kind of accuracy we demand.

Assign $m = 10\ kg, \gamma = 2\ kg/s$

- Let mass $m = 10\ kg$ and $\gamma = 2\ kg/sec$. It is also known $g = 9.8m/sec^2$. (*"kg" stands for Kilogram and "m" for meter.*)
- For such an object

$$10\frac{dv}{dt} = 98 - 2v \quad or \quad \frac{dv}{dt} = 9.8 - .2v \qquad (2)$$

- Always, keep track of the units. Here Kg is the unit of mass, the unit of distance/length used is meter, time is measured second.
- The solution $v = v(t)$, need/would not be unique.

Direction field or Slope field

A basic concept in this course is the Direction field.

- For any point (t, v), $\frac{dv}{dt}$ is the slope of the tangent of the solution (among many), passing through the point (t, v).
- For example, for $v = 5m/sec$ the slope

$$\frac{dv}{dt} = 9.8 - .2v = 9.8 - 1 = 8.8$$

- Graphical representation of the tangents, of the solutions, is called the Direction field or Slope field of v.

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- We make a table

$v \setminus time$	1	2	3	4	5	6	7	8	9	10
40	1.8	1.8	1.8	1.8	1.8	1.8	1.8	1.8	1.8	1.8
41	1.6	1.6	1.6	1.6	1.6	1.6	1.6	1.6	1.6	1.6
42	1.4	1.4	1.4	1.4	1.4	1.4	1.4	1.4	1.4	1.4
43	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2
44	1	1	1	1	1	1	1	1	1	1
45	.8	.8	.8	.8	.8	.8	.8	.8	.8	.8

- Display a small tangent segment of slope, as in the table, in the $tv-$plane. This display will be the direction field.

Direction Field: Definition

- A wide class of DEs are given in the form

$$\frac{dy}{dt} = f(t, y) \quad \text{where } f \ \text{ is a function of } \ t, y. \qquad (3)$$

  We solve for $y = y(t)$, as a function of t.
- $f(t, y)$ is, sometimes, called the rate function.
- Evaluate $\frac{dy}{dt} = f(t, y)$ and make a table, for a number of points (t, y) on the $ty-$plane.
- The Direction Field is obtained, by drawing a short line segment on these points on the grid, with slope $f(t, y)$.

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- These line segments represent the tangent of the solution (among many), that passes through the respective point.
- Looking at the direction field, one can visualize the graph of the solutions, without solving the equation. In deed, it would not always be possible to write down an analytic solution, to a given DE.
- Use some computational aid (TI-84) to compute $\frac{dy}{dt} = f(t, y)$ to make a table. Better still, use some software (e. g. MATLAB) to construct a direction field.

Equilibrium Solution

When $\frac{dy}{dt} = f(t, y) = 0$, the corresponding solution is called the Equilibrium Solution. The corresponding points (t, y) are called the Equilibrium points.

- At the equilibrium points, the direction fields are horizontal.
- The solution of the DE reaches a max or min at these points.

In most/all cases, in this section, $f(t, y) = f(y)$, are independent of t. This will not be the case in future.

Population Growth Model

Another Important example of a DE is that of Population Growth.

- If $p = p(t)$ is the size of the population, at time t. Then the growth rate is $\frac{dp}{dt}$.
- A simplistic model (reasonably functional) of growth is assumes that the growth is proportional to the size p. So,

$$\frac{dp}{dt} = rp \tag{4}$$

where r is a constant. The population is growing or deflating, depending on whether r is positive or negative. If $r = .5$, then $\frac{dp}{dt} = .5p$.

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- An improved version will account for loss or gain due to death, immigation and other reasons. Again, a simplistic version will assume that this loss is constant. If the loss is 450 per unit time, then

$$\frac{dp}{dt} = .5p - 450 \tag{5}$$

The equilibrium occurs when $\frac{dp}{dt} = .5p - 450 = 0$. Which is when $p = 900$.