Chapter 3: Second Order ODE
§3.5 Complex roots of the CE

Satya Mandal, KU

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Homogeneous LSODEs

- Recall a Homogeneous LSODEs has one of the following two forms:

$$\mathcal{L}(y) = y'' + p(t)y' + q(t)y = 0 \qquad (1)$$

Or $$\mathcal{L}(y) = P(t)y'' + Q(t)y' + R(t)y = 0 \qquad (2)$$

where $p(t), q(t), P(t), Q(t), R(t)$ are functions of t.

- The Trivial Solution: For any homogeneous equation (1, 2), $y = 0$ is a solution.

Complex solutions to real

Sometimes the equation 1 (or 2), would have complex solutions, while we are interested only in real solution. The following theorem helps.

Theorem 3.5.1: Consider the homogeneous equation (1), where $p(t), q(t)$ are real valued functions of t.
Let $y = \varphi(t) = u(t) + iv(t)$ be a complex solution of the ODE (1), where $u(t)$ is the real part and $v(t)$ is the imaginary part of y.
Then, both $y = u(t)$, $y = v(t)$ are solutions of (1).
Proof: Use linearity.

Complex roots of the CE

Consider a 2^{nd}-Order Homogeneous linear ODE, with constant coefficients:

$$\mathcal{L}(y) = ay" + by' + cy = 0 \qquad a, b, c \in \mathbb{R} \tag{3}$$

$$\text{The CE of (3) is :} \quad ar^2 + br + c = 0 \tag{4}$$

- In §3.2, 3.4 we dealt with the situations, when (4), respectively, had unequal or repeated real roots.
- In this section, we deal with the case, when the CE (4) would have complex roots.

Continued

- The CE (4) would have complex root, when $b^2 - 4ac < 0$. The roots are

$$\begin{cases} r_1 = \lambda + i\mu = \frac{-b+\sqrt{b^2-4ac}}{2a} \\ r_2 = \lambda - i\mu = \frac{-b-\sqrt{b^2-4ac}}{2a} \end{cases} \quad \text{where} \quad i = \sqrt{-1}$$

  We say, r_1 and r_2 are conjugate of each other.
- As in §3.2 (3) has two solutions:

$$\begin{cases} y_1(t) = e^{r_1 t} = \exp[(\lambda + i\mu)t] = e^{\lambda t} e^{i\mu t} \\ y_2(t) = e^{r_2 t} = \exp](\lambda - i\mu)t] = e^{\lambda t} e^{-i\mu t} \end{cases} \qquad (5)$$

- However, (5) involves complex exponentiation $e^{i\mu t}, e^{-i\mu t}$.

Complex Exponentiation

- For real numbers θ, we define $e^{i\theta} = \cos\theta + i\sin\theta$.
- For complex numbers $z = \rho + i\theta$ define

$$e^z := e^{\rho + i\theta} := e^{\rho} e^{i\theta} = e^{\rho}(\cos\theta + i\sin\theta)$$

- All the rules of exponentiation that you are familiar with work, with this definition of e^z. In particular

$$e^{z+w} = e^z e^w \qquad \text{for all} \quad z, w \in \mathbb{C}.$$

- *Justifications for defining complex exponentiation e^z this way, is dealt with in the Complex Analysis Courses.*

Solution of (3)

- So the solution (5) of (3) reduces to:

$$\begin{cases} y_1(t) = e^{\lambda t}(\cos \mu t + i \sin \mu t) \\ y_2(t) = e^{\lambda t}(\cos \mu t - i \sin \mu t) \end{cases} \tag{6}$$

- By Theorem 3.5.1 both the real and complex parts (of y_1 or y_2) are solution of (3). We get two real solutions:

$$\begin{cases} u(t) = e^{\lambda t} \cos \mu t \\ v(t) = e^{\lambda t} \sin \mu t \end{cases} \tag{7}$$

Continued

- Wronskian of u, u is

$$W(u,v)(t) = \begin{vmatrix} u(t) & v(t) \\ u'(t) & v'(t) \end{vmatrix}$$

$$= \begin{vmatrix} e^{\lambda t}\cos\mu t & e^{\lambda t}\sin\mu t \\ \lambda e^{\lambda t}\cos\mu t - e^{\lambda t}\mu\sin\mu t & \lambda e^{\lambda t}\sin\mu t + e^{\lambda t}\mu\cos\mu t \end{vmatrix}$$

$$= e^{2\lambda t}\begin{vmatrix} \cos\mu t & \sin\mu t \\ \lambda\cos\mu t - \mu\sin\mu t & \lambda\sin\mu t + \mu\cos\mu t \end{vmatrix} = \mu e^{2\lambda t}$$

Continued

- So, (since $\mu \neq 0$), Wronskian $W(u, v)(t) = \mu e^{2\lambda t} \neq 0$.
- So, u, v form a fundamental set of solutions of (3).
- So, the general (real) solution of (3) has the form

$$y = c_1 u(t) + c_2 v(t) = c_1 e^{\lambda t} \cos \mu t + c_2 e^{\lambda t} \sin \mu t \qquad (8)$$

  where c_1, c_2 are arbitrary constants.
- We can write the same as

$$y = e^{\lambda t}(c_1 \cos \mu t + c_2 \sin \mu t) \qquad (9)$$

Behavior of the solution

If the CE (4) has complex roots or $\mu \neq 0$, then, the solution (9) has two factors:

- The exponential factor:

$$E(t) = e^{\lambda t}$$

  Depending on the sign of λ this part will "blow up" to ∞ or "decay" to the $x-$axis (horizontal asymptote).

- The periodic factor.

$$\Phi(t) = c_1 \cos \mu t + c_2 \sin \mu t \quad \text{with periodicity} = \frac{2\pi}{\mu}$$

- Further,

$$-(|c_1| + |c_2|) \leq \Phi(t) \leq (|c_1| + |c_2|)$$

The graph of $\Phi(t)$ contributes to a steady oscillation.

- The behavior of the solution y (as in (9)) will be a combination of (1) the exponential rise/decay due to $E(t)$ and (2) the periodic oscillation due to $\Phi(t)$.

Continued

So, the nature of the solutions is summarized as follows:

- If $\lambda = 0$ then the solution would be a steady oscillation.
- If $\lambda > 0$, it will be unsteady oscillation.
- If $\lambda < 0$ the oscillation will stabilize with time.

Example 1

Consider the IVP:

$$\begin{cases} y'' - 4y' + 8y = 0 \\ y(\pi/8) = 0 \\ y'(\pi/8) = e^{\pi/4} \end{cases}$$

- Solve the problem
- Sketch the graph
- Describe the nature of the solution, as $t \to \infty$

Solution

- The CE: $r^2 - 4r + 8 = 0$
- Roots of the CE: $r_1 = 2 + 2i$, $r_2 = 2 - 2i$.
- By solution (9), the general solution

$$y = e^{\lambda t}(c_1 \cos \mu t + c_2 \sin \mu t) = e^{2t}(c_1 \cos 2t + c_2 \sin 2t)$$

- The answer to the last part: the solution would be an unsteady oscillation; because the exponential part is e^{2t}

Continued

- Before we use the initial values, compute

$$y' = 2e^{2t}(c_1\cos 2t + c_2\sin 2t) + e^{2t}(-2c_1\sin 2t + 2c_2\cos 2t)$$

- Initial value conditions:

$$\begin{cases} y(\pi/8) = 2e^{\pi/4}\left(\frac{c_1}{\sqrt{2}} + \frac{c_2}{\sqrt{2}}\right) = 0 \\ y'(\pi/8) = 2e^{\pi/4}\left(\frac{c_1}{\sqrt{2}} + \frac{c_2}{\sqrt{2}}\right) + e^{\pi/4}\left(-\frac{2c_1}{\sqrt{2}} + \frac{2c_2}{\sqrt{2}}\right) = e^{\pi/4} \end{cases}$$

$$\begin{cases} c_1 + c_2 = 0 \\ \frac{4}{\sqrt{2}}c_2 = 1 \end{cases} \implies \begin{cases} c_1 = -\frac{1}{2\sqrt{2}} \\ c_2 = \frac{1}{2\sqrt{2}} \end{cases}$$

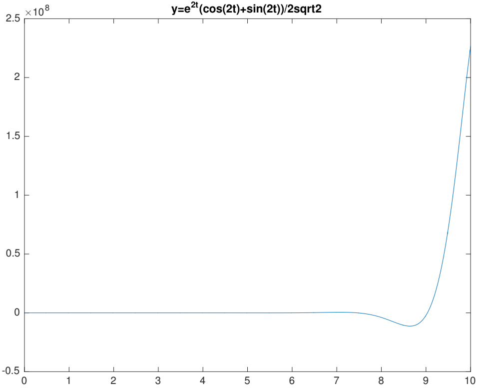
Continued

- So, the solution is

$$y = e^{2t}\left(-\frac{1}{2\sqrt{2}}\cos 2t + \frac{1}{2\sqrt{2}}\sin 2t\right)$$

- Repeat: $y = y(t)$ has an unsteady/unstable oscillation.

Graph of $y = y(t)$:



Example 2 (Dampened Oscillation)

Consider the IVP:

$$\begin{cases} y'' + 4y' + 5y = 0 \\ y(\pi/4) = 2 \\ y'(\pi/4) = -4 \end{cases}$$

- Solve the problem
- Sketch the graph
- Describe the nature of the solution, as $t \to \infty$

Solution

- The CE: $r^2 + 4r + 5 = 0$
- Roots of the CE: $r_1 = -2 + i$, $r_2 = -2 - i$.
- By solution (9), the general solution

$$y = e^{\lambda t}(c_1 \cos \mu t + c_2 \sin \mu t) = e^{-2t}(c_1 \cos t + c_2 \sin t)$$

- The answer to the last part: the solution will be an stabilized/dampened/ decaying oscillation; because the exponential part is e^{-2t}

Continued

- Before we use the initial values, compute

$$y' = -2e^{-2t}(c_1 \cos t + c_2 \sin t) + e^{-2t}(-c_1 \sin t + c_2 \cos t)$$

- Initial value conditions:

$$\begin{cases} y(\pi/4) = e^{-\pi/2}\left(\frac{c_1}{\sqrt{2}} + \frac{c_2}{\sqrt{2}}\right) = 2 \\ y'(\pi/4) = -2e^{-\pi/2}\left(\frac{c_1}{\sqrt{2}} + \frac{c_2}{\sqrt{2}}\right) + e^{-\pi/2}\left(-\frac{c_1}{\sqrt{2}} + \frac{c_2}{\sqrt{2}}\right) = -4 \end{cases}$$

$$\begin{cases} c_1 + c_2 = 2\sqrt{2}e^{\pi/2} \\ -4 + e^{-\pi/2}\left(-\frac{c_1}{\sqrt{2}} + \frac{c_2}{\sqrt{2}}\right) = -4 \end{cases} \Rightarrow \begin{cases} c_1 + c_2 = 2\sqrt{2}e^{\pi/2} \\ c_1 = c_2 \end{cases}$$

Continued

- So, $c_1 = c_2 = \sqrt{2}e^{\pi/2}$
- So, the solution is

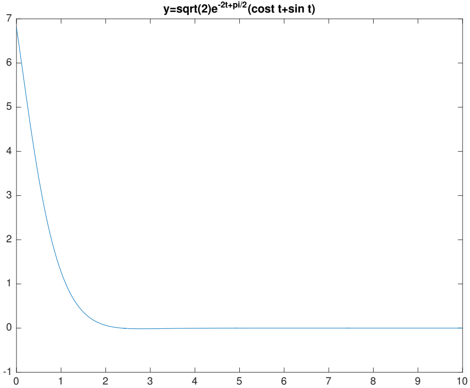
$$y = e^{-2t}\left(c_1 \cos t + c_2 \sin t\right)$$

$$= e^{-2t}\left(\sqrt{2}e^{\pi/2} \cos t + \sqrt{2}e^{\pi/2} \sin t\right)$$

$$= \sqrt{2}e^{-2t+\pi/2}\left(\cos t + \sin t\right)$$

- Repeat: the $y = y(t)$ has stabilized/dampened/ decaying oscillation.

Graph of $y = y(t)$: The exponential part $E(t) = e^{-2t+\pi/2}$ dampens (flattens) the graphs very quickly.



Example 3

Consider the IVP:

$$\begin{cases} y'' + 9y = 0 \\ y(0) = 0 \\ y'(0) = 1 \end{cases}$$

- Solve the problem
- Sketch the graph
- Describe the nature of the solution, as $t \to \infty$

Solution

- The CE: $r^2 + 9 = 0$
- Roots of the CE: $r_1 = 3i$, $r_2 = -3i$.
- By solution (9), the general solution

$$y = e^{\lambda t}(c_1 \cos \mu t + c_2 \sin \mu t) = c_1 \cos 3t + c_2 \sin 3t$$

- Answer to the last part: the solution will be a STABLE oscillation; because there is no exponential part.

Continued

- Before we use the initial values, compute

$$y' = -3c_1 \sin 3t + 3c_2 \cos 3t$$

- Initial value conditions:

$$\begin{cases} y(0) = c_1 = 0 \\ y'(0) = 3c_2 = 1 \end{cases} \implies \begin{cases} c_1 = 0 \\ c_2 = \frac{1}{3} \end{cases}$$

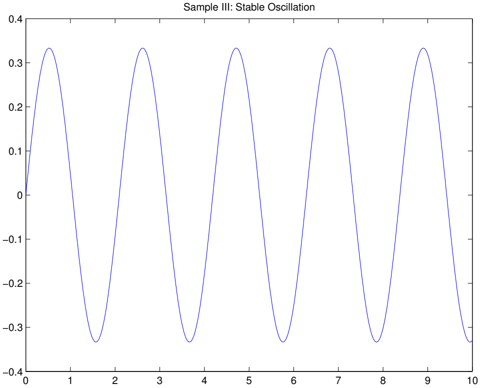
Continued

- So, the solution is

$$y = \frac{1}{3}\sin 3t$$

- Repeat: $y = y(t)$ has an STABLE oscillation.

Graph of $y = y(t)$:


Sample III: Stable Oscillation
0.4
0.3
0.2
0.1
0
−0.1
−0.2
−0.3
−0.4
0
1
2
3
4
5
6
7
8
9
10

On the Matlab Graph

- It took some trial and error to get a good graph.
- Following commands were used to get this graph:
  - t=[0:.01:10];
  - y=sin(3*t)/3;
  - plot(t,y), title('Sample III: Stable Oscillation')

Example 4

Consider the IVP:

$$\begin{cases} y" + \pi^2 y = 0 \\ y(1) = 1 \\ y'(1) = 1 \end{cases}$$

- Solve the problem
- Sketch the graph
- Describe the nature of the solution, as $t \to \infty$

Solution

- The CE: $r^2 + \pi^2 = 0$
- Roots of the CE: $r_1 = \pi i$, $r_2 = -\pi i$.
- By solution (9), the general solution

$$y = e^{\lambda t}(c_1 \cos \mu t + c_2 \sin \mu t) = c_1 \cos \pi t + c_2 \sin \pi t$$

- Answer to the last part: the solution will be a STABLE oscillation; because there is no exponential part.

Continued

- Before we use the initial values, compute

$$y' = -\pi c_1 \sin \pi t + \pi c_2 \cos \pi t$$

- Initial value conditions:

$$\begin{cases} y(1) = -c_1 = 1 \\ y'(1) = -\pi c_2 = 1 \end{cases} \implies \begin{cases} c_1 = -1 \\ c_2 = -\frac{1}{\pi} \end{cases}$$

Continued

- So, the solution is

$$y = c_1 \cos \pi t + c_2 \sin \pi t = -\cos \pi t - \frac{1}{\pi} \sin \pi t$$

- Repeat: $y = y(t)$ has an STABLE oscillation.

Graph of $y = y(t)$: