Chapter 3: Second Order ODE
§3.4 Repeated roots of the CE

Satya Mandal, KU

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Homogeneous LSODEs

- Recall a Homogeneous LSODEs has forms:

$$\mathcal{L}(y) = y" + p(t)y' + q(t)y = 0 \qquad (1)$$

where $p(t), q(t), g(t)$ are functions of t.

- Then, (1) can also be written as:

$$\mathcal{L}(y) = P(t)y" + Q(t)y' + R(t)y = 0 \qquad (2)$$

where $P(t), Q(t), R(t), G(t)$ are functions of t.

- The Trivial Solution: For any homogeneous equation (1, 2), $y = 0$ is a solution.

Repeated equal roots of the CE

- Recall the Homogeneous LSODEs, with constant coefficients:

$$\mathcal{L}(y) = ay'' + by' + cy = 0, \quad \text{with} \quad a, b, c \in \mathbb{R} \qquad (3)$$

$$\text{The CE of}(3): \quad ar^2 + br + c = 0 \qquad (4)$$

- In §3.2, we dealt with the case when (4) had two distinct real roots. In this section, we deal with case when CE has repeated roots. This will be the case, when $b^2 - 4ac = 0$.

Continued

- The equal root is $r = -\frac{b}{2a}$.
- Then, $y_1 = e^{-\frac{bt}{2a}}$ is a solution of (3), which can be checked by substitution in (3).
- In the next frame, we directly check, $y_2(t) = ty_1(t)$ is also a solution of (3).

Continued

- Substituting $y = y_2$ in (3): $\mathcal{L}(y_2) =$

$$ay_2'' + by_2' + cy_2 = a(2y_1' + ty_1'') + b(y_1 + ty_1') + c(ty_1)$$

$$= t(ay_1'' + by_1' + cy_1) + (2ay_1' + by_1) = t * 0 + (2ay_1' + by_1)$$

$$= 2a\left(-\frac{b}{2a}e^{-\frac{bt}{2a}}\right) + b\left(e^{-\frac{bt}{2a}}\right) = 0.$$

  This establishes that y_2 is a solution of (3).
- So, we have two solutions of (3):

$$\begin{cases} y_1 = e^{-\frac{bt}{2a}} \\ y_2 = ty_1 = te^{-\frac{bt}{2a}} \end{cases} \tag{5}$$

Fundamental Pair

We investigate, if y_1, y_2 form a Fundamental pair of solutions.

- Compute the Wronskian:

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-\frac{bt}{2a}} & te^{-\frac{bt}{2a}} \\ -\frac{b}{2a}e^{-\frac{bt}{2a}} & e^{-\frac{bt}{2a}} + t\left(-\frac{b}{2a}e^{-\frac{bt}{2a}}\right) \end{vmatrix}$$

$$= e^{-\frac{bt}{a}} \begin{vmatrix} 1 & t \\ -\frac{b}{2a} & 1 - \frac{bt}{2a} \end{vmatrix} = e^{-\frac{bt}{a}} \neq 0$$

- Since Wronskian $W(y_1, y_2) \neq 0$, $y_1 = e^{-\frac{bt}{2a}}, y_2 = te^{-\frac{bt}{2a}}$ form a fundamental pair solutions.

The General Solution

- By §3.3, any solution (the general solution) of (3) can be written as: $y = c_1 y_1 + c_2 y_2$ That is

$$y = c_1 e^{rt} + c_2 t e^{rt} \tag{6}$$

OR

$$y = (c_1 + c_2 t) e^{rt} \tag{7}$$

where $r = -\frac{b}{2a}$ is the double root of the CE and c_1, c_2 are arbitrary constants.

Example 1

Consider the IVP:

$$\begin{cases} y'' + 8y' + 16y = 0 \\ y(0) = 1 \\ y'(0) = 1 \end{cases}$$

Solve this IVP and give the graph of the solution.

Solution

- The CE $r^2 + 8r + 16 = 0$.
- Roots of the CE $r = \frac{-8 \pm \sqrt{64 - 4 * 16}}{2} = -4$, which is a double root.
- By (7), the general solution is

$$y = (c_1 + c_2 t)e^{rt} = (c_1 + c_2 t)e^{-4t} \tag{8}$$

- The derivative:

$$y' = c_2 e^{-4t} - 4(c_1 + c_2 t)e^{-4t} \tag{9}$$

Continued

- The initial conditions:

$$\begin{cases} y(0) = c_1 = 1 \\ y'(0) = c_2 - 4c_1 = 1 \end{cases} \Longrightarrow \begin{cases} c_1 = 1 \\ c_2 = 5 \end{cases}$$

- So, the solution is

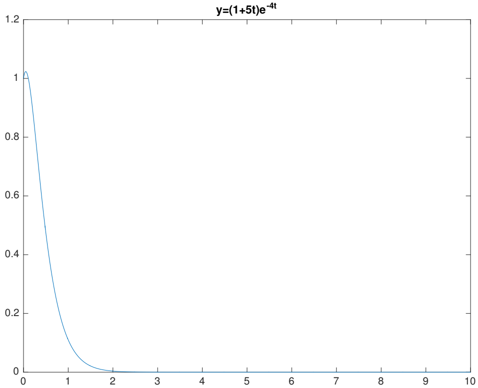
$$y = (1 + 5t)e^{-4t}$$

The Limit

We also compute the Limit:

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty}\left((1+5t)e^{-4t}\right) = 0$$

Graph of $y = y(t)$:



Example IA

Change the initial condition in (I) and consider the IVP:

$$\begin{cases} y'' + 8y' + 16y = 0 \\ y(1) = e^{-4} \\ y'(1) = e^{-4} \end{cases}$$

Solve this IVP and give the graph of the solution.

Solution

- The general solution remains the same as in (8) and the derivative y' is also as in (13)
- The initial conditions:

$$\begin{cases} y(1) = (c_1 + c_2)e^{-4} = e^{-4} \\ y'(1) = c_2 e^{-4} - 4(c_1 + c_2)e^{-4} = e^{-4} \end{cases} \Longrightarrow$$

$$\begin{cases} c_1 + c_2 = 1 \\ -4c_1 - 3c_2 = 1 \end{cases} \Longrightarrow \begin{cases} c_1 = -4 \\ c_2 = 5 \end{cases}$$

Continued

- From the general solution (8):

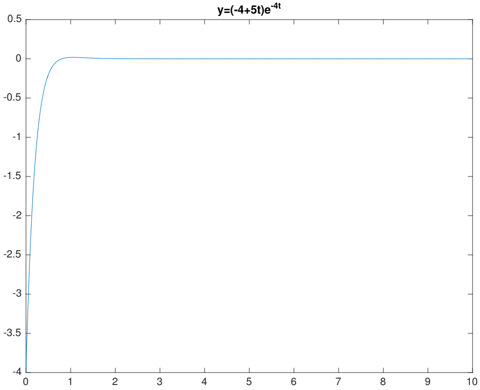
$$y = (c_1 + c_2 t)e^{-4t} = (-4 + 5t)\, e^{-4t}$$

The Limit

We also compute the Limit:

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \left(\left(-4+5t\right)e^{-4t}\right) = 0$$

Graph of $y = y(t)$:



In Example IB, we change the sign of $y'(1)$.

Example IB

Change the initial condition in (I), again and consider the IVP:

$$\begin{cases} y'' + 8y' + 16y = 0 \\ y(1) = e^{-4} \\ y?(1) = -e^{-4} \end{cases}$$

Solve this IVP and give the graph of the solution.

Solution

- The general solution remains the same as in (??) and the derivative y' is also as in (13)
- The initial conditions:

$$\begin{cases} y(1) = (c_1 + c_2)e^{-4} = e^{-4} \\ y'(1) = c_2 e^{-4} - 4(c_1 + c_2)e^{-4} = -e^{-4} \end{cases} \Longrightarrow$$

$$\begin{cases} c_1 + c_2 = 1 \\ -4c_1 - 3c_2 = -1 \end{cases} \Longrightarrow \begin{cases} c_1 = -2 \\ c_2 = 3 \end{cases}$$

Continued

- From the general solution (8):

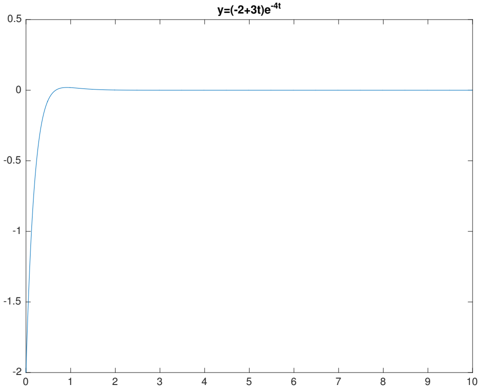
$$y = (c_1 + c_2 t)e^{-4t} = (-2 + 3t)\, e^{-4t}$$

The Limit

We also compute the Limit:

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \left((-2+3t)\, e^{-4t}\right) = 0$$

Graph of $y = y(t)$:



In Example IC, I will have $y'(1) = 0$.

Example IC

Change the initial condition in (I), again and consider the IVP:

$$\begin{cases} y'' + 8y' + 16y = 0 \\ y(1) = e^{-4} \\ y'(1) = 0 \end{cases}$$

Solve this IVP and give the graph of the solution.

Solution

- The general solution remains the same as in (??) and the derivative y' is also as in (13)
- The initial conditions:

$$\left\{\begin{array}{l} y(1) = (c_1 + c_2)e^{-4} = e^{-4} \\ y'(1) = c_2 e^{-4} - 4(c_1 + c_2)e^{-4} = 0 \end{array}\right. \Longrightarrow$$

$$\left\{\begin{array}{l} c_1 + c_2 = 1 \\ -4c_1 - 3c_2 = 0 \end{array}\right. \Longrightarrow \left\{\begin{array}{l} c_1 = -3 \\ c_2 = 4 \end{array}\right.$$

Continued

- From the general solution (8):

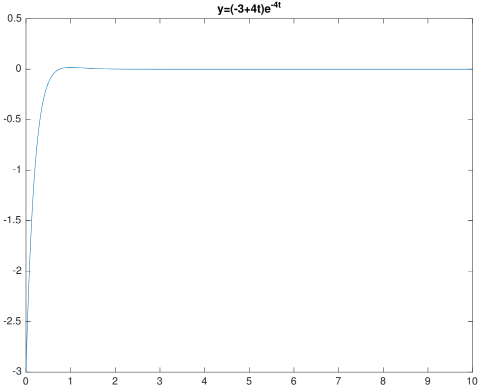
$$y = (c_1 + c_2 t)e^{-4t} = (-3 + 4t)\, e^{-4t}$$

The Limit

We also compute the Limit:

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \left(\left(-3+4t\right)e^{-4t}\right) = 0$$

Graph of $y = y(t)$:



Example 2

Consider the IVP:

$$\begin{cases} 4y'' - 20y' + 25y = 0 \\ y(0) = 1 \\ y'(0) = 1 \end{cases}$$

Solve this IVP and give the graph of the solution.

Solution

- The CE $4r^2 - 20r + 25 = 0$.
- Roots of the CE $r = \frac{20 \pm \sqrt{20^2 - 4*4*25}}{8} = \frac{5}{2}$, which is a double root.
- By (7), the general solution is

$$y = (c_1 + c_2 t)e^{rt} = (c_1 + c_2 t)e^{\frac{5t}{2}} \quad (10)$$

- The derivative:

$$y' = c_2 e^{\frac{5t}{2}} + \frac{5}{2}(c_1 + c_2 t)e^{\frac{5t}{2}} \quad (11)$$

Continued

- The initial conditions:

$$\begin{cases} y(0) = c_1 = 1 \\ y'(0) = c_2 + \frac{5}{2}c_1 = 1 \end{cases} \Longrightarrow \begin{cases} c_1 = 1 \\ c_2 = -\frac{3}{2} \end{cases}$$

Continued

- From the general solution (11):

$$y = (c_1 + c_2 t)e^{\frac{5t}{2}} = \left(1 - \frac{3}{2}t\right) e^{\frac{5t}{2}}$$

The Limit

We also compute the Limit:

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \left(\left(1 - \frac{3}{2}t\right) e^{\frac{5t}{2}} \right) = -\infty$$

Graph of $y = y(t)$:

To be Inserted

Example 3

Consider the IVP:

$$\begin{cases} 25\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + y = 0 \\ y(0) = 1 \\ y'(0) = 2 \end{cases}$$

Solve this IVP and give the graph of the solution.

Solution

- The CE $25r^2 + 10r + 1 = 0$.
- Roots of the CE $r = \frac{-10 \pm \sqrt{100-4*25}}{50} = -\frac{1}{5}$, which is a double root.
- By (7), the general solution is

$$y = (c_1 + c_2 t)e^{rt} = (c_1 + c_2 t)e^{-\frac{t}{5}} \tag{12}$$

- The derivative:

$$y' = c_2 e^{-\frac{t}{5}} - \frac{1}{5}(c_1 + c_2 t)e^{-\frac{t}{5}} \tag{13}$$

Continued

- The initial conditions:

$$\left\{\begin{array}{l} y(0)=c_1=1 \\ y'(0)=c_2-\frac{1}{5}c_1=2 \end{array}\right. \Longrightarrow \left\{\begin{array}{l} c_1=1 \\ c_2=\frac{11}{5} \end{array}\right.$$

Continued

- From the general solution (12):

$$y = (c_1 + c_2 t)e^{-\frac{t}{5}} = \left(1 + \frac{11}{5}t\right) e^{-\frac{t}{5}}$$

The Limit

We also compute the Limit:

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \left(\left(1 + \frac{11}{5} t \right) e^{-\frac{t}{5}} \right) = 0$$

Graph of $y = y(t)$: