Chapter 3: Second Order ODE
§3.8 Elements of Particle Dynamics

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Objective

The objective of this section is to explain that any second degree linear ODE represents the motion of a particle. This emanates from Newton's Laws of Motion that

$$\text{Force} = \text{Mass} \times \text{Acceleration} \tag{1}$$

Consider a 2^{nd}-order linear ODE, with constant coefficients:

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = g(t) \quad \text{with} \quad a, b, c \in \mathbb{R}, \ \ a \neq 0 \tag{2}$$

Without loss of generality, we can assume that $a > 0$.

Continued

Now suppose the following:

1. Let $y = y(t)$ be the position, at time t, of a particle of mass $a = m$, moving in a straight line. Then, the acceleration of the particle, at time t, is $\frac{d^2y}{dt^2}$. So, total force acting on the particle, at time t is

$$F(t) = a\frac{d^2y}{dt^2}.$$

By Newton's Law (1), this would equal to the total force acting on the particle.

Forces Acting on the Particle

Depending on the situation, the forces acting on the particle, could be classified, as follows:

- Forces $f_1(t)$ that is proportional to the velocity $\frac{dy}{dt}$ (*for example, drag*). So, $f_1(t) = -b\frac{dy}{dt}$, for some constant $-b$.
- Forces $f_0(t)$ that is proportional to the position/distance y of the particle. So, $f_0(t) = -cy$, for some constant $-c$.
- Other external forces that are not one of the above, which we denote by $g(t)$. Gravitational pull could be one such example.

Agreement

So, the total force, at time t is:

$$F(t) = f_1(t) + f_0(t) + g(t) = -b\frac{dy}{dt} - cy + g(t)$$

$$\text{By Newton's Law (1)} \quad a\frac{d^2y}{dt^2} = -b\frac{dy}{dt} - cy + g(t)$$

which is precisely the ODE (2):

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = g(t)$$

Conclusion

The 2^{nd}-order ODE

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = g(t) \quad \text{with} \quad a, b, c \in \mathbb{R}, \; a > 0 \qquad (3)$$

represents motions of a particle of mass $m := a$. Depending on the actual situation, we may have

$$b = 0 \qquad \text{or} \qquad c = 0 \qquad \text{or} \qquad g(t) = 0$$

Simple Harmonic Motion (SHM)

The ODE (3), is said to represent Simple Harmonic Motion, if $b = 0$, $g(t) = 0$, with $c > 0$. Changing notations, the Simple Harmonic Motion is represented by the Equation:

$$a\frac{d^2y}{dt^2} + \kappa y = 0 \quad \text{with} \quad 0 < a, \kappa < \infty \tag{4}$$

$$\text{With} \quad \omega = \sqrt{\frac{\kappa}{a}}, \quad \text{roots of the CE} \quad r = \pm\omega i$$

A fundamental set of solutions are

$$\begin{cases} y_1 = \cos\omega t \\ y_2 = \sin\omega t \end{cases}$$

Continued: SHM

$$\text{So, a general solution is} \quad y = A\cos\omega t + B\sin\omega t$$

$$\text{With} \quad \begin{cases} A = R\cos\delta \\ B = R\sin\delta \\ R = \sqrt{A^2 + B^2} \end{cases} \qquad y = R\cos(\omega t - \delta) \qquad (5)$$

Further, note one could "use" sin in (5), because

$$y = R\cos(\omega t - \delta) = R\sin\left(\omega t - \delta + \frac{\pi}{2}\right),$$

Math to Mechanics: SHM

The solution of SHM (4), in the latter form (5)

$$y = R\cos(\omega t - \delta) \qquad \text{where} \quad \omega = \sqrt{\frac{\kappa}{a}} \tag{6}$$

is much preferred in Mechanics and Engineering. The unknown constants R, δ can be computed from initial conditions. We define:

$$\begin{cases} R = \textbf{Amplitude} \\ \omega = \textbf{Frequency} \\ T := \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{\kappa}{a}} = \textbf{Periodicity} \\ \delta = \textbf{Phase} \end{cases}$$

Simple Harmonic Oscillation

Simple Harmonic Motion is also referred to as
Simple Harmonic Oscillation.

Simple Pendulum

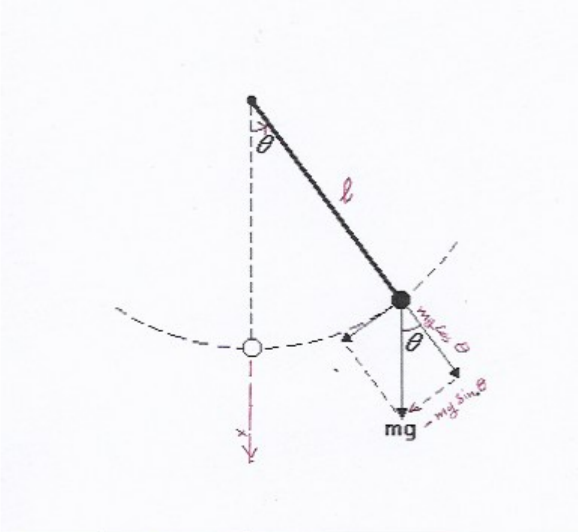
The motion of a "simple pendulum" would be an example of Simple Harmonic Motion (up to an approximation). Most of us are familiar with Pendulum Clocks and, Pendulums. A "simple pendulum" is an idealized physical structure, defined as follows, for purpose of studying the motion of a pendulum.

Definition. A Simple Pendulum consists of a point mass m (*to be called the "bob"*), hanging from a string of length ℓ. The other end of the string is fixed at a pivot point O. The bob is displaced by an initial angle θ_0, and released from rest, to oscillate freely. Assume, the string has no mass, there is no friction at the pivot point O and there is no resistance due to air or otherwise.

Modeling; Polar Coordinate Syatem

To model the motion of the simple pendulum, use polar coordinate system (r, θ), as in the diagram, in the next frame. The pivot point O is the origin and vertical line downward through O is x-axis. The positive direction, for θ would be the anti-clockwise direction.

Notation The length of the string ℓ being fixed, let $(\ell, \theta(t))$ denote the position of the bob, t seconds after it is released from rest. We study the function $\theta = \theta(t)$.


θ
ℓ
θ
mg cos θ
mg sin θ
mg
x

The Model of Angular Motion

The table of actions on the bob, and their components:

Components :	Tangential
Gravitaty :	$-mg\sin\theta$
Acceleration :	$\ell\frac{d^2\theta}{dt^2}$

By Newton's Law, the model for the motion of the simple pendulum:

$$-mg\sin\theta = m\ell\frac{d^2\theta}{dt^2} \quad \Longrightarrow \quad \frac{d^2\theta}{dt^2} = -\frac{g}{\ell}\sin\theta \qquad (7)$$

Approximation to The Model

- Pendulum oscillated within small angle θ. And, for small angles $\sin\theta \approx \theta$, approximately.
- So, the model (8) is approximated by the model:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{\ell}\theta \qquad \text{Or} \qquad \frac{d^2\theta}{dt^2} + \frac{g}{\ell}\theta = 0 \tag{8}$$

This establishes that the angular motion of the Pendulum is, approximately, a Simple Harmonic Motion.

Solution of SHM

$$\text{By (6)}: \qquad \begin{cases} \text{The Periodicity}: \quad \omega = \sqrt{\frac{g}{\ell}} \\ \theta(t) = R\cos\left(\sqrt{\frac{g}{\ell}}t - \delta\right) \end{cases}$$

The Initial Conditions:

$$\begin{cases} \theta(0) = R\cos(\delta) = \theta_0 \\ \theta'(0) = R\sqrt{\frac{g}{\ell}}\sin(\delta) = 0 \end{cases} \Longrightarrow \begin{cases} R = \theta_0 \\ \delta = 0 \end{cases}$$

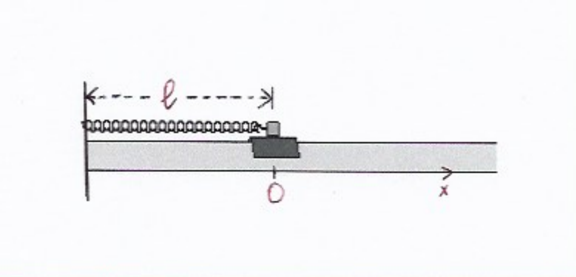
$$\text{So,} \qquad \begin{cases} \theta(t) = \theta_0\cos\left(\sqrt{\frac{g}{\ell}}t\right) \\ \text{Amplitude} := \theta_0 \\ \text{Periodicity} := \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{\ell}{g}} \end{cases}$$

A Simple Spring-Mass System

A Simple Spring-Mass System is another example of SHM.

Definition. A Simple Spring-Mass System consists of an object, of mass m, resting on a frictionless rail, and a spring. One end of the string is tied to the wall, above a frictionless rail (without touching), and other end to the object. Further,

1. At equilibrium, the length of this sytem is ℓ.
2. Object is pulled from the equilibrium point and releases from rest. The object starts sliding back and fourth.



The Spring Mass-System: In Equilibrium Position

- The origin "O" is the position of the object, at equilibrium. The Object is pulled (or compressed) to the position x_0, and released from rest.
- $x = x(t)$ would denote the position of the object, at time t, after release from rest. So, $x(0) = x_0$ and $x'(0) = 0$.

The Model

- At time t, the only force acting on the body is due to the pull toward the position of equilibrium O, due to spring action. (*Use Hooke's Law!*) So, $F(t) = -\kappa x(t)$, where $\kappa > 0$ is the spring constant. The negative sign accounts for the fact that $F(t)$ acts opposite to the positive direction of x.
- By Newton's Law

$$m\frac{d^2x}{dt^2} = F(f) \Longrightarrow m\frac{d^2x}{dt^2} + \kappa x(t) = 0$$

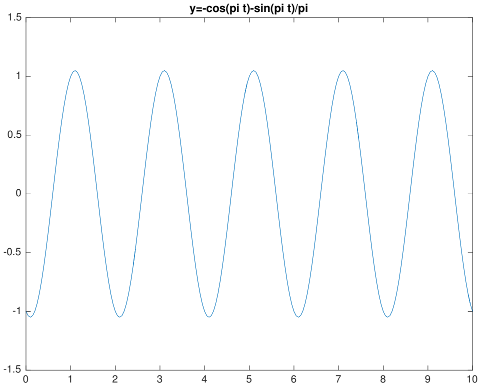
This is exactly the Simple Harmonic Motion Model (4).

Exercise

Exercise. Given $x(0) = x_0$ and $x'(0) = 0$, write down the Equation of $x = x(t)$. In Particular, give the Amplitude, periodicity and Frequency.

SHM: The Typical Graph

A Typical Graph of the solution of (4),
Simple Harmonic Motion:



Damped Harmonic Motion: Definition

Refer back to the Equation 3. It was argued that a 2^{nd}-order ODE (3) represents the motion of particle, of mass $m = a$.

Definition. The motion of a particle, as in (3), is referred to as Damped Harmonic Motion (DHM), if

$$g(t) = 0, \quad a > 0, b > 0, c > 0 \quad \text{and} \quad b^2 - 4ac < 0$$

Changing notations, a DHM is given by

$$a\frac{d^2y}{dt^2} + \gamma\frac{dy}{dt} + \kappa y = 0 \quad \begin{cases} a > 0, \gamma > 0, \kappa > 0 \\ \text{and} \quad \gamma^2 - 4a\kappa < 0 \end{cases} \qquad (9)$$

DHM: Qualifications

- The qualification "Damped" refers to the term $\gamma \frac{dy}{dt}$, which represents the force due to damp, because of resistance, friction etc. The model assumes that this is proportional to the velocity $\frac{dy}{dt}$ and γ is the dampening constant.
- The qualification "Damped" is a also justified by the solution of (9), as was dealt with in Section 3.5, as explained in the next frame.
- The qualification "Harmonic", corresponds to the assumption, $\gamma^2 - 4a\kappa < 0$. This leads to complex roots of the CE, and hence a periodic component of the solution.

DHM: Example

A Spring-Mass System described above, assumes the ideal condtion that the rail was friction less. However, in reality, the rail would provide some resistance. A model of the Sprint-Mass System would be a damped harmonic motion, if

1. The model incorporates the resistance due to friction.
2. It assumes, that the resistance is proportional to velocity.
3. The resistance is small enough, so that $\gamma^2 - 4a\kappa < 0$.

DHM: Solution

- The CE of the ODE (9) and its roots:

$$ar^2 + \gamma r + \kappa = 0 \quad \Longrightarrow \quad r = -\frac{\gamma}{2a} \pm \frac{\sqrt{4a\kappa - \gamma^2}}{2a} i$$

$$\text{With} \quad \lambda = \frac{b}{2a}, \quad \omega = \frac{\sqrt{4a\kappa - \gamma^2}}{2a}, \quad r = -\lambda \pm \omega i$$

- So, a fundamental pair of solutions of the DHM (9) is

$$\begin{cases} y_1 = e^{-\lambda t} \cos \omega t \\ y_1 = e^{-\lambda t} \sin \omega t \end{cases}$$

DHM: The General Solution

So, the general solutions is (with A, B constants):

$$y = Ay_1 + By_2 = e^{-\lambda t}\left(A\cos\omega t + \sin\omega t\right)$$

$$\text{With } \begin{cases} A = R\cos\delta \\ B = R\sin\delta \\ R = \sqrt{A^2 + B^2} \end{cases} \qquad y = Re^{-\lambda t}\cos\left(\omega t - \delta\right) \qquad (10)$$

One can compute R and δ, from initial conditions. Also, as in the case of Simple Harmonic motion, we could write

$$y = Re^{-\lambda t}\cos\left(\omega t - \delta\right) = Re^{-\lambda t}\sin\left(\omega t - \delta + \frac{\pi}{2}\right)$$

So, we have choices of both cos or sin.

DHM: The Limit

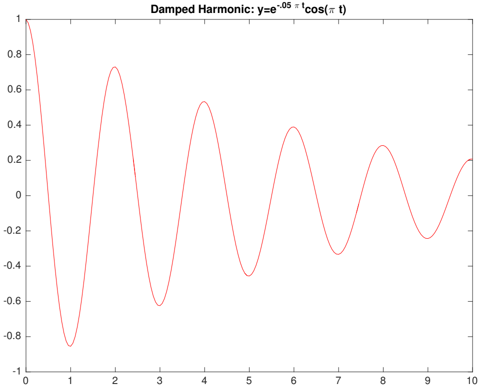
Note $\lambda = \frac{b}{2a} > 0$. So,

$$\lim_{t\to\infty} y = \lim_{t\to\infty} \left(Re^{-\lambda t} \cos\left(\omega t - \delta\right)\right) = 0$$

This, again, justifies the qualification "Damped".

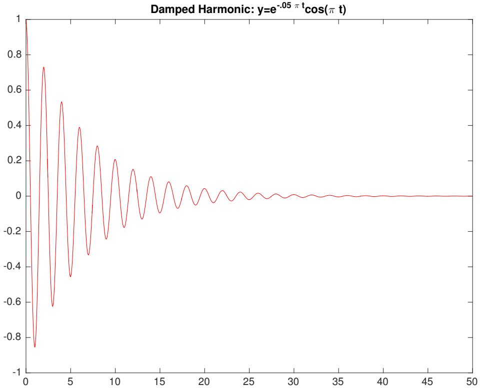
DHM: The Typical Graph

A Typical Graph of Damped Harmonic Motion (10):



DHM: The Typical Graph

The same graph, of (10), with $0 \leq t \leq 50$ range:



Unsteady Harmonic Motion: Definition

As opposed to Damped harmonic motion and in analogy to the terminology "unsteady Oscillation" in Section 3.5, we discuss Unsteady Harmonic Motion. Again, refer back to the Equation 3, and it represents the motion of particle, of mass $m = a$.

Definition. The motion of a particle, as in (3), is referred to as Unsteady Harmonic Motion (UHM), if

$$g(t) = 0, \quad a > 0, b < 0, c > 0 \quad \text{and} \quad b^2 - 4ac < 0$$

Changing notations, a UHM is given by

$$a\frac{d^2y}{dt^2} - \gamma\frac{dy}{dt} + \kappa y = 0 \quad \begin{cases} a > 0, \gamma > 0, \kappa > 0 \\ \text{and} \quad \gamma^2 - 4a\kappa < 0 \end{cases} \tag{11}$$

UHM: Solution

Repeat the steps above, to get a solutions of the ODE (11):

- The CE of the ODE (11) and its roots:

$$ar^2 - \gamma r + \kappa = 0 \quad \Longrightarrow \quad r = \frac{\gamma}{2a} \pm \frac{\sqrt{4a\kappa - \gamma^2}}{2a} i$$

$$\text{With} \quad \lambda = \frac{b}{2a}, \quad \omega = \frac{\sqrt{4a\kappa - \gamma^2}}{2a}, \quad r = \lambda \pm \omega i$$

- So, a fundamental pair of solutions of the DHM (9) is

$$\begin{cases} y_1 = e^{\lambda t} \cos \omega t \\ y_1 = e^{\lambda t} \sin \omega t \end{cases}$$

DHM: The General Solution

So, the general solutions is (with A, B constants):

$$y = Ay_1 + By_2 = e^{\lambda t}\left(A\cos\omega t + \sin\omega t\right)$$

$$\text{With } \begin{cases} A = R\cos\delta \\ B = R\sin\delta \\ R = \sqrt{A^2 + B^2} \end{cases} \qquad y = Re^{\lambda t}\cos\left(\omega t - \delta\right) \qquad (12)$$

One can compute R and δ, from initial conditions. Also, as before, we could write

$$y = Re^{\lambda t}\cos\left(\omega t - \delta\right) = Re^{-\lambda t}\sin\left(\omega t - \delta + \frac{\pi}{2}\right)$$

So, we have choices of both cos or sin.

DHM: The Limit

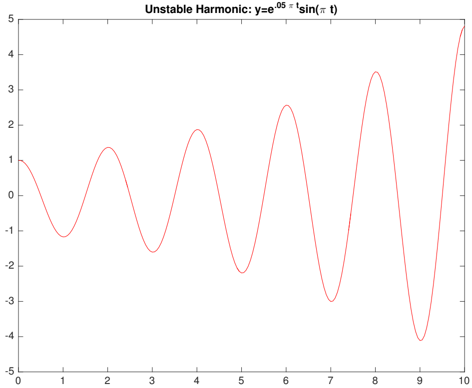
Note $\lambda = \frac{b}{2a} > 0$. So,

$$\lim_{t\to\infty} y = \lim_{t\to\infty} \left(Re^{\lambda t} \cos(\omega t - \delta) \right) = DNE$$

The Exponential part $e^{\lambda t}$ keeps blowing up the periodic part $R\cos(\omega t - \delta)$.

DHM: The Typical Graph

A Typical Graph of Unsteady Harmonic Motion (12):



Unforced and Forced Motions

Again, refer to the ODE (3):

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = g(t) \quad \text{with} \quad a, b, c \in \mathbb{R}, \ a > 0 \qquad (13)$$

This represents the motion of a particle, with mass $m := a$.

Definition. We have two defintion:

- The ODE (13) is said to represents an Unforced Motion, if $g(t) = 0$.
- The ODE (13) is said to represents an Forced Motion, if $g(t) \neq 0$.

Unforced Motion

We comment:

- According to the defintion, an unforced motion is represented by a Homogeneous Linear Equation, with constant coefficients:

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0 \quad \text{with} \quad a, b, c \in \mathbb{R}, \; a > 0 \quad (14)$$

This was dealt with in Section 3.2, 3.4, 3.5.

Hormonic Motion

We considered Three Types of Unforced ODEs:

- Simple Harmonic Motion.
- Dampled Harmonic Motion.
- Unsteady Harmonic Motion.

The common denominator, among these three Harmonic Motions, was that the CE of (14) had complex roots. That means,

$$b^2 - 4ac < 0$$

The complex root, imposes a periodic oscillation to the solution. The case of complex roots was dealt with in § 3.5.

Classification Hormonic Motion

Let (14) represent a Harmonic Motion. So, $b^2 - 4ac < 0$. The real part of the roots of the CE is

$$\lambda = -\frac{b}{2a}. \qquad \text{Also,} \quad a > 0. \qquad \text{Therefore,}$$

- If $b = 0$, (14) represents a Simple Harmonic Motion.
- If $b > 0$, (14) represents a Damped Harmonic Motion.
- If $b < 0$, (14) represents a Unsteady Harmonic Motion.

Forced Motion

According to the definition, an unforced motion is represented by a Homogeneous Linear Equation, with constant coefficients:

$$a\frac{d^2y}{dt^2}+b\frac{dy}{dt}+cy=g(t)\neq 0 \quad \text{with} \quad a,b,c\in\mathbb{R},\ a>0 \quad (15)$$

This was dealt with in Section 3.6, 3.7

Literature on Forced Motion

There is some discussions in the literature (Textbooks and Internet), regarding Forced Motion. Many consider, following two cases of (15):

$$\begin{cases} a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = F_0 \cos\omega_0 t \\ a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = F_0 \sin\omega_0 t \end{cases} \quad \text{with} \quad a, b, c \in \mathbb{R}, \;\; a > 0$$

(*This ω_0 should not be mixed up with the notation ω, used above for periodicity.*) We solve some such problems in §3.6, 3.7.

The Problem Set and Homework

- I expect you to read this section carefully and be able to derive some of what discussed above, in the Takehome part of the exams.
- We would refrain from the customary practice, of adding a set of examples or problems, at the end of this section. We would do likewise, regarding Homework on this section. As is mentioned above, we solved or assigned problems on these concepts, in §3.5, 3.6, 3.7.

Continued

- *The reason for this departure from the customary practice is two fold. The problem sets on this topic in the literature appears a little artificial, to me. Some of problems are, essentially same as those in §3.5, 3.6, 3.7, encased within a story on Mechanics. Other set of problems, ask to compute Amplitude, Periodicity etc., which may belong in the Mechanics classes.*