Chapter 5: System of 1^{st}-Order Linear ODE §5.6 Complex Eigenvalues

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20 March 2018

Complex Eigenvalues

- We continue to consider homogeneous linear systems with constant coefficients:

$$\mathbf{y}' = \mathbf{A}\mathbf{y} \qquad \mathbf{A} \text{ is an n} \times \text{n matrix with constant entries} \tag{1}$$

- In §5.5, we considered the situation when all the eigenvalues of **A**, were real and distinct. In this section, we consider when some of the eigen values are complex.
- As in §5.4, solutions of (1) will be denoted by

$$\mathbf{y}^{(1)}(t), \cdots, \mathbf{y}^{(n)}(t).$$

Principle of superposition

- Recall the Principle of superposition and the converse (§5.4): IF $\mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(n)}$ are solution of (1), then, any constant linear combination

$$\mathbf{y} = c_1\mathbf{y}^{(1)} + \cdots + c_n\mathbf{y}^{(n)} \tag{2}$$

  is also a solution of the same system (1).
- The converse is also true, if Wronskian $W \neq 0$.
- Further, if r is an eigenvalue of $\mathbf{A}$ and ξ is an eigenvector of A, corresponding to r, then

$$\mathbf{y} = \xi e^{rt} \quad \text{is a solution of } (1) \tag{3}$$

Complex eigenvalues and vectors

- Now, suppose **A** has a complex eigenvalue $r_1 = \lambda + i\mu$ and $\xi^{(1)}$ is an eigenvector, for r_1. That means

$$(\mathbf{A} - (\lambda + i\mu)I)\xi^{(1)} = \mathbf{0}. \qquad (4)$$

- Apply conjugation to (4):

$$(A - (\lambda - i\mu)I)\overline{\xi^{(1)}} = \mathbf{0} \qquad \text{This means :}$$

  - $r_2 = \overline{r_1} = \lambda - i\mu$ an eigenvalue of **A**. And,
  - $\xi^{(2)} = \overline{\xi^{(1)}}$ is an eigenvector of **A**, corresponding to r_2.

Continued: Two conjugate complex Solutions

- Two eigen values $r_1, r_2 = \overline{r_1}$ and the corresponding eigenvalues gives two solutions of (1):

$$\mathbf{y}^{(1)} = \xi^{(1)} e^{r_1 t}, \qquad \mathbf{y}^{(2)} = \xi^{(2)} e^{r_2 t} \tag{5}$$

- Write $\xi^{(1)} = \mathbf{a} + i\mathbf{b}$, where $\mathbf{a}, \mathbf{b}$ real real vectors. Then,

$$\mathbf{y}^{(1)} = (\mathbf{a} + i\mathbf{b})e^{(\lambda+i\mu)t} = (\mathbf{a} + i\mathbf{b})[e^{\lambda t}(\cos \mu t + i \sin \mu t)]$$

$$= e^{\lambda t}(\mathbf{a} \cos \mu t - \mathbf{b} \sin \mu t) + i e^{\lambda t}(\mathbf{a} \sin \mu t + \mathbf{b} \cos \mu t)$$

Continued: Two Real Solutions

- Both real and imaginary part of $\mathbf{y}^{(1)}$ are solutions of (1), as follows:

$$\begin{cases} \mathbf{u} = e^{\lambda t}(\mathbf{a}\cos\mu t - \mathbf{b}\sin\mu t) \\ \mathbf{v} = e^{\lambda t}(\mathbf{a}\sin\mu t + \mathbf{b}\cos\mu t) \end{cases} \tag{6}$$

- These real solutions $\mathbf{u}, \mathbf{v}$ fit in very well as a part of a fundamental set of n solutions. There will be too many cases to make this statement precise, in complete details. However, we make a statement in the following frame.
- Often, we will consider systems of 2 or 3 equations. So, following statement will suffice, in most cases.

As part of Fundamental set

Theorem 5.6.1 Consider the homogenous linear system (1): $\mathbf{y}' = A\mathbf{y}$, where A is an $n \times n$ matrix, with real entries.

- Suppose $r_1 = \lambda + i\mu$, $r_1 = \lambda - i\mu$ are two conjugate eigenvalues of $\mathbf{A}$. As above, let $\xi^{(1)} = \mathbf{a} + i\mathbf{b}$ is an eigenvector of r_1. Accordingly, the conjugate $\xi^{(2)} = \mathbf{a} + i\mathbf{b}$ is an eigenvector of r_2.
- Let $\mathbf{u}, \mathbf{v}$ be as in (6).

Then, there are (real) solutions $\mathbf{y}^{(3)}, \ldots, \mathbf{y}^{(n)}$ of (1), such that $\mathbf{u}, \mathbf{v}, \mathbf{y}^{(3)}, \ldots, \mathbf{y}^{(n)}$ forms a fundamental set of solutions of (1).

Continued

Further, the solutions $\mathbf{y}^{(3)}, \ldots, \mathbf{y}^{(n)}$ are determined by the eigenvalues $r_1, r_2, r_3, \ldots, r_n$, (real or complex) and their multiplicities.

Hence, any solution $\mathbf{x}$ has the form (2):

$$\mathbf{x} = c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{y}^{(3)} + \cdots + c_n\mathbf{y}^{(n)} \tag{7}$$

Example 1

Find the general solution (real valued) of the equation:

$$\mathbf{y}' = \begin{pmatrix} -3 & 5 \\ -1 & 1 \end{pmatrix} \mathbf{y} \tag{8}$$

- Eigenvalues of the coef. matrix **A**, are: given by

$$\begin{vmatrix} -3-r & 5 \\ -1 & 1-r \end{vmatrix} = 0 \quad r = -1+i, -1-i$$

Eigenvectors

Analytically, eigenvectors for $r = -1 + i$ is given by $(\mathbf{A} - rI)\xi = \mathbf{0}$, which is

$$\begin{pmatrix} -3 - (-1 + i) & 5 \\ -1 & 1 - (-1 + i) \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The second row is $2 + i$-times the first row. It follows:

$$\begin{pmatrix} -2 - i & 5 \\ -1 & 2 - i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow$$

Continued

$$\begin{pmatrix} 0 & 0 \\ -1 & 2-i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow$$

With $\xi_2 = 1$, an eigenvector of $r = -1 + i$ is

$$\xi^{(1)} = \begin{pmatrix} 2-i \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + i \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

The solution

- So, the real and the imaginary part of $\xi^{(1)}$ are:

$$\mathbf{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

- With $r = -1 + i$, we have $\lambda = -1, \mu = 1$. By (6),

$$\begin{cases} \mathbf{u} = e^{-t}\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix} \cos t - \begin{pmatrix} -1 \\ 0 \end{pmatrix} \sin t\right) \\ \mathbf{v} = e^{-t}\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix} \sin t + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \cos t\right) \end{cases}$$

Continued

So, the general solution of (8)

$$\mathbf{y} = c_1\mathbf{u} + c_2\mathbf{v}$$

$$= c_1 e^{-t}\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}\cos t - \begin{pmatrix} -1 \\ 0 \end{pmatrix}\sin t\right) +$$

$$c_2 e^{-t}\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}\sin t + \begin{pmatrix} -1 \\ 0 \end{pmatrix}\cos t\right)$$

Continued

- $\mathbf{y} =$

$$c_1 e^{-t} \begin{pmatrix} 2\cos t + \sin t \\ \cos t \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 2\sin t - \cos t \\ \sin t \end{pmatrix}$$

Example 2

Find the general solution (real valued) of the equation:

$$\mathbf{y}' = \begin{pmatrix} 1 & 2 & 3 \\ -2 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{y} \qquad (9)$$

- Eigenvalues of the coef. matrix **A**, are:

$$\begin{vmatrix} 1-r & 2 & 3 \\ -2 & 1-r & 2 \\ 0 & 0 & 1-r \end{vmatrix} = 0$$

$$(1-r)\begin{vmatrix} 1-r & 2 \\ -2 & 1-r \end{vmatrix} = 0$$

So, $r = 1, 1 \pm 2i$

Eigenvectors

- Eigenvectors for $r = 1$ is given by $(\mathbf{A} - rI)\mathbf{x} = \mathbf{0}$, which is

$$\begin{pmatrix} 1-1 & 2 & 3 \\ -2 & 1-1 & 2 \\ 0 & 0 & 1-1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 2 & 3 \\ -2 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Use TI-84 (rref):

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1.5 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

With $\xi_3 = 2$, an eigenvector of $r = 1$ is: $\xi^{(1)} = \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$.

The corresponding solution $\mathbf{y}^{(1)} = \xi^{(1)} e^{rt} = \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} e^{t}$

Eigenvectors

- Eigenvectors for $r = 1 + 2i$ is given by $(\mathbf{A} - rI)\xi = \mathbf{0}$, which is

$$\begin{pmatrix} 1-(1+2i) & 2 & 3 \\ -2 & 1-(1+2i) & 2 \\ 0 & 0 & 1-(1+2i) \end{pmatrix} \xi = \mathbf{0}$$

$$\begin{pmatrix} -2i & 2 & 3 \\ -2 & -2i & 2 \\ 0 & 0 & -2i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

So,

$$\begin{cases} -2i\xi_1 + 2\xi_2 + 3\xi_3 = 0 \\ -2\xi_1 - 2i\xi_2 + 2\xi_3 = 0 \\ -2i\xi_3 = 0 \end{cases} \begin{cases} -2i\xi_1 + 2\xi_2 = 0 \\ -2\xi_1 - 2i\xi_2 = 0 \\ \xi_3 = 0 \end{cases} \begin{cases} -i\xi_1 + \xi_2 = 0 \\ 0 = 0 \\ \xi_3 = 0 \end{cases}$$

With $\xi_1 = 1$, an eigenvector of $r = 1 + 2i$ is:

$$\xi^{(2)} = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} i$$

Solutions corresponding to $r = 1 \pm 2i$

By (6) two real solutions, corresponding to $r = 1 \pm 2i$ are:

$$\begin{cases} \mathbf{u} = e^{\lambda t}(\mathbf{a}\cos\mu t - \mathbf{b}\sin\mu t) \\ \mathbf{v} = e^{\lambda t}(\mathbf{a}\sin\mu t + \mathbf{b}\cos\mu t) \end{cases}$$

$$\begin{cases} \mathbf{u} = e^t\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\cos 2t - \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\sin 2t\right) = e^t\begin{pmatrix} \cos 2t \\ -\sin 2t \\ 0 \end{pmatrix} \\ \mathbf{v} = e^t\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\sin 2t + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\cos 2t\right) = e^t\begin{pmatrix} \sin 2t \\ \cos 2t \\ 0 \end{pmatrix} \end{cases}$$

The general solution

Combining $\mathbf{y}^{(1)}, \mathbf{u}, \mathbf{v}$, by (7), the general solution of (9) is

$$\mathbf{x} = c_1\mathbf{x}^{(1)} + c_2\mathbf{u} + c_3\mathbf{v}$$

$$= c_1 \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} e^t + c_2 e^t \begin{pmatrix} \cos 2t \\ -\sin 2t \\ 0 \end{pmatrix} + c_3 e^t \begin{pmatrix} \sin 2t \\ \cos 2t \\ 0 \end{pmatrix}$$

titleExample 3 Solve the IVP

$$\mathbf{y}' = \begin{pmatrix} 1 & -3 \\ 2 & 3 \end{pmatrix} \mathbf{y}, \qquad \mathbf{y}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{10}$$

- Eigenvalues of the coef. matrix **A**, are: given by

$$\begin{vmatrix} 1-r & -3 \\ 2 & 3-r \end{vmatrix} = 0 \Longrightarrow (1-r)(3-r)+6=0$$

So,

$$r^2-4r+9=0 \Longrightarrow r=\frac{4\pm\sqrt{16-36}}{2}$$

So,

$$r = 2 \pm \sqrt{5}i$$

Eigenvectors

Analytically, eigenvectors for $r = 2 + \sqrt{5}i$ is given by $(\mathbf{A} - rI)\xi = \mathbf{0}$, which is

$$\begin{pmatrix} 1-(2+\sqrt{5}i) & -3 \\ 2 & 3-(2+\sqrt{5}i) \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

It follows:

$$\begin{pmatrix} -1-\sqrt{5}i & -3 \\ 2 & 1-\sqrt{5}i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow$$

Continued

$$\begin{pmatrix} 0 & 0 \\ 2 & 1-\sqrt{5}i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Taking $\xi_2 = 2$, an eigen vector is

$$\xi^{(1)} = \begin{pmatrix} -(1-\sqrt{5}i) \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} + i \begin{pmatrix} \sqrt{5} \\ 0 \end{pmatrix}$$

Two Real Solutions

With $r = 2 + \sqrt{5}i$, we have $\lambda = 2, \mu = \sqrt{5}$. By (6), we have tow real solutions:

$$\begin{cases} \mathbf{u} = e^{2t}\left(\begin{pmatrix} -1 \\ 2 \end{pmatrix} \cos\sqrt{5}t - \begin{pmatrix} \sqrt{5} \\ 0 \end{pmatrix} \sin\sqrt{5}t\right) \\ \mathbf{v} = e^{2t}\left(\begin{pmatrix} -1 \\ 2 \end{pmatrix} \sin\sqrt{5}t + \begin{pmatrix} \sqrt{5} \\ 0 \end{pmatrix} \cos\sqrt{5}t\right) \end{cases}$$

Continued

We simplify:

$$\begin{cases} \mathbf{u} = e^{2t} \begin{pmatrix} -\cos\sqrt{5}t - \sqrt{5}\sin\sqrt{5}t \\ 2\cos\sqrt{5}t \end{pmatrix} \\ \mathbf{v} = e^{2t} \begin{pmatrix} -\sin\sqrt{5}t + \sqrt{5}\cos\sqrt{5}t \\ 2\sin\sqrt{5}t \end{pmatrix} \end{cases}$$

The General solution

$$\text{So, the general solutions is } \mathbf{y} = c_1\mathbf{u} + c_2\mathbf{v}$$

$$= c_1 e^{2t} \begin{pmatrix} -\cos\sqrt{5}t - \sqrt{5}\sin\sqrt{5}t \\ 2\cos\sqrt{5}t \end{pmatrix}$$

$$+c_2 e^{2t} \begin{pmatrix} -\sin\sqrt{5}t + \sqrt{5}\cos\sqrt{5}t \\ 2\sin\sqrt{5}t \end{pmatrix} =$$

$$e^{2t} \begin{pmatrix} -\cos\sqrt{5}t - \sqrt{5}\sin\sqrt{5}t & -\sin\sqrt{5}t + \sqrt{5}\cos\sqrt{5}t \\ 2\cos\sqrt{5}t & 2\sin\sqrt{5}t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Use Initial Values

Using initial conditions:

$$\begin{pmatrix} -1 & \sqrt{5} \\ 2 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

So, $c_1 = \frac{1}{2}$, $c_2 = \frac{3}{2\sqrt{5}}$

Answer

The particular solutions is: $\mathbf{y} =$

$$e^{2t}\begin{pmatrix} -\cos\sqrt{5}t - \sqrt{5}\sin\sqrt{5}t & -\sin\sqrt{5}t + \sqrt{5}\cos\sqrt{5}t \\ 2\cos\sqrt{5}t & 2\sin\sqrt{5}t \end{pmatrix}\begin{pmatrix} \frac{1}{2} \\ \frac{3}{2\sqrt{5}} \end{pmatrix}$$