Chapter 5: System of 1^{st}-Order Linear ODE
§5.1 Introduction

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System of ODE

- Like in Linear Algebra, the word "system" refers to a system of more than one unknown variables and, possibly, equations.
- As before t will denote the independent variable and $y_1, \ldots, y_n$ will denote (unknown) dependent variables.

System of 1^{st}-Order ODE

- A system of first order DE looks like:

$$\begin{cases} \frac{dy_1}{dt} & = F_1(t, y_1, \ldots, y_n) \\ \frac{dy_2}{dt} & = F_2(t, y_1, \ldots, y_n) \\ \cdots & \cdots \\ \frac{dy_m}{dt} & = F_m(t, y_1, \ldots, y_n) \end{cases} \tag{1}$$

  This is a system of m-equations, in n unknown variables. In this course, we only consider systems of n equations in n variables. That means, we would have $m = n$.

- In addition, we may have initial conditions

$$y_1(t_0) = y_1^0, y_2(t_0) = y_2^0, \cdots, y_n(t_0) = y_n^0. \tag{2}$$

Combining (4), (2), a system of 1^{st}-order initial value problem (IVP) consist of the following:

$$\begin{cases} \frac{dy_1}{dt} & = F_1(t, y_1, \ldots, y_n) \\ \frac{dy_2}{dt} & = F_2(t, y_1, \ldots, y_n) \\ \cdots & \cdots \\ \frac{dy_n}{dt} & = F_n(t, y_1, \ldots, y_n) \end{cases} \qquad \begin{cases} y_1(t_0) = y_1^0 \\ y_2(t_0) = y_2^0 \\ \cdots \\ y_n(t_0) = y_n^0 \end{cases} \tag{3}$$

Existence and Uniqueness Theorem

Theorem 5.1.1 Consider the IVP (3). Assume on this region

$$R = \left\{ (t, y_1, \ldots, y_n) : \begin{array}{c} a_0 < t < b_0 \\ a_1 < y_1 < b_1 \\ a_2 < y_2 < b_2 \\ \cdots \\ a_n < y_n < b_n \end{array} \right\}$$

all the partial derivatives $\frac{\partial y_i}{\partial y_j}$ are continuous, and $(y_1^0, \ldots, y_n^0) \in R$. Then, the IVP (3) has a unique solution $y_1 = \varphi_1(t), \ldots, y_n = \varphi_n(t)$.

System of 1^{st}-order Linear ODE

We would mainly discuss Linear Systems, as follows.
Definition. A system 1^{st}-order linear ODE looks like:

$$\left\{\begin{array}{llllll} y_1' & = p_{11}(t)y_1 & +p_{12}(t)y_2 & +\cdots & +p_{1n}(t)y_n & +g_1(t) \\ y_2' & = p_{21}(t)y_1 & +p_{22}(t)y_2 & +\cdots & +p_{2n}(t)y_n & +g_2(t) \\ \cdots & & \cdots & & \cdots & \\ y_n' & = p_{n1}(t)y_1 & +p_{n2}(t)y_2 & +\cdots & +p_{nn}(t)y_n & +g_n(t) \end{array}\right. \quad (4)$$

The equation (4) is called Homogeneous, if
$g_1 = g_2 = \cdots = g_n = 0$.

Existence and Uniqueness Theorem

We state the Existence and Uniqueness Theorem, for linear systems (4).

Theorem 5.1.2 Consider the system (4). Assume $p_{11}, p_{12}, \ldots, p_{nn}; g_1, \ldots, g_n$ are continuous on an open interval $I : \alpha < t < \beta$. Let $t_0 \in I$. Consider the IVP (4), together with initial condition

$$y_1(t_0) = y_1^0, y_2(t_0) = y_2^0, \cdots, y_n(t_0) = y_n^0$$

This IVP has a unique solution on I.

Proof. Follows from Theorem 5.1.1.

- We will work on solving (4) in latter sections.

Reduction of Linear second order ODEs

To motivate use of systems of 1^{st}-Order Linear ODE, we reduce a 2^{nd}-order Linear ODE to one such.

- Consider a 2^{nd}-order linear ODE, with constant coefficients:

$$ay'' + by' + cy = g(t) \qquad a \neq 0 \tag{5}$$

- Write $y_1 = y, y_2 = y_1'$. Then (5) reduces to

$$\begin{cases} y_1' & = y_2 \\ y_2' & = -\frac{c}{a}y_1 \quad -\frac{b}{a}y_2 \quad +\frac{g(t)}{a} \end{cases} \tag{6}$$