# §5.2 Algebra of Matrices

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# Why Matrices?

Algebra of Matrices would be the main tool to study and solve System of $1^{st}$-order Linear ODE. So, we provide a background of the Algebra of Matrices.

# Definition

A matrix **A** of size $m \times n$ is defined as an array, with $m$ rows and $n$ columns:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{13} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix} \text{ Also written as } \mathbf{A} = (a_{ij}).$$

$a_{ij}$s are called entries of **A**.

# Continued

- A matrix of size $m \times 1$ is called column vector. A matrix of size $1 \times n$ is called row vector.
- Often, in lower level courses only matrices of real numbers are considered. But most of it works in a more generality.
- In this course, we will also consider matrices $\mathbf{A} = (a_{ij})$ where the entries $a_{ij}$ are complex numbers.
- Further, we will consider matrices $\mathbf{A} = (a_{ij})$ where the entries $a_{ij} = a_{ij}(t)$ are functions of $t$.

# Transpose and conjugate

- Given a matrix $\mathbf{A} = (a_{ij})$ the transpose matrix $\mathbf{A}^T = (a_{ji})$ is obtained by writing the columns of $\mathbf{A}$ and rows as columns.
- If $\mathbf{A} = (a_{ij})$ is a matrix of complex numbers, then the conjugate matrix $\overline{\mathbf{A}} = (\overline{a_{ij}})$ is obtained by replacing each entry $a_{ij}$ by its conjugate $\overline{a_{ij}}$.
- The transpose conjugate matrix $A^* = \overline{\mathbf{A}}^T$ is obtained by taking conjugate and then transpose.

# Equality and Zero

- Two matrices $A = (a_{ij}), B = (b_{ij})$ are equal, if they have same size $(m \times n)$ and

$$a_{ij} = b_{ij} \qquad for \quad 1 \leq i \leq m, 1 \leq j \leq n.$$

- The symbol $\mathbf{0}$ denotes the matrix whose entries are 0.

# Addition

If $A = (a_{ij}), B = (b_{ij})$ are two matrices of equal size $(m \times n)$, then their **sum** is defined to be the $m \times n$ matrix given by

$$A + B = (a_{ij} + b_{ij}).$$

*So, the sum is obtained by adding the respective entries.*
If the sizes of two matrices are different, then the sum is NOT defined.

Definition and Notations
Matrix Addition
Scalar Multiplication
Example
Matrix Multiplication
Example

# Scalar Multiplication

If $A = (a_{ij})$ is a $m \times n$ matrix and $c$ is a complex number, then the **scalar multiplication** of $A$ by $c$ is the $m \times n$ matrix given by

$$cA = (ca_{ij}).$$

# Example of scalar multiplication

Let

$$A = \begin{pmatrix} 1 & 1 & -3 \\ 10 & 7 & -3 \end{pmatrix} \Longrightarrow 11A = \begin{pmatrix} 11 & 11 & -33 \\ 110 & 77 & -33 \end{pmatrix}$$

Also,

$$(1-i)A = \begin{pmatrix} 1-i & 1-i & -3+3i \\ 10-10i & 7-7i & -3+3i \end{pmatrix}$$

# Matrix Multiplication

Suppose $A = (a_{ij})$ is a matrix of size $m \times n$ and $B = (b_{ij})$ is a matrix of size $n \times p$. The the **product** $AB$ is an $m \times p$ matrix

$$AB = (c_{ij}) \quad where \ \ c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

# Matrix Multiplication

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{pmatrix}$$

$$= \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \cdots & \cdots & \cdots & \cdots \\ c_{m1} & c_{m2} & \cdots & c_{mp} \end{pmatrix} \quad c_{12} = a_{11}b_{12} + a_{12}b_{22} + \cdots + a_{1n}b_{n2}$$

# Example of matrix multiplication

Let

$$A = \begin{pmatrix} 1 & 1 & -3 \\ 10 & 7 & -3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 2 & 1 \end{pmatrix},$$

Since number of columns of $A$ and number of rows of $B$ are same, the product $AB$ is defined.

We have

$$AB = \begin{pmatrix} 1*1+1*1+(-3)*2 & 1*1+1*0+(-3)*1 \\ 10*1+7*1+(-3)*2 & 10*1+7*0+(-3)*1 \end{pmatrix}$$

$$= \begin{pmatrix} -4 & -2 \\ 11 & 7 \end{pmatrix}$$

**Remark.** $BA$ is ALSO defined, which will be a $3 \times 3$ matrix.

# Algebra of Matrices

Let $A, B, C$ be $m \times n$ matrices and $c, d$ be scalars. Then,

| | |
|---|---|
| $A + B = B + A$ | *Commutativity of addition* |
| $A + (B + C) = (A + B) + C$ | *Associativity of addition* |
| $(cd)A = c(dA)$ | *Associativity of scalar multiplicatio* |
| $c(A + B) = cA + cB$ | *a Distributive property* |
| $(c + d)A = cA + dA$ | *a Distributive property* |

# Continued

Let $A, B, C$ be matrices and $c$ is a constant. Assume all the matrix products below are defined. Then

$$
\begin{array}{ll}
A(BC) = (AB)C & \textit{Associativity Matrix Product} \\
A(B+C) = AB + AC & \textit{A Distributive Property} \\
(A+B)C = AC + BC & \textit{Distributive Property} \\
c(AB) = (cA)B = A(cB) &
\end{array}
$$

Proofs would be routine checking, which we would skip.

- Matrix multiplication is not necessarily commutative. That means, often $AB \neq BA$

# Vector Multiplication

- By a vector, we mean a column vector.
- Consider two vectors of (complex) numbers:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix} \quad \text{and} \quad \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \cdots \\ y_n \end{pmatrix}$$

- Define a product $\mathbf{x}^T\mathbf{y} := \sum_{i=1}^{n} x_i y_i$. This is extension of dot product.

# Continued

- The inner product or scalar product between $\mathbf{x}, \mathbf{y}$ is defined as

$$(\mathbf{x}, \mathbf{y}) := \sum_{i=1}^{n} x_i \overline{y_i}$$

- If $\mathbf{x}, \mathbf{y}$ are vectors of real numbers, then $\mathbf{x}^T\mathbf{y} = (\mathbf{x}, \mathbf{y})$.
- For another vector $\mathbf{z}$ and $\alpha \in \mathbb{C}$ we have

$$(\mathbf{x}, \mathbf{y}) = \overline{(\mathbf{y}, \mathbf{x})}, \ (\mathbf{x}, \mathbf{y} + \mathbf{z}) = (\mathbf{x}, \mathbf{y}) + (\mathbf{x}, \mathbf{z})$$

$$(\alpha\mathbf{x}, \mathbf{y}) = \alpha(\mathbf{x}, \mathbf{y}), \quad (\mathbf{x}, \alpha\mathbf{y}) = \overline{\alpha}(\mathbf{x}, \mathbf{y})$$

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# Continued

- Then, $(\mathbf{x}, \mathbf{x}) := \sum_{i=1}^{n} x_i \overline{x_i} = \sum_{i=1}^{n} |x_i|^2$ is a nonnegative real number.
- The length or magnitude of $\mathbf{x}$ is defined as $\| \mathbf{x} \| := \sqrt{(\mathbf{x}, \mathbf{x})}$.
- It follows $\| \mathbf{x} \| = 0 \iff \mathbf{x} = \mathbf{0}$.
- We say $\mathbf{x}, \mathbf{y}$ are orthogonal, if $(\mathbf{x}, \mathbf{y}) = 0$.

# The Identity Matrix

- For a positive integer, $\mathbf{I_n}$ would denote the square matrix of order $n$ whose main diagonal (left to right) entries are 1 and rest of the entries are zero.
- So,

$$\mathbf{I_1} = (1), \quad \mathbf{I_2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{I_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- $\mathbf{I_n}$ is called the identity matrix of order $n$. Often, we write $I = I_n$.

# Continued

- For matrices $\mathbf{A}, \mathbf{B}$, we have $\mathbf{A}I = \mathbf{A}$ and $I\mathbf{B} = \mathbf{B}$.

# Inverse of a Matrix

- A squate matrix $\mathbf{A}$ is said to invertible, if there is a matrix $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{BA} = I$.
- Such a matrix $\mathbf{B}$ is unique, when there is one. In that case, denote $\mathbf{A}^{-1} := \mathbf{B}$. So,

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = I.$$

- An invertible matrix $\mathbf{A}$ is also called nonsingular. If a matrix is not invertible, it is said to be a singular matrix.

# Good News

- Both TI-84 and Matlab can compute inverses of matrices.

# Analytic method

- The linear algebra course is not a prerequisite for this course.
- However, to give an analytic method to compute inverses of a matrix, we need to define detertminat of a square matrix. This will be done in a separate note.

# Minors, Cofactors and Inverses

- Let $A = (a_{ij})$ be an $n \times n$-matrix.
- Let $C_{ij}$ denote the cofactor of the the $(i, j)^{ij}$-entry of $A$.
- Let $C = (C_{ij})$ be the cofactor matrix of $A$.
- Let $Adj(A) = C^t$ denote the transpose of $C$.
- If $\det(A) \neq 0$ then $A^{-1} = \frac{Adj(A)}{\det(A)}$

# Matrix of Functions

We also consider matrices whose entries are functions of $t$ and perform usual operation on them.

- We write

$$\mathbf{A}(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1}(t) & a_{m2}(t) & \cdots & a_{mn}(t) \end{pmatrix}$$

- Similiary, we write column/row matrices $\mathbf{X}(t)$.

# Operations on matrices of functions

- Given a matrix of functions $\mathbf{A}(t) = (a_{ij}(t))$, define

$$\frac{d\mathbf{A}(t)}{dt} = \mathbf{A}'(t) = \left(\frac{da_{ij}(t)}{dt}\right)$$

- Similarly, define

$$\int \mathbf{A}(t)dt = \left(\int a_{ij}(t)dt\right), \quad \int_a^b \mathbf{A}(t)dt = \left(\int_a^b a_{ij}(t)dt\right)$$

# Continued

By routine checking:

- $\frac{d(c\mathbf{A})}{dt} = \frac{cd(\mathbf{A})}{dt}$ for any matrix $c$ of constants.
- $\frac{d(\mathbf{A}+\mathbf{B})}{dt} = \frac{d\mathbf{A}}{dt} + \frac{d\mathbf{B}}{dt}$
- $\frac{d(\mathbf{AB})}{dt} = \frac{d\mathbf{A}}{dt}\mathrm{B} + \mathrm{A}\frac{d\mathbf{B}}{dt}$