III Second Order DE
§3.2 Wronskian and Solutions of Homogeneous LSODEs

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Recall, second order DE (SODE) has the form

\[
\frac{d^2y}{dt^2} = f \left( t, y, \frac{dy}{dt} \right) \tag{1}
\]

This is also written as

\[
y'' = f(t, y, y')
\]
A linear SODE (LSODE), is are often written as:

\[
\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t) \tag{2}
\]

This is also written as

\[
y'' + p(t)y' + q(t)y = g(t)
\]

where \( p(t), q(t), g(t) \) are functions of \( t \).
Another form of LSODE (2) is:

\[ P(t) \frac{d^2 y}{dt^2} + Q(t) \frac{dy}{dt} + R(t)y = G(t) \quad (3) \]

where \( P(t), Q(t), R(t), G(t) \) are functions of \( t \). This is also written as

\[ P(t)y'' + Q(t)y' + R(t)y = G(t) \]
Homogeneous Equations

- The DEs (2, 3) would be called **homogeneous**, if \( g(t) = 0 \) or \( G(t) = 0 \). So, a homogeneous equation looks like:

\[
y'' + p(t)y' + q(t)y = 0 \quad \text{or} \quad P(t)y'' + Q(t)y' + R(t)y = 0
\]

(4)

- (The Trivial Solution): The constant \( y = 0 \) is a solution of any such homogeneous equation (4). (*This property is analogous to that of system of homogeneous linear equations in algebra.*)

- In §3.1 we considered LSODEs with constant coefficients.
Derivative as an operator

- It is helpful to think of the derivative $D = \frac{d}{dt}$ as an operator.
- Given any differentiable function $\varphi(t)$, $D = \frac{d}{dt}$ operates on $\varphi(t)$ and produces the derivative $D(\varphi) = \frac{d\varphi}{dt}$.
- $D$ sends $\varphi \mapsto D(\varphi) = \frac{d\varphi}{dt}$.
- We extend this idea of "operators" in the next frame, in the context of linear second order DEs (LSODE).
Suppose $p(t), q(t)$ are two continuous functions on an open interval $I = (\alpha, \beta)$, which means: $\alpha < t < \beta$. We define a differential operator $L$, which operates on all twice differentiable functions $\varphi(t)$ on $I$ as follows:

$$L(\varphi) := \frac{d^2 \varphi}{dt^2} + p \frac{d \varphi}{dt} + q \varphi$$

This is also written as $L(\varphi) := \varphi'' + p\varphi' + q\varphi$.

We also write $L = D^2 + pD + q$ where $D = \frac{d}{dt}$.
Such operators are like "functions". Given any twice differentiable functions $\varphi$, the "operation" $\mathcal{L}$ operates on $\varphi$ and produces a new function $\mathcal{L}(\varphi) := \varphi'' + p\varphi' + q\varphi$.

\[ \mathcal{L} \text{ associates } \varphi \rightarrow \mathcal{L}(\varphi) := \varphi'' + p\varphi' + q\varphi \]
Example: \( \mathcal{L} = D^2 + 2e^t D + \sqrt{t} \) is a differential operator.

When it operates on \( \varphi(t) = t^3 + \sin t \), then \( \mathcal{L}(t^3 + \sin t) \)

\[
= D^2(t^3 + \sin t) + 2e^t D(t^3 + \sin t) + \sqrt{t}(t^3 + \sin t)
= (6t - \sin t) + (3t^3 + \cos t) + \sqrt{t}(t^3 + \sin t)
\]

Example: \( \mathcal{L} = D^2 + \sin(2t)D + \ln t \) is a differential operator.

When it operates on \( \varphi(t) = e^{2t} \), then

\[
\mathcal{L}(e^{2t}) = D^2(e^{2t}) + \sin(2t)D(e^{2t}) + \ln t(e^{2t})
= 4e^{2t} + \sin(2t)(2e^{2t}) + \ln t(e^{2t})
\]
Properties and Plan

- **Properties**: Let \( \mathcal{L} = D^2 + pD + q \) be a differential operator. Then:
  - It is additive, in the sense
    \[
    \mathcal{L}(\varphi_1 + \varphi_2) = \mathcal{L}(\varphi_1) + \mathcal{L}(\varphi_2)
    \]
  - Then for any scalar \( a \in \mathbb{R} \) we have \( \mathcal{L}(a\varphi) = a\mathcal{L}(\varphi) \).
  - Putting them together for scalars \( a, b \in \mathbb{R} \) we have:
    \[
    \mathcal{L}(a\varphi_1 + b\varphi_2) = a\mathcal{L}(\varphi_1) + b\mathcal{L}(\varphi_2)
    \]  \( (6) \)

- **Plan**: Get used to the idea (jargon) of such operators \( \mathcal{L} \). Use this jargon to express LSODEs.
Consequence of the Properties

The **Principle of Superposition**:

- **Theorem 3.2.2**: Suppose $\mathcal{L} = D^2 + pD + q$ is a differential operator. Consider the homogeneous LSODE $\mathcal{L}(y) = 0$. Suppose $y_1(t), y_2(t)$ are two solutions of this DE. Then, for any constants $c_1, c_2 \in \mathbb{R}$, the linear combination $c_1y_1 + c_2y_2$ is also a solution of this equation.

**Proof.** By (6) we have

$$\mathcal{L}(c_1y_1 + c_2y_2) = c_1\mathcal{L}(y_1) + c_2\mathcal{L}(y_2) = c_1 \cdot 0 + c_2 \cdot 0 = 0.$$ 

The proof is complete.
Given any equation (in math or life), existence of a solution is not guaranteed. If and when, there is a solution, there is no guarantee that the solution will be unique. We seek conditions, under which, there is such a guarantee:
Theorem 3.2.1. Consider the initial value problem (IVP):

\[
\begin{align*}
\frac{d^2y}{dt^2} + p(t)y' + q(t)y &= g(t) \\
y(t_0) &= y_0 \\
y'(t_0) &= y'_0
\end{align*}
\] (7)

If \( p(t), q(t) \) are continuous on an open interval \( I \) then, (7) has a solution \( y = \varphi(t) \) such that

- \( y = \varphi(t) \) is defined on all over \( I \),
- The solution \( y = \varphi(t) \) is unique.
Solving IVP

In this section, we consider IVPs, among homogeneous LSODE

\[
\begin{cases}
\mathcal{L}(y) = y'' + p(t)y' + q(t)y = 0 \\
y(t_0) = y_0 \\
y'(t_0) = y'_0
\end{cases}
\]

(8)

◮ Theorem 3.1.1 ensures, there is a unique solution of (8).
◮ Suppose \( y_1, y_2 \) are two solutions of the DE (first line).
  ◦ We explore: Whether some linear combination \( y = c_1y_1 + c_2y_2 \) would be the solution of (8). If yes, how to compute \( c_1, c_2 \)?
◮ Answer: It depends on how "independent" \( y_1, y_2 \) are? This is similar to linear algebra, to be clarified in class.
Assume \( y = c_1y_1 + c_2y_2 \) is the solution of (8) and explore.

- Differentiate: \( y' = c_1y'_1 + c_2y'_2 \).
- Use the initial conditions:
  \[
  \begin{cases}
  c_1y_1(t_0) + c_2y_2(t_0) = y_0 \\
  c_1y'_1(t_0) + c_2y'_2(t_0) = y'_0
  \end{cases}
  \quad (9)
  \]
- We will try to use Cramer’s rule. Write
  \[
  W = \begin{vmatrix}
  y_1(t_0) & y_2(t_0) \\
  y_1'(t_0) & y_2'(t_0)
  \end{vmatrix}
  \quad (10)
  \]
This determinant will be called the Wronskian.
If $W \neq 0$, then by Cramer's rule:

$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y'_0 & y'_2(t_0) \end{vmatrix}}{W}, \quad c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y'_1(t_0) & y'_0 \end{vmatrix}}{W}$$

Conclusion: If Wronskian $W \neq 0$, then $y = c_1y_1 + c_2y_2$ is a solution of (8), where $c_1, c_2$ is given by (11).

Note that Wronskian can be defined as a function of $t$:

$$W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y'_1(t) & y'_2(t) \end{vmatrix} = y_1(t)y'_2(t) - y_2(t)y'_1(t)$$

(12)
Definitions and Remarks

Consider the homogeneous LSODE

\[ \mathcal{L}(y) = y'' + p(t)y' + q(t)y = 0 \quad (13) \]

and also the IVP (8)

- Two solutions \( y_1, y_2 \) of (13) would be called fundamental solution, whenever Wronskian \( W(t) \neq 0 \).
- A linear combination \( y = c_1 y_1 + c_2 y_2 \), where \( c_1, c_2 \) are arbitrary constants, would be called a general solution of (13).
Most Importantly:

- **Theorem 3.2.4.** If $W(t) \neq 0$ for some $t$ on $I$, then every solution of (13) is of the form $y = c_1y_1 + c_2y_2$. 
Sample I (Ex. 5)

Find the Wronskian of \( y_1 = e^t \sin t, y_2 = e^t \cos t \).

- Find Derivatives:
  \[
  y'_1 = e^t \sin t + e^t \cos t = e^t(\sin t + \cos t),
  \]
  \[
  y'_2 = e^t \cos t - e^t \sin t = e^t(\cos t - \sin).
  \]

- So, the Wronskian

\[
W(t) = \begin{vmatrix}
  y_1 & y_2 \\
  y'_1 & y'_2
\end{vmatrix} = \begin{vmatrix}
  e^t \sin t & e^t \cos t \\
  e^t(\sin t + \cos t) & e^t(\cos t - \sin t)
\end{vmatrix} = -e^{2t}
\]
Find the longest interval in which the IVP has unique solution:

\[
\begin{cases}
(t - 1)y'' - 3ty' + 4y = 0 \\
y(-2) = 2 \\
y'(-2) = 1
\end{cases}
\]

First, write the DE in the form (8):

\[
y'' - \frac{3t}{t - 1}y' + \frac{4}{t - 1}y = 0
\]
Continued

- **Continuity**: $p(t) = -\frac{3t}{t-1}$ and $q(t) = \frac{4}{t-1}$ are continuous on $(-\infty, 1)$ and $(1, \infty)$
- **Answer**: By theorem 3.2.1, since the initial $t = -2$ is in $(-\infty, 1)$, the answer is $(-\infty, 1)$. 
Find a set of fundamental solutions for the Homogeneous LSODE

\[ y'' + 4y' + 3y = 0, \quad t_0 = 1. \]

- Coefficients are constant. So, we use methods of §3.1.
- The CE: \( r^2 + 4r + 3 = 0 \). So, \( r_1 = -3 \), \( r_2 = -1 \)
- So, \( y_1 = e^{r_1 t} = e^{-3t} \), \( y_1 = e^{r_2 t} = e^{-t} \) are two solutions.
- The Wronskian:

\[
W(t) = \begin{vmatrix}
y_1 & y_2 \\
y'_1 & y'_2
\end{vmatrix} = \begin{vmatrix}
e^{-3t} & e^{-t} \\
-3e^{-3t} & -e^{-t}
\end{vmatrix} = 2e^{-4t}
\]
Finally: At $t = t_0 = 1$, we have
$W(t_0) = W(1) = 2e^{-4} \neq 0$.

So, $y_1 = e^{-3t}, y_1 = e^{r_2t} = e^{-t}$ form a fundamental set of solutions.
Consider the DE $y'' + 4y = 0$. Consider the functions $y_1(t) = \cos 2t$, $y_2(t) = \sin 2t$. (1) Verify, if $y_1, y_2$ are solutions of this DE, (2) If yes, do they form a fundamental set of solutions?

► Check, if $y_1(t) = \cos 2t$ is a solution.

$$y_1' = -2 \sin 2t, \quad y_1'' = -4 \cos 2t \implies$$

$$y_1'' + 4y_1 = -4 \cos 2t + 4 \cos 2t = 0$$

So, $y_1(t) = \cos 2t$ is a solution of this DE. Similarly, so is $y_2(t) = \sin 2t$. 
The Wronskian:

\[ W(t) = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \begin{vmatrix} \cos 2t & \sin 2t \\ -2 \sin 2t & 2 \cos 2t \end{vmatrix} = 2 \]

Finally: In deed, \( W(t) = 2 \neq 0 \).

Therefore, \( y_1 = \cos 2t \), \( y_1 = \sin 2t \) form a fundamental set of solutions.
Abel’s Theorem

- **Theorem 3.2.7.** Suppose $y_1, y_2$ are two solutions of (13) and $p, q$ are continuous on an open interval $I$. Then,
  - The Wronskian
    \[
    W(y_1, y_2)(t) = c \exp \left[ - \int p(t) dt \right]
    \]  
    (14)

  where $c$ is constant, independent of $t$.
  - Either $W(y_1, y_2)(t) = 0$ for all $t$ in $I$ (case $c = 0$) or $W(y_1, y_2)(t) \neq 0$ for all $t$ in $I$. 

Consider the homogeneous LSODE:

\[(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1) = 0\]

Compute the Wronskian of two solutions of this equation, without solving.

- We did not learn any technique to solve such an equation. We will use Abel’s formula (14).
- To use use (14), we write the equation in the form (13):

\[y'' - \frac{2x}{1 - x^2} y' + \frac{\alpha(\alpha + 1)}{1 - x^2} = 0\]
So, \( p(x) = -\frac{2x}{1-x^2} \).

By (14), the Wronskian

\[
W = c \exp \left( - \int p(x) \, dx \right) = c \exp \left( + \int \frac{2x}{1 - x^2} \, dx \right)
\]

\[
= c \exp \left( - \ln |1 - x^2| \right) = \frac{c}{|1 - x^2|}
\]
§3.2 Assignments and Homework

- Read Example 1, 2 (Finding domain of solutions).
- Read Example 4, 5 (Determining fundamental solutions)
- Read Example 7 (Wronskian using Abel’s theorem)
- Homework: §3.2 See Homework site!