Math 221	**Exam 1 (Takehome part)**	Satya Mandal
Fall 21	Maximum 96 (for TWO parts)	**Due**: With In-class Exam

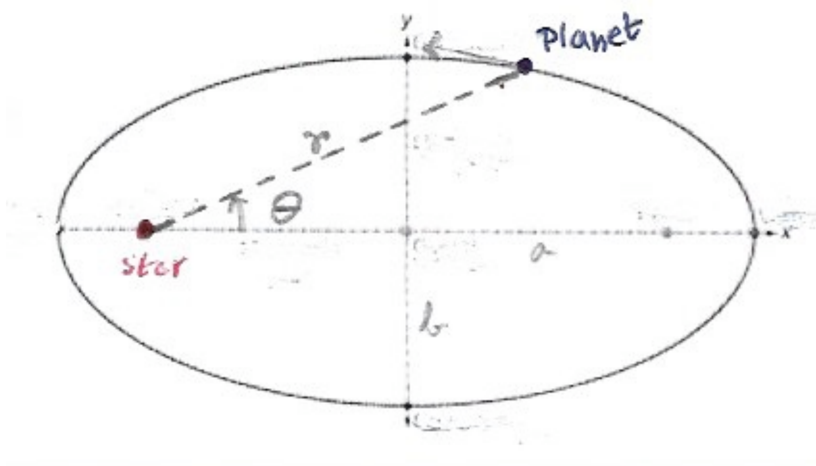
Name	
Student ID	

Information

1. *Each problem, in either part, is worth 8 points.*
2. *During the In Class exam, you are allowed to consult this part.*
3. *For In Class Exam, you can bring a standard size paper with formulas.*

1. Solve the IVP
$$\frac{dy}{dt} + \frac{4t^3}{1+t^4}y = \cos(5t + t^5) \qquad y(0) = \frac{1}{5}$$
2. Give a general solution of the Homogeneous ODE
$$\frac{dy}{dx} = \frac{x^3 - 3xy^2}{y^3 - 3x^2y}$$
3. Give a general solution of the ODE, in the Bernoulli's form:
$$\frac{dy}{dt} + y\cos t = y^2 \sin t \cos t$$
4. A body of mass m is ejected vertically upward, from earth. Reproduce the exposition given in the lecture notes, and compute the escape velocity of the body.
5. Consider the ODE
$$\mathbf{(y\cos x + 2xe^y) + (\sin x + x^2e^y - 1)\frac{dy}{dx} = 0}$$
Prove that the equation is EXACT and give a general solution of this equation.
6. Consider the ODE
$$\frac{dy}{dt} = (y+1)(y-1)(-3+\cos t)$$
   (a) Determine the Equilibrium Solutions.
   (b) Classify them as Stable or Unstable Equilibrium. (*Avoid analytic solution.*)
7. Consider the initial value problem $y' = \frac{4-ty}{1+y^2}$, $y(0) = -2$. Use Euler method to approximate y ate $t = 1$, with $h = .025$. (Submit the table of output from MS Excel or Matlab).

8. State and interpret the second Law of Kepler on planetary motion, to derive a differential equation of angular motion of the equation, and solve. Consult the Wiki site. Use polar coordinates:



(a) First, state Kepler's second Law. Then, interpret it to derive the differential equation.

(b) Consider the equation of the orbit of the planet $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Write $r(\theta)$ in terms of a, b, θ. Eliminate r from the second equation and solve (numerically, if analytic solution is not possible).

(c) Write down equatuion of the angular velocity θ, if possible.

Guidance: Look at the Wiki site and derive the differential equation

$$\frac{dA}{dt} = \frac{r^2}{2}\frac{d\theta}{dt}$$

Write $e = \sqrt{a^2 - b^2}$. Recall, $(0, -e)$ and $(0, -e)$ are the foci.
In our diagram, $(0, -e)$ is the position of the **star**.
The origin of the new polar coordinate system is at the star $(-e, 0)$.
The polar coordinate of the planet is $(r\cos\theta - \sqrt{a^2 - b^2}, r\sin\theta) = (r\cos\theta - e, r\sin\theta)$.
So,

$$\frac{(r\cos\theta - e)^2}{a^2} + \frac{r^2\sin^2\theta}{b^2} = 1 \Longrightarrow \frac{b^2\left(r^2\cos^2\theta - 2er\cos\theta + e^2\right) + a^2r^2\sin^2\theta}{a^2b^2} = 1 \Longrightarrow$$

$$r^2\left(b^2\cos^2\theta + a^2\sin^2\theta\right) - r(2eb^2\cos\theta) + b^2e^2 - a^2b^2 = 0 \Longrightarrow$$

$$r^2\left(a^2 - e^2\cos^2\theta\right) - r(2eb^2\cos\theta) + b^4 = 0 \Longrightarrow$$

$$r = \frac{2eb^2\cos\theta \pm \sqrt{4e^2b^4\cos^2\theta - 4b^4(a^2 - e^2\cos^2\theta)}}{2(a^2 - e^2\cos^2\theta)} = \frac{2eb^2\cos\theta \pm 2ab^2}{2(a^2 - e^2\cos^2\theta)}$$

$$= \frac{b^2}{a \pm e\cos\theta}$$

This is correct, checked!
Note $r(0) = a + e$. This settles that

$$r = r(\theta) = \frac{b^2}{a - e\cos\theta}$$

So,

$$\kappa = \frac{dA}{dt} = \frac{r^2}{2}\frac{d\theta}{dt} = \frac{b^4}{2(a - e\cos\theta)^2}\frac{d\theta}{dt}$$

So,

$$\frac{2\kappa t}{b^4} = \int \frac{1}{(a - e\cos\theta)^2} d\theta$$