Math 221: Online Lecture Guidance

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1 §4.1 Higher Order ODE
General Overview

I want you to read this section (§ 4.1) from the Lecture notes.
We give usual definitions of Higher Orders ODEs.
Other than that we make a point that the theory of higher order (linear) ODEs are remarkably similar to that of second order linear ODEs.
For this reason, many Instructors and Textbooks skip this chapter.
I decided to provide a flavor.

2 4.2 Linear Homogeneous ODE with constant coefficients

As in the last Chapter 3, after discussion theory of Linear ODEs, we solve Linear ODEs with constant coefficients.

1. **Definition** A Homogeneous Linear ODE with constant coefficient, is defined as follws:
$$\mathcal{L}(y) = a_n \frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = 0 \qquad (1)$$
with $a_0, a_1, \cdots, a_n \in \mathbb{R}$ and $a_n \neq 0$.

2. As in we did for 2^{nd}-order ODEs, by substituting $y = e^{rt}$ in (1) we get
$$\mathcal{L}(e^{rt}) = e^{rt}\left(a_n r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0\right) = 0$$

3. It follows, $y = e^{rt}$ is a solution of (1) if and only if
$$a_n r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0 = 0 \tag{2}$$

4. So, solving the ODE (1) reduces to solving the polynomial equation (2). This Equation (2) is called the characteristic equation (CE) of (1).

   The polynomial $\quad \rho(r) := a_n r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0 \quad (3)$
   is called the characteristic polynomial of (1). So, the characteristic equation can be written as
$$\rho(r) = 0$$

5. From Fundamental Theorem of Algebra (*which I mentioned in class*), we can write
$$\rho(r) = (r - r_1)^{k_1}(r - r_2)^{k_2}\cdots(r - r_m)^{k_m} \quad \text{with } k_i \geq 1,$$
   $k_1 + \cdots + k_m = n$, where $r_1, \ldots, r_m \in \mathbb{C}$ are distinct (with some $r_i \in \mathbb{R}$).

6. If r_1 is real, then r_1 spits out the following k_1 solutions of (1):
$$\begin{cases} y = e^{r_1 t} \\ y = te^{r_1 t} \\ y = t^2 e^{r_1 t} \\ \cdots \\ y = t^{k_1 - 1}e^{r_1 t} \end{cases} \qquad \text{This can be checked by substitution in (1).}$$
   Likewise, for any real root r_i.

7. If r_1 is complex (i.e. $r_1 \notin \mathbb{R}$), then its conjugate $\overline{r_1}$ is also a root of $\rho(r)$. Without loss of generality $r_2 = \overline{r_1}$. The pair $\begin{cases} r_1 = \lambda_1 + \mu_1 i \\ \overline{r_1} = r_2 = \lambda_1 - \mu_2 i \end{cases}$ spits out $2k_1$ solutions of (1):
$$\begin{cases} y = e^{\lambda_1 t}\cos\mu_1 t & y = e^{\lambda_1 t}\sin\mu_1 t \\ y = te^{\lambda_1 t}\cos\mu_1 t & y = te^{\lambda_1 t}\sin\mu_1 t \\ y = t^2 e^{\lambda_1 t}\cos\mu_1 t & y = t^2 e^{\lambda_1 t}\sin\mu_1 t \\ \cdots & \cdots \\ y = t^{k_1-1}e^{\lambda_1 t}\cos\mu_1 t & y = t^{k_1-1}e^{\lambda_1 t}\sin\mu_1 t \end{cases}$$

Likewise, for each pair of complex roots $r_i, \overline{r}_i$ of $\rho(r)$.

8. The process explained in the above, give total of n real solutions (1):

$$y = y_1(t), y = y_2, \ldots, y = y_n$$

9. The list of n solutions above form a Fundamental Set of Solutions of (1), which can be checked by checking that the Wronskian (see (11)

$$W(y_1, \ldots, y_n) \neq 0$$

So, the general solution of (1) is:

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n \qquad \text{where} \quad c_i \in \mathbb{R} \tag{4}$$

10. **Solving Examples:** Unlike quadratic formula, there no formula to compute the roots of polynomials $\rho(r)$ with $\deg(\rho(r)) \geq 3$.
Main trick to solve polynomial equation $\rho(r) = 0$ is, first guess a root α and factor $\rho(r) = (r - \alpha)p(r)$.

11. We solve a few simple problems, in this section, only to provide a flavor. It seems I am only copying and pasting from the lecture notes, which makes no further sense. So, read all four examples from the lecture notes.

3 §4.3 Nonhomogeneous Linear ODE

I am just thinking what would I have done, if I were lecturing in Snow 301. When I have extra comments I do that. Otherwise, I mostly read from my lecture notes.

1. No homework was assigned on this section. We only gave general overview of how to solve Higher Order Linear equations, and this is remarkably similar to 2^{nd}-order linear ODEs.

2. A Nonhomogeneous Linear ODE of order n can be written as:

$$\mathcal{L}(y) = g(t) \quad \text{with} \quad g(t) \neq 0, \quad \text{where} \tag{5}$$

$$\begin{cases} \mathcal{L} := \frac{d^n}{dt^n} + p_{n-1}(t)\frac{d^{n-1}}{dt^{n-1}} + \cdots + p_1(t)\frac{d}{dt} + p_0(t) \\ \text{OR} \\ \mathcal{L} := P_n(t)\frac{d^n}{dt^n} + P_{n-1}(t)\frac{d^{n-1}}{dt^{n-1}} + \cdots + P_1(t)\frac{d}{dt} + P_0(t) \end{cases} \tag{6}$$

   We usually assume that $p_i(t)$, $P_i(t)$, $g(t)$ are continuous on an open interval I.

3. The Homogeneous Linear Equation corresponding to (5) or (8) is

$$\mathcal{L}(y) = 0 \tag{7}$$

4. Theorem 4.3.2 A Let Y_p be a solution of (5) $\mathcal{L}(y) = g(t)$, to be called a "particular solution". As was for the 2^{nd}-order ODEs, any solution of Y of (5) can be written as

$$Y = Y_p + y_h \quad \text{where } y_h \text{ is a solutions of (7)}$$

3.1 With Constant Coefficients

In Chapter 3, after introducing general theory of 2^{nd}-order Linear ODEs, we solve Linear ODEs with constant coefficients. We do the same here.

1. **Definition** A nonHomogeneous Linear ODE (5) is said to have constant coefficient, if $p_i(t), P_i(t)$ are constant functions. So, a linear Homogeneous ODE, of order n, with constant coefficients looks like

$$\mathcal{L}(y) = a_n \frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = g(t) \tag{8}$$

with $a_0, a_1, \cdots, a_n \in \mathbb{R}$, $a_n \neq 0$ and $g(t) \neq 0$.

2. Theorem 4.3.2 B Let Y_p be a solution of (8) $\mathcal{L}(y) = g(t)$, to be called a "particular solution". So, the general solution of Y of (8) can be written as

$$Y = Y_p + y_h \quad \text{where } y_h \text{ is a solutions of (7)}$$

Now, let $y = y_1, y = y_2, \ldots, y = y_n$ be a fundamental set of solutions (7) $\mathcal{L}(y) = 0$. Then,

$$\begin{cases} y_h = \sum_{i=1}^n c_i y_i(t) & \text{where } c_1, c_2, \ldots, c_n \text{ are arbitrary constants.} \\ Y = Y_p + y_h = Y_p + \left(\sum_{i=1}^n c_i y_i(t)\right) \end{cases}$$

3. In § 4.2 we provided a flavor of how to solve homogeneous liner equations. So, we need to provide a way to compute a particular solution. As in chapter 3, we comment of two methods:

   (a) Method of Variation of Parameters.

   (b) Method of Undetermined Coefficients.

   We will give formula for the fist method.

3.2 Method of Variation of Parameters

Theorem 4.3.3: Consider former of the two forms of the nonhomogeneous Linear ODE (5) or (8), of order n. That means,

$$\begin{cases} \mathcal{L}(y) = g(t), & \text{with} \\ \mathcal{L} := \frac{d^n}{dt^n} + p_{n-1}(t)\frac{d^{n-1}}{dt^{n-1}} + \cdots + p_1(t)\frac{d}{dt} + p_0(t) \end{cases} \tag{9}$$

1. Assume $p_i(t)$, $g(t)$ are continuous on an open interval I.

2. Let $y = y_1, y = y_2, \ldots, y = y_n$ be a fundamental set of solutions of the homogeneous ODE $\mathcal{L}(y) = 0$.

   Then: A particular solution of (9) is given by

$$Y = \sum_{i=1}^{n} y_i(t) \int \frac{\omega_i(t) g(t) dt}{W(t)} \qquad \text{where} \tag{10}$$

   (a) $W(t) := W(y_1, y_2, \ldots, y_n)$ is Wronskian of $y_1, y_2, \ldots, y_n$.

   (b) And, $\omega_i(t)$ denotes the cofactor of $y_i^{(n-1)}$ in the Wronskian matrix.

   where

$$W(t) = \begin{vmatrix} y_1(t) & y_2(t) & y_3 & \cdots & y_n(t) \\ y_1'(t) & y_2'(t) & y_3 & \cdots & y_n'(t) \\ y_1^{(2)}(t) & y_2^{(2)}(t) & y_3^{(2)} & \cdots & y_n^{(2)}(t) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & y_3^{(n-1)} & \cdots & y_n^{(n-1)}(t) \end{vmatrix} \quad t \in I \tag{11}$$