Chapter 1: System of Linear Equations
§ 1.1 Introduction to linear Equations

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Goals

Refresh your memory regarding Systems of Linear Equations:

- Define a System of Linear of equations (a "System").
- Define homogeneous Systems.
- Row-echelon form of a linear system.
- Gaussian elimination method of solving a system.

The word "System" usually, refers to more than one equations, in more then one variables.

Definitions

A linear equation in n (unknown) variables has the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

Here $x_1, \ldots, x_n$ are unknown variables and $a_1, a_2, \ldots, a_n, b$ known are real numbers. We say b is the constant term and a_i is the coefficient of x_i.

For real numbers $s_1, \ldots, s_n$, if

$$a_1s_1 + a_2s_2 + \cdots + a_ns_n = b$$

we say that

$$x_1 = s_1, x_2 = s_2, \ldots, x_n = s_n$$

is a solution of this equation.

Example 1.1.1 A

An example of a linear equation in two unknowns is

$$2x + 7y = 5.$$

A solution of this equation is $x = -1, y = 1$.
The equation has many more solutions. **All the points in the graph of this equation (which is a line), would be a solution, of this equation.**

Example 1.1.1 B

An example of a linear equation in three unknowns is

$$2x + y + \pi z = \pi.$$

A solution of this equation is $x = 0, y = 0, z = 1$.
The equation has many more solutions. **All the points in the graph of this equation (which is a plane in 3-space) would be a solution, of this equation.**

Linear Systems

By a **System of Linear Equations** in n variables $x_1, x_2, \ldots, x_n$ we mean a collection of linear equations in these variables. A system of m linear equations in these n variables can be written as

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3 \\ \cdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m \end{cases} \qquad (1)$$

where a_{ij} and b_i are all real numbers. Such a linear system is called a **homogeneous linear system** if

$$b_1 = b_2 = \cdots = b_m = 0.$$

A Solution

A solution to the system (1) is a sequence of n numbers $s_1, \ldots, s_n$ that is solution to all these m equations of (1). More precisely, we say that

$$x_1 = s_1, x_2 = s_2, \cdots, x_n = s_n$$

is a solution of the system (1).
Subsequently, we will see that a system (1) may not have nay solution, exactly one solution, or infinitely many solutions.

Example 1.1.2 A

In two variables, here is an example of a system of two equation:

$$\begin{cases} 2x + y = 3 \\ x - 9y = -8 \end{cases}$$

Clearly, $x = 1, y = 1$ is the (only) solution to this system. *Geometrically, the solution of this system is given by the point where the graphs (two lines) of these two equations meet.*

Example 1.1.2 B

Also note that the system

$$\begin{cases} 2x + y = 3 \\ 2x + y = 7 \end{cases}$$

does not have any solution. Such a system would be called an **inconsistent** system. *Geometrically, is these two equations in the system represent two parallel lines (they never meet)*.

Example 1.1.2 C

In three variables, the following is an example of a system of two equation:

$$\begin{cases} 2x + y + 2z = 3 \\ x - 9y + 2z = -8 \end{cases}$$

Clearly, $x = 1, y = 1, z = 0$ is a solution to this system. This system has many more solutions. For example,

$$x = 11, y = 0, z = -19/2$$

is also a solution of this system. *Geometrically, solutions are given by the points where the graphs (two planes) in 3-space of these two equations meet.*

Classification Theorem

Given a linear system (1) in n variables, precisely one the the following three is true:

- The system has NO solution (inconsistent system).
- The system has exactly one solution (consistent system).
- The system has infinitely many solutions (consistent system).

Equivalent Systems

Two systems of linear equations (like (1)) are called

equivalent,

if they have precisely the same set of solutions.
Following operations on a system produces equivalent systems:

- Interchange two equations.
- Multiply an equation by a nonzero constant.
- Add a multiple of an equation to another one.

These three operations are sometimes known as
basic or elementary operations.

Row Echelon Form

A linear system in **row-echelon form**, looks like

$$\left\{\begin{array}{r} x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\ x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \\ x_3 + \cdots + a_{3n}x_n = b_3 \\ \cdots \\ \cdots \end{array}\right. \tag{2}$$

More precisely,

- you drop one or more variables in each successive equation (step). (*So, some variable may disappear, at this step.*)
- The coefficient of the "leading variable" in each equation is 1.
- Zero rows appear at the bottom.

Example 1.1.3

Here are three systems in Roe-Echelon Form: In two variables x, y this would (sometimes) look like

$$\begin{cases} x + a_{12}y = b_1 \\ \qquad\quad y = b_2 \end{cases}$$

In three variables x, y, z this would (sometimes) look like

$$\begin{cases} x + a_{12}y + a_{13}z = b_1 \\ \qquad\quad y + a_{23}z = b_2 \\ \qquad\qquad\qquad z = b_3 \\ \qquad\qquad\qquad 0 = b_4 \end{cases} \quad \text{OR} \quad \begin{cases} x + a_{12}y + a_{13}z = b_1 \\ \qquad\qquad\qquad z = b_3 \end{cases}$$

The Reduction Theorem

Theorem:

- Any system (1) of linear equations, is equivalent to a linear system in row-echelon form (like (2)). (*We say any system (1) can be reduced to a system in row-echelon from (2).*)
- This can be achieved by a sequence of applications of the three basic elementary operation described above.
- This process is known as **Gaussian elimination**.

Example 1.1.4

Reduce the following system to a row echelon form and solve:

$$\begin{cases} x - 5y = 3 & Eqn - 1 \\ -8x + 40y = 14 & Eqn - 2 \end{cases} \qquad (3)$$

Add 8 times Enq-1 to Eqn-2:

$$\begin{cases} x - 5y = 3 & Eqn - 1 \\ 0 = 34 & Eqn - 3 \end{cases}$$

The Eqn-3 is absurd. So, the system has no solution. The system is inconsistent.

Example 1.1.5

Reduce the following system to a row echelon form and solve:

$$\begin{cases} 9x - 4y = 5 & Eqn-1 \\ 3x + 2y = 0 & Eqn-2 \end{cases}$$

Multiply Eqn-2 by 3:

$$\begin{cases} 9x - 4y = 5 & Eqn-1 \\ 9x + 6y = 0 & Eqn-3 \end{cases}$$

Subtract Eqn-1 from the Eqn-3

$$\begin{cases} 9x - 4y = 5 & Eqn-1 \\ 10y = -5 & Eqn-4 \end{cases}$$

Continued

$$\text{Divide Eqn} - 4 \text{ by } 10: \begin{cases} 9x - 4y = 5 & Eqn - 1 \\ y = -\frac{1}{2} & Eqn - 5 \end{cases}$$

$$\text{Divide Eqn} - 1 \text{ by } 9: \begin{cases} x - \frac{4}{9}y = \frac{5}{9} & Eqn - 6 \\ y = -\frac{1}{2} & Eqn - 5 \end{cases}$$

This is the row-echelon form. Now substitute $y = -\frac{1}{2}$ in Eqn-6

$$x - \frac{4}{9}\left(-\frac{1}{2}\right) = \frac{5}{9} \qquad or \qquad x = \frac{1}{3}.$$

$$\text{So, the solution is } \; x = \frac{1}{3}, \quad y = -\frac{1}{2}.$$

Example 1.1.6

$$\begin{cases} \frac{x_1+3}{4} + \frac{x_2-1}{3} = 1 & Eqn-1 \\ x_1 - \frac{x_2}{2} = 6 & Eqn-2 \end{cases}$$

multiply Eqn-1 by 12, Eqn-2 by 2 and simplify:

$$\begin{cases} 3x_1 + 4x_2 = 7 & Eqn-3 \\ 2x_1 - x_2 = 12 & Eqn-2 \end{cases}$$

Add $-\frac{2}{3}$ times Eqn-3 to Eqn-2:

$$\begin{cases} 3x_1 + 4x_2 = 7 & Eqn-3 \\ \frac{-11}{3}x_2 = \frac{22}{3} & Eqn-4 \end{cases}$$

Continued

$$\text{Multiply Eqn} - 4 \text{ by } \frac{-3}{11} \quad \begin{cases} 3x_1 + 4x_2 = 7 & Eqn - 3 \\ \quad\quad x_2 = -2 & Eqn - 5 \end{cases}$$

$$\text{Multiply Eqn} - 3 \text{ by } \frac{1}{3} \quad \begin{cases} x_1 + \frac{4}{3}x_2 = \frac{7}{3} & Eqn - 6 \\ \quad\quad x_2 = -2 & Eqn - 5 \end{cases}$$

Theabove is the row-echelon form of the system. Substitute $x_2 = -2$ in Eqn-6 and get $x_1 = \frac{8}{3} + \frac{7}{3} = 5$.

$$x_1 = 5, \quad x_2 = -2.$$

So, the system is consistent and has unique solution

Example 1.1.7

Deduce an equivalent row-echelon form and solve the following system:

$$\begin{cases} 2x_1 + 4x_2 - x_3 = 7 & Eqn-1 \\ x_1 - 11x_2 + 4x_3 = 3 & Eqn-2 \\ 10x_1 - 6x_2 + 4x_3 = 3 & Eqn-3 \end{cases}$$

First, switch Eqn-1 and Eqn-2:

$$\begin{cases} x_1 - 11x_2 + 4x_3 = 3 & Eqn-2 \\ 2x_1 + 4x_2 - x_3 = 7 & Eqn-1 \\ 10x_1 - 6x_2 + 4x_3 = 3 & Eqn-3 \end{cases}$$

Continued

Subtract 2 times Eqn-2 from Eqn-1 and 10 times Eqn-2 from Eqn-3:

$$\begin{cases} x_1 - 11x_2 + 4x_3 = 3 & Eqn-2 \\ 26x_2 - 9x_3 = 1 & Eqn-4 \\ 104x_2 - 36x_3 = -27 & Eqn-5 \end{cases}$$

Subtract 4 times Eqn-4 from Eqn-5:

$$\begin{cases} x_1 - 11x_2 + 4x_3 = 3 & Eqn-2 \\ 26x_2 - 9x_3 = 1 & Eqn-4 \\ 0 = -31 & Eqn-6 \end{cases}$$

The system is inconsistent because Eqn-6 is absurd.

Continued

To obtain the row-echelon form, we divid Eqn-4 by 26:

$$\begin{cases} x_1 - 11x_2 + 4x_3 = 3 & Eqn - 3 \\ x_2 - \frac{9}{26}x_3 = \frac{1}{26} & Eqn - 7 \\ 0 = -31 & Eqn - 6 \end{cases}$$

Example 1.1.8

Deduce an equivalent row-echelon form and solve the following system:

$$\begin{cases} x_1 \qquad\qquad + 4x_3 = \quad 13 & Eqn-1 \\ 2x_1 - x_2 \quad + .5x_3 = \quad 3.5 & Eqn-2 \\ 2x_1 - 2x_2 - 7x_3 = -19 & Eqn-3 \end{cases}$$

Subtract 2 times Eqn-1 from Eqn-2 and subtract 2 times Eqn-1 from Equn-3:

$$\begin{cases} x_1 \qquad\qquad + 4x_3 = \quad 13 & Eqn-1 \\ \quad -x_2 - 7.5x_3 = -22.5 & Eqn-4 \\ \quad -2x_2 - 15x_3 = -45 & Eqn-5 \end{cases}$$

Continued

Subtract 2 times Eqn-4 from Eqn-5:

$$\begin{cases} x_1 \qquad\qquad + 4x_3 = \quad 13 & \qquad Eqn-1 \\ \quad -x_2 - 7.5x_3 = -22.5 & \qquad Eqn-4 \\ \qquad\qquad\qquad 0 = 0 & \qquad Eqn-6 \end{cases}$$

Multiply Eqn-4 by -1 and we get

$$\begin{cases} x_1 \qquad\qquad + 4x_3 = \quad 13 & \qquad Eqn-1 \\ \qquad x_2 + 7.5x_3 = 22.5 & \qquad Eqn-7 \\ \qquad\qquad\qquad 0 = 0 & \qquad Eqn-6 \end{cases}$$

The above is the row-echelon form of the system.

Continued

The system is consistent. Since the echelon form has actually two equations and number of variables are higher (*three in this problem*), the system has infinitely many solutions. For any value (parameter) $x_3 = t$, we have

$$x_2 = 22.5 - 7.5t \qquad \textit{and} \qquad x_1 = 13 - 4t.$$

So, a parametric solution of this system is

$$\begin{cases} x_1 = 13 - 4t, \\ x_2 = 22.5 - 7.5t, \\ x_3 = t. \end{cases} \qquad t \in \mathbb{R}.$$

Example 1.1.9

Deduce an equivalent row-echelon form and solve the following system:

$$\begin{cases} x_1 & & & +3x_4 & = 4 & \quad Eqn-1 \\ & 6x_2 & -3x_3 & -3x_4 & = 0 & \quad Eqn-2 \\ & 3x_2 & & -2x_4 & = 1 & \quad Eqn-3 \\ 2x_1 & -x_2 & +4x_3 & & = 5 & \quad Eqn-4 \end{cases}$$

Subtract 2 time Eqn-1 from Eqn-4:

$$\begin{cases} x_1 & & & +3x_4 & = 4 & \quad Eqn-1 \\ & 6x_2 & -3x_3 & -3x_4 & = 0 & \quad Eqn-2 \\ & 3x_2 & & -2x_4 & = 1 & \quad Eqn-3 \\ & -x_2 & +4x_3 & -6x_4 & = -3 & \quad Eqn-5 \end{cases}$$

Continued

Multiply Eqn-2 by $\frac{1}{6}$:

$$\left\{\begin{array}{rrrrl} x_1 & & & +3x_4 & = 4 \qquad Eqn-1 \\ & x_2 & -.5x_3 & -.5x_4 & = 0 \qquad Eqn-6 \\ & 3x_2 & & -2x_4 & = 1 \qquad Eqn-3 \\ & -x_2 & +4x_3 & -6x_4 & = -3 \qquad Eqn-5 \end{array}\right.$$

Subtract 3 times Eqn-6 from Eqn-3 and add Eqn-2 to Eqn-5:

$$\left\{\begin{array}{rrrrl} x_1 & & & +3x_4 & = 4 \qquad Eqn-1 \\ & x_2 & -.5x_3 & -.5x_4 & = 0 \qquad Eqn-6 \\ & & 1.5x_3 & -.5x_4 & = 1 \qquad Eqn-7 \\ & & 3.5x_3 & -6.5x_4 & = -3 \qquad Eqn-8 \end{array}\right.$$

Continued

Multiply Eqn-7 by $\frac{2}{3}$:

$$\begin{cases} x_1 & & & +3x_4 & = 4 & \qquad Eqn-1 \\ & x_2 & -.5x_3 & -.5x_4 & = 0 & \qquad Eqn-6 \\ & & x_3 & -\frac{1}{3}x_4 & = \frac{2}{3} & \qquad Eqn-9 \\ & & 3.5x_3 & -6.5x_4 & = -3 & \qquad Eqn-8 \end{cases}$$

Subtract 3.5 times Eqn-9 from Eqn-8:

$$\begin{cases} x_1 & & & +3x_4 & = 4 & \qquad Eqn-1 \\ & x_2 & -.5x_3 & -.5x_4 & = 0 & \qquad Eqn-6 \\ & & x_3 & -\frac{1}{3}x_4 & = \frac{2}{3} & \qquad Eqn-9 \\ & & & -\frac{16}{3}x_4 & = -\frac{16}{3} & \qquad Eqn-10 \end{cases}$$

Continued

Multiply Eqn-10 by $-\frac{3}{16}$:

$$\begin{cases} x_1 & & & +3x_4 & = 4 & & Eqn-1 \\ & x_2 & -.5x_3 & -.5x_4 & = 0 & & Eqn-6 \\ & & x_3 & -\frac{1}{3}x_4 & = \frac{2}{3} & & Eqn-9 \\ & & & x_4 & = 1 & & Eqn-11 \end{cases}$$

The above is a row-echelon form of the system. By back-substitution:

$$x_4 = 1, \quad x_3 = \frac{2}{3} + \frac{1}{3} = 1, \quad x_2 = 1, \quad x_1 = 1.$$