Chapter 1: System of Linear Equations § 1.3 Application of Linear systems (Read Only)

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Summer 2017: Fall 18 Update

Goals

In this section, we do a few applications of linear systems, as follows.

- Fitting polynomials,
- Network analysis,
- Kirchoff's Laws for electrical networks

Invincibility of Linear Algebra

System of linear equations is much easier to handle than nonlinear systems. (I do not mean for this class only, I mean for expert mathematicians and scientists.) In fact, it is really very difficult to handle nonlinear systems. That is why, there is a wide range of applications of linear systems.

Number of points needed

Recall the facts:

- there is exactly one line $y = c + mx$ that passes through two given points.
- there is exactly one parabola $y = ax^2 + bx + c$ that passes through three given points.
- More generally, given $n + 1$ points in the plane, there is exactly one polynomial

$$p(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + a_n x^n \quad \text{of degree } n$$

so that the graph $y = p(x)$ will pass through these points.

Method to fit polynomial

Suppose a collection of data is represented by n points:

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n).$$

Assume the $x-$coordinates $x_1, x_2, \ldots, x_n$ are distinct.
We determine a UNIQUE polynomial

$$p(x) = a_0 + a_1 x_1 + a_2 x^2 + \cdots + a_{n-1} x^{n-1} \quad \text{with} \quad \deg(p) \leq n-1$$

so that the graph of $y = p(x)$ passes through these points.

- Given n such points, to determine $p(x)$ we need to find the coefficients $a_0, a_1, \ldots, a_{n-1}$.
- Since (x_i, y_i) passes through the graph of $y = p(x)$, we have $y_i = p(x_i)$.

Continued

More explicitly,

$$\left\{\begin{array}{llllll} a_0 & +a_1x_1 & +a_2x_1^2 & +\cdots & +a_{n-1}x_1^{n-1} & = y_1 \\ a_0 & +a_1x_2 & +a_2x_2^2 & +\cdots & +a_{n-1}x_2^{n-1} & = y_2 \\ a_0 & +a_1x_3 & +a_2x_3^2 & +\cdots & +a_{n-1}x_3^{n-1} & = y_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_0 & +a_1x_n & +a_2x_n^2 & +\cdots & +a_{n-1}x_n^{n-1} & = y_n \end{array}\right. \qquad (1)$$

This is a linear system of n equations, with n unknowns (variables) $a_0, a_1, a_2, \ldots, a_{n-1}$.

Continued

The augmented matrix of this linear system is:

$$\begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} & y_1 \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} & y_2 \\ 1 & x_3 & x_3^2 & \cdots & x_3^{n-1} & y_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} & y_n \end{pmatrix}$$

and the coefficients matrix is

$$\begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ 1 & x_3 & x_3^2 & \cdots & x_3^{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{pmatrix}.$$

This matrix is called **Vandermonde-matrix** in $x_1, x_2, \ldots, x_n$.

Continued

- Since $x_1, \ldots, x_n$ are assumed to be distinct, it is known that the linear system (1), has a unique solution.
- We can reduce the augmented matrix to row echelon form and solve for $a_0, a_1, \ldots, a_{n-1}$.

Example 1.3.1

Determine the polynomial function (of degree 2) that passes through the points $(2,4),(3,6),(4,10)$.

Solution: Let $p(x) = a + bx + cx^2$. Since these points pass through the graph of $y = p(x) = a + bx + cx^2$, we have

$$\left\{\begin{array}{llll} a & +b2 & +c2^2 & =4 \\ a & +b3 & +c3^2 & =6 \\ a & +b4 & +c4^2 & =10 \end{array}\right. \quad or \quad \left\{\begin{array}{llll} a & +2b & +4c & =4 \\ a & +3b & +9c & =6 \\ a & +4b & +16c & =10 \end{array}\right.$$

Continued

The augmented matrix of this system is:

$$\begin{pmatrix} 1 & 2 & 4 & 4 \\ 1 & 3 & 9 & 6 \\ 1 & 4 & 16 & 10 \end{pmatrix}$$

Now we reduce the matrix to the row-echelon form. To do this subtract row-1 from row-2 and row-3:

$$\begin{pmatrix} 1 & 2 & 4 & 4 \\ 0 & 1 & 5 & 2 \\ 0 & 2 & 12 & 6 \end{pmatrix}$$

Now, subtract 2 times row-2 from row-3:

$$\begin{pmatrix} 1 & 2 & 4 & 4 \\ 0 & 1 & 5 & 2 \\ 0 & 0 & 2 & 2 \end{pmatrix}$$

Continued

Divide the last row by 2:

$$\begin{pmatrix} 1 & 2 & 4 & 4 \\ 0 & 1 & 5 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

The matrix is in row-echelon form. The linear system corresponding to this matrix is:

$$\begin{cases} a & +2b & +4c & = 4 \\ & b & +5c & = 2 \\ & & c & = 1. \end{cases}$$

$$\text{So} \quad c = 1, \quad b = 2 - 5 = -3, \quad a = 4 - 4 + 6 = 6$$

Continued

So

$$p(x) = a + bx + cx^2 = 6 - 3x + x^2.$$

You can use TI to graph it, and check that the graph passes through the given three points.

Example 1.3.2

Here is some US census population data:

Year	1980	1990	2000
population y	227	249	281

Here population is given in millions.

- Fit a quadratic polynomial passing through these points.
- Use it to predict population in year 2010 and 2020.

Solution: Let t be the variable time and set $t = 0$ for the year 1980. The table reduces to

t	0	10	20
y	227	249	281

Continued

Let $p(t) = a + bt + ct^2$ be the polynomial that fits this data.

Since the data points pass through the graph of $y = p(t) = a + bt + ct^2$, we have

$$\begin{cases} a & +b0 & +c0^2 & = 227 \\ a & +b10 & +c10^2 & = 249 \\ a & +b20 & +c20^2 & = 281 \end{cases}$$

$$\begin{cases} a & & & = 227 \\ a & +10b & +100c & = 249 \\ a & +20b & +400c & = 281 \end{cases}$$

Continued

The augmented matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 227 \\ 1 & 10 & 100 & 249 \\ 1 & 20 & 400 & 281 \end{pmatrix}$$

Now use TI-84 (or you can hand reduce) to reduce the matrix to Gauss-Jordan form:

$$\begin{pmatrix} 1 & 0 & 0 & 227 \\ 0 & 1 & 0 & 1.7 \\ 0 & 0 & 1 & .05 \end{pmatrix}$$

$$\text{So,} \quad a = 227, b = 1.7, c = 0.05$$

Continued

$$\text{So,} \quad y = p(t) = 227 + 1.7t + .05t^2.$$

This answers part (1). For part (2), for year 2010, we have $t = 30$ and predicted population is

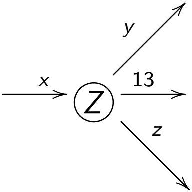
$$p(30) = 227 + 1.7 * 30 + .05 * 30^2 = 323 \ mi.$$

Similarly, for year 2020, wehave $t = 40$ and predicted population is

$$p(30) = 227 + 1.7 * 40 + .05 * 40^2 = 375 \ mi.$$

Basic Network

A network consists of junctions and branches. Following is an example of network:



Such network systems are used to model variety of situations, including in economics, traffic, telephone signal and electrical engineering.

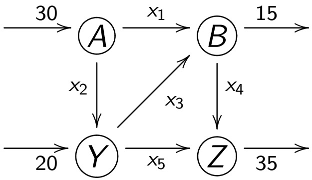
Continued

Such models assumes that **the total flow into a junction is equal to total flow out of the junction.** Accordingly, above network is represented by

$$x = y + 13 + z.$$

Example 1.3.3

The flow of traffic through a network of telephone towers is shown in the following figure:



- Solve this system for x_1, x_2, x_3, x_4, x_5.
- Find the traffic flow when $x_2 = 20$ and $x_3 = 5$.
- Find the traffic flow when $x_2 = 15$ and $x_3 = 0$.

Continued

Solution: From junction A, we get

$$x_1 + x_2 = 30$$

From junction B, we get

$$x_1 + x_3 = 15 + x_4 \qquad OR \qquad x_1 + x_3 - x_4 = 15$$

From junction Y, we get

$$x_2 + 20 = x_3 + x_5 \qquad OR \qquad x_2 - x_3 - x_5 = -20$$

From junction Z, we get

$$x_4 + x_5 = 35.$$

Continued

We will write the system in a better way:

$$\begin{cases} x_1 + x_2 = 30 \\ x_1 + x_3 - x_4 = 15 \\ x_2 - x_3 - x_5 = -20 \\ x_4 + x_5 = 35 \end{cases}$$

To solve this linear system, we write the augmented matrix:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 30 \\ 1 & 0 & 1 & -1 & 0 & 15 \\ 0 & 1 & -1 & 0 & -1 & -20 \\ 0 & 0 & 0 & 1 & 1 & 35 \end{pmatrix}$$

Continued

Reduce this matrix to row-echelon form. Subtract row 1 from row 2:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 30 \\ 0 & -1 & 1 & -1 & 0 & -15 \\ 0 & 1 & -1 & 0 & -1 & -20 \\ 0 & 0 & 0 & 1 & 1 & 35 \end{pmatrix}$$

Add second row to third:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 30 \\ 0 & -1 & 1 & -1 & 0 & -15 \\ 0 & 0 & 0 & -1 & -1 & -35 \\ 0 & 0 & 0 & 1 & 1 & 35 \end{pmatrix}$$

Continued

Add third roe to fourth:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 30 \\ 0 & -1 & 1 & -1 & 0 & -15 \\ 0 & 0 & 0 & -1 & -1 & -35 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Multiply second row by -1 and third row by -1:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 30 \\ 0 & 1 & -1 & 1 & 0 & 15 \\ 0 & 0 & 0 & 1 & 1 & 35 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The matrix is in row-echelon form.

Continued

The corresponding linear system is given by:

$$\begin{cases} x_1 & +x_2 & & & & = 30 \\ & x_2 & -x_3 & +x_4 & & = 15 \\ & & & x_4 & +x_5 & = 35 \\ & & & & 0 & = 0 \end{cases}$$

$$\text{With } x_2 = t, x_3 = s, \quad \begin{cases} x_1 = 300 - t \\ x_2 = t, \\ x_3 = s, \\ x_4 = 15 - t + s, \\ x_5 = 35 - x_4 = 150 + t - s. \end{cases}$$

Continued

This answers (1). For (2) $t = x_2 = 20, s = x_3 = 5$. So,

$$x_1 = 10, \quad x_2 = 20, \quad x_3 = 5, \quad x_4 = 0, \quad x_5 = 30.$$

For (3) $t = x_2 = 15, s = x_3 = 0$. So,

$$x_1 = 15, \quad x_2 = 15, \quad x_3 = 0, \quad x_4 = 0, \quad x_5 = 35.$$

Kirchhoff's Laws

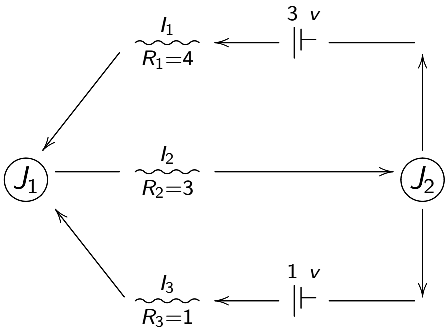
Systems of Linear equations is also used in electrical network. Analysis of electrical network is guided by two properties known as **Kirchhoff's Laws:**

- All the current flowing into a junction must flow out of it.
- The sum of the products IR (I is current and R is resistance) around a closed path is equal to the total voltage.

A battery is denoted by $|\vdash \quad or \quad \dashv|$ and the resistance is denoted by $\quad \sim\!\sim\!\sim\!\sim\!\sim \quad$.

Example 1.3.4

Consider the electrical circuit.



(*The circuit should be connected, I could not draw a better one.*)

Continued

Use Kirchhoff-Law to determine I_1, I_2, I_3.

Solution: Apply (1) of Kirchhoff-Law to junction J_1, we have

$$I_1 \quad +I_3 \quad = I_2 \quad Eqn-1$$

Applying the same to J_2 wil give the same equation. So, we will not write it.
Now apply (2) of Kirchhoff-Law

$$\begin{cases} R_1 I_1 & +R_2 I_2 & & = 3 \\ & R_2 I_2 & +R_3 I_3 & = 1 \end{cases} \quad OR$$

$$\begin{cases} 4I_1 & +3I_2 & & = 3 & Eqn-2 \\ & 3I_2 & +I_3 & = 1 & Eqn-3 \end{cases}$$

Continued

The the network system is given by

$$\begin{cases} I_1 & -I_2 & +I_3 & = 0 & Eqn-1 \\ 4I_1 & +3I_2 & & = 3 & Eqn-2 \\ & 3I_2 & +I_3 & = 1 & Eqn-3 \end{cases}$$

The augmented matrix is:

$$\begin{pmatrix} 1 & -1 & 1 & 0 \\ 4 & 3 & 0 & 3 \\ 0 & 3 & 1 & 1 \end{pmatrix}$$

Now, we reduce this matrix to row-echelon form.

Continued

To dothis, first subtract 4 time first reo from second:

$$\begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 7 & -4 & 3 \\ 0 & 3 & 1 & 1 \end{pmatrix}$$

Divide row two by 7:

$$\begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -\frac{4}{7} & \frac{3}{7} \\ 0 & 3 & 1 & 1 \end{pmatrix}$$

Continued

Subtract 3 times rwo two from row three:

$$\begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -\frac{4}{7} & \frac{3}{7} \\ 0 & 0 & \frac{19}{7} & -\frac{2}{7} \end{pmatrix}$$

Divide row three by $\frac{19}{7}$:

$$\begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -\frac{4}{7} & \frac{3}{7} \\ 0 & 0 & 1 & -\frac{2}{19} \end{pmatrix}$$

Continued

Now, we further reduce it to Gauss-Jordan form. To do this, add second row to first:

$$\begin{pmatrix} 1 & 0 & \frac{3}{7} & \frac{3}{7} \\ 0 & 1 & -\frac{4}{7} & \frac{3}{7} \\ 0 & 0 & 1 & -\frac{2}{19} \end{pmatrix}.$$

Now subtract $\frac{3}{7}$ times third row from first:

$$\begin{pmatrix} 1 & 0 & 0 & \frac{9}{19} \\ 0 & 1 & -\frac{4}{7} & \frac{3}{7} \\ 0 & 0 & 1 & -\frac{2}{19} \end{pmatrix}.$$

Continued

Now, add $\frac{4}{7}$ time third roe to second:

$$\begin{pmatrix} 1 & 0 & 0 & \frac{9}{19} \\ 0 & 1 & 0 & \frac{7}{19} \\ 0 & 0 & 1 & -\frac{2}{19} \end{pmatrix}.$$

The corresponding linear system s given by,

$$\left\{ \begin{array}{cccc} I_1 & & & = \frac{9}{19} \\ & I_2 & & = \frac{7}{19} \\ & & I_3 & = -\frac{2}{19} \end{array} \right.$$