Determinant: §3.1 The Determinant of a Matrix

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Summer 17
We will define determinant of SQUARE matrices, inductively, using the definition of Minors and cofactors.

We will see that determinant of triangular matrices is the product of its diagonal elements.

Determinants are useful to compute the inverse of a matrix and solve linear systems of equations (Cramer’s rule).
Overview of the definition

▶ Given a square matrix $A$, the determinant of $A$ will be defined as a number, to be denoted by $det(A)$ or $|A|$.

▶ Given such a matrix $A$ of size $n \times n$, it is possible to give a direct definition (or a formula) of $det(A)$. Unfortunately, this may be beyond the scope at this level.

▶ Therefore, we define inductively. That means, we first define determinant of $1 \times 1$ and $2 \times 2$ matrices. Use this to define determinant of $3 \times 3$ matrices. Then, use this to define determinant of $4 \times 4$ matrices and so.
For a $1 \times 1$ matrix $A = [a]$ define $det(A) = |A| = a$.

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

define $det(A) = |A| = ad - bc$. 
Example 3.1.1

Let

\[ A = \begin{bmatrix} 2 & 17 \\ 3 & -2 \end{bmatrix} \]

then

\[
\text{det}(A) = |A| = 2 \times (-2) - 17 \times 3 = -53
\]
Example 3.1.2

Let

\[ A = \begin{bmatrix} 3 & 27 \\ 1 & 9 \end{bmatrix} \]

then \( \det(A) = |A| = 3 \times 9 - 1 \times 27 = 0.\)
First, we define **Minors** and **Cofactors** of $3 \times 3$ matrices. Let

\[
A = \begin{bmatrix}
    a_{11} & a_{12} & a_{13} \\
    a_{21} & a_{22} & a_{23} \\
    a_{31} & a_{32} & a_{33}
\end{bmatrix}
\]

Then, the **Minor** $M_{ij}$ of $a_{ij}$ is defined to be the determinant of the $2 \times 2$ matrix obtained by deleting the $i^{th}$ row and $j^{th}$ column.
For example

\[ M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \]

Like wise

\[ M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \quad M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}, \quad M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}. \]
Cofactors of $3 \times 3$ matrices

Let $A$ the $3 \times 3$ matrix as in the above frame. Then, the **Cofactor** $C_{ij}$ of $a_{ij}$ is defined, by some sign adjustment of the minors, as follows:

$$C_{ij} = (-1)^{i+j} M_{ij}$$

For example, using the above frame

$$C_{11} = (-1)^{1+1} M_{11} = M_{11} = a_{22}a_{33} - a_{23}a_{33}$$

$$C_{23} = (-1)^{2+3} M_{23} = - M_{23} = -(a_{11}a_{32} - a_{12}a_{31})$$

$$C_{32} = (-1)^{3+2} M_{32} = - (a_{11}a_{23} - a_{13}a_{21}).$$
Determinant of $3 \times 3$ matrices

Let $A$ be the $3 \times 3$ matrix as above. Then the determinant of $A$ is defined by

$$\det(A) = |A| = a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13}$$

This definition may be called ”definition by expansion by cofactors, along the first row”. It is possible to define the same by expansion by second of third row, which we will be discussed later.
Example 3.1.3

Let

\[
A = \begin{vmatrix}
2 & 1 & 1 \\
3 & -2 & 0 \\
-2 & 1 & 1
\end{vmatrix}
\]

Compute the minors \(M_{11}, M_{12}, M_{13}\), the cofactors \(C_{11}, C_{12}, C_{13}\) and the determinant of \(A\).
Solution:

Then minors

\[ M_{11} = \begin{vmatrix} -2 & 0 \\ 1 & 1 \end{vmatrix}, \quad M_{12} = \begin{vmatrix} 3 & 0 \\ -2 & 1 \end{vmatrix}, \quad M_{13} = \begin{vmatrix} 3 & -2 \\ -2 & 1 \end{vmatrix} \]

Or

\[ M_{11} = -2, \quad M_{12} = 3, \quad M_{13} = -1 \]
Continued

So, the cofactors

\[ C_{11} = (-1)^{1+1} M_{11} = -2, \quad C_{12} = (-1)^{1+2} M_{12} = -3, \]
\[ C_{13} = (-1)^{1+3} M_{13} = -1 \]

So,

\[ |A| = a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13} = 2*(-2) + 1*(-3) + 1*(-1) = -8 \]
Example 3.1.4

Let

\[ A = \begin{vmatrix} 2 & 1 & 1 \\ 2 & -1 & 1 \\ 4 & 0 & 5 \end{vmatrix} \]

Compute the determinant of \( A \).
Solution:

Then minors

\[ M_{11} = \begin{vmatrix} -1 & 1 \\ 0 & 5 \end{vmatrix}, \quad M_{12} = \begin{vmatrix} 2 & 1 \\ 4 & 5 \end{vmatrix}, \quad M_{13} = \begin{vmatrix} 2 & -1 \\ 4 & 0 \end{vmatrix} \]

Or

\[ M_{11} = -5, \quad M_{12} = 6, \quad M_{13} = 4 \]
Continued

So, the cofactors

\[ C_{11} = (-1)^{1+1} M_{11} = -5, \quad C_{12} = (-1)^{1+2} M_{12} = -6, \]
\[ C_{13} = (-1)^{1+3} M_{13} = 4 \]

So,

\[ |A| = a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13} = 2(-5) + 1(-6) + 1(4) = -12 \]
Example 3.1.5

Let

\[
A = \begin{vmatrix}
1 & 1 & 1 \\
1 & 2 & -1 \\
2 & 3 & 0
\end{vmatrix}
\]

Compute the determinant of \( A \).
Solution:

Then minors

\[ M_{11} = \begin{vmatrix} 2 & -1 \\ 3 & 0 \end{vmatrix}, \quad M_{12} = \begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix}, \quad M_{13} = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} \]

Or

\[ M_{11} = 3, \quad M_{12} = 2, \quad M_{13} = -1 \]
So, the cofactors

\[ C_{11} = (-1)^{1+1} M_{11} = 3, \quad C_{12} = (-1)^{1+2} M_{12} = -2, \]
\[ C_{13} = (-1)^{1+3} M_{13} = -1 \]

So,

\[ |A| = a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13} = 1 \times 3 + 1 \times (-2) + 1 \times (-1) = 0 \]
Alternative Method for $3 \times 3$ matrices:

\[ A = \begin{bmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{bmatrix} \]

Form the $3 \times 5$ matrix by augmenting $1^{st}$, $2^{nd}$ columns to $A$:

\[
\begin{bmatrix}
  a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\
  a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\
  a_{31} & a_{32} & a_{33} & 31 & a_{32}
\end{bmatrix}
\]
Then $|A|$ can be computed as follows:

- add the product of all three entries in the three left to right diagonal
- add the product of all three entries in the three right to left diagonal
- Then, $|A|$ is the difference.

**Exercise:** Compute the determinant of the matrices in Example 1.3.3-5, using this method.
Inductive process of definition

► We defined determinant of matrices size $3 \times 3$, using the determinant of $2 \times 2$ matrices.

► Now, we do the same for $4 \times 4$ matrices. This means first define minors, which would be determinant of $3 \times 3$ matrices. Then, define Cofactors by adjusting the sign of the Minors. Then, use the cofactors of define the determinant of the $4 \times 4$ matrix.

► Then, we can define minors, cofactors and determinant of $5 \times 5$ matrices. The process continues.
Minors of $n \times n$ Matrices

We assume that we know how to define determinant of $(n - 1) \times (n - 1)$ matrices. Let

\[
A = \begin{bmatrix}
    a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
    a_{21} & a_{22} & a_{13} & \cdots & a_{2n} \\
    a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}
\end{bmatrix}
\]

be a square matrix of size $n \times n$. The minor $M_{ij}$ of $a_{ij}$ is defined to be the determinant of the $(n - 1) \times (n - 1)$ matrix obtained by deleting the $i^{th}$ row and $j^{th}$ column.
Cofactors and Determinant of $n \times n$ Matrices

Let $A$ be a $n \times n$ matrix.

- Define

$$C_{ij} = (-1)^{i+j} M_{ij} \quad \text{which is called the cofactor of } a_{ij}.$$ 

- Define

$$det(A) = |A| = \sum_{j=1}^{n} a_{1j} C_{1j} = a_{11} C_{11} + a_{12} C_{12} + \cdots + a_{1n} C_{1n}$$ 

This would be called a definition by expansion by cofactors, along first row.

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Determinant: §3.1 The Determinant of a Matrix
Theorem 3.1.1

Theorem 3.1.1 Let $A$ be an $n \times n$ matrix as above. Then,

- $|A|$ can be computed by expanding by any row ($i^{th}$ row):
  \[
  |A| = \sum_{j=1}^{n} a_{ij} C_{ij} = a_{i1} C_{i1} + a_{i2} C_{i2} + \cdots + a_{in} C_{in}
  \]

- $|A|$ can ALSO be computed by expanding by any column ($j^{th}$ column):
  \[
  |A| = \sum_{i=1}^{n} a_{ij} C_{ij} = a_{1j} C_{1j} + a_{2j} C_{2j} + \cdots + a_{nj} C_{nj}
  \]

- Proof. Proof is needed, which we skip.
The general definition, without using induction

This may be beyond the scope at this level. However, I will attempt to explain. Let $A$ be a $n \times n$ matrix, as above.

- **Permutation:** Any rearrangement $\sigma$ of the set $\{1, 2, \ldots, n\}$ is called a permutation of $\{1, 2, \ldots, n\}$. So, $\sigma$ would look like $\sigma = i_1, i_2, \ldots, i_n$ where $i_j \in \{1, 2, \ldots, n\}$ and each one appears only once.

- To each such permutation $\sigma$ a sign (signature) $\pm 1$ is attached, which we will not explain. $\text{sign}(\sigma) = 1$ or $-1$. 
We have

\[ |A| = \sum_{\text{all } \sigma} \text{sign}(\sigma) a_{1j_1} a_{2j_2} a_{3j_3} \cdots a_{nj_n} \]

where \( \sigma = j_1, j_2, \ldots, j_n \) and \( \pm \) is the sign of \( \sigma \).

In fact, we take products of \( n \) entries, such that exactly one factor comes from each row and each column, then adjust the sign of such products and add.
Definitions. Let $A$ be a $n \times n$ matrix.

- We say $A$ is **Upper Triangular** matrix, all entries of $A$ below the main diagonal (left to right) are zero. In notations, if $a_{ij} = 0$ for all $i > j$.

- We say $A$ is **Lower Triangular** matrix, all entries of $A$ above the main diagonal (left to right) are zero. In notations, if $a_{ij} = 0$ for all $i < j$. 
Theorem 3.1.2

Theorem 3.1.2 Let $A$ be a triangular matrix of order $n$. Then $|A|$ is product of the main-diagonal entries. Notationally,

$$|A| = a_{11}a_{22} \cdots a_{nn}.$$ 

Proof. The proof is easy when $n = 1, 2$. We prove it when $n = 3$. Let use assume $A$ is lower triangular. So,

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$
We expand by the first row:

\[ |A| = a_{11} C_{11} + 0 C_{12} + 0 C_{13} = a_{11} C_{11} \]

\[ = a_{11} (-1)^{1+1} \begin{vmatrix} a_{22} & 0 \\ a_{32} & a_{33} \end{vmatrix} = a_{11} a_{22} a_{33} \]

For upper triangular matrices, we can prove similarly, by column expansion. For higher order matrices, we can use mathematical induction.
Example 1.3.6

Compute the determinant, by expansion by cofactors, of

\[ A = \begin{bmatrix} x & y & z \\ 1 & 4 & 4 \\ 1 & 0 & 2 \end{bmatrix} \]

Solution.

- The cofactors

\[ C_{11} = (-1)^{1+1} \begin{vmatrix} 4 & 4 \\ 0 & 2 \end{vmatrix} = 8, \quad C_{12} = (-1)^{1+2} \begin{vmatrix} 1 & 4 \\ 1 & 2 \end{vmatrix} = 2 \]
\[ C_{13} = (-1)^{1+3} \left| \begin{array}{cc} 1 & 4 \\ 1 & 0 \end{array} \right| = -4 \]

\[ \text{So, } |A| = a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13} \]

\[ = x \times 8 + y \times 2 + z \times (-4) = 8x + 2y - 4z \]
Example 3.1.7.

Let

\[
A = \begin{bmatrix}
3 & 7 & -3 & 13 \\
0 & -7 & 2 & 17 \\
0 & 0 & 4 & 3 \\
0 & 0 & 0 & 5
\end{bmatrix}
\]

Compute \( \text{det}(A) \).

Solution. This is an upper triangular matrix. So, \( |A| \) is the product of the diagonal entries. So

\[
|A| = 3 \times (-7) \times 4 \times 5 = -420.
\]
Example 3.1.8.

Let \( A = \begin{bmatrix} 3 & 7 & -3 & 13 \\ 0 & -7 & 2 & 17 \\ 3 & 7 & 1 & 16 \\ -3 & -7 & 3 & -8 \end{bmatrix} \) Compute \( \det(A) \).

**Solution.** We expand, by first row. First, we compute the minors, and cofactors, of the elements in the first row.

\[
M_{11} = \begin{vmatrix} -7 & 2 & 17 \\ 7 & 1 & 16 \\ -7 & 3 & -8 \end{vmatrix} = 756 \quad C_{11} = (-1)^{1+1} M_{11} = 756
\]
The Determinant of a SQUARE Matrix

Determinant of $3 \times 3$ matrices

Determinant of Matrices of Higher Order

More Problems

Continued

\[
M_{12} = \begin{vmatrix} 0 & 2 & 17 \\ 3 & 1 & 16 \\ -3 & 3 & -8 \end{vmatrix} = 156 \quad C_{12} = (-1)^{1+2}M_{12} = -156
\]

\[
M_{13} = \begin{vmatrix} 0 & -7 & 17 \\ 3 & 7 & 16 \\ -3 & -7 & -8 \end{vmatrix} = 168 \quad C_{13} = (-1)^{1+3}M_{13} = 168
\]

\[
M_{14} = \begin{vmatrix} 0 & -7 & 2 \\ 3 & 7 & 1 \\ -3 & -7 & 3 \end{vmatrix} = 84 \quad C_{14} = (-1)^{1+4}M_{14} = -84
\]
So,

$$\det(A) = a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13} + a_{14} C_{14}$$

$$= 3 \times 756 + 7 \times (-156) - 3 \times 168 + 13 \times (-84) = -420$$
Example 3.1.9

Let \( A = \begin{bmatrix}
3 & 7 & 1 & 21 \\
3 & 0 & 3 & 38 \\
3 & 7 & 1 & 16 \\
-3 & -7 & 3 & -8
\end{bmatrix} \)

Compute \( \det(A) \).

Solution. We expand, by first row. First, we compute the minors, and cofactors, of the elements in the first row.

\[
M_{11} = \begin{vmatrix}
0 & 3 & 38 \\
7 & 1 & 16 \\
-7 & 3 & -8
\end{vmatrix} = 896 \quad C_{11} = (-1)^{1+1} M_{11} = 896
\]
The Determinant of a SQUARE Matrix

Determinant of $3 \times 3$ matrices

Determinant of Matrices of Higher Order

More Problems

Continued

\[ M_{12} = \begin{vmatrix} 3 & 3 & 38 \\ 3 & 1 & 16 \\ -3 & 3 & -8 \end{vmatrix} = 216 \quad C_{12} = (-1)^{1+2}M_{12} = -216 \]
\[ M_{13} = \begin{vmatrix} 3 & 0 & 38 \\ 3 & 7 & 16 \\ -3 & -7 & -8 \end{vmatrix} = 168 \quad C_{13} = (-1)^{1+3}M_{13} = 168 \]
\[ M_{14} = \begin{vmatrix} 3 & 0 & 3 \\ 3 & 7 & 1 \\ -3 & -7 & 3 \end{vmatrix} = 84 \quad C_{14} = (-1)^{1+4}M_{14} = -84 \]
So,

$$\det(A) = a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13} + a_{14} C_{14}$$

$$= 3 \times 896 + 7 \times (-216) + 1 \times 168 + 21 \times (-84) = -420$$
Example 3.1.10

Solve \[
\begin{vmatrix}
x + 1 & 1 \\
-1 & x - 1
\end{vmatrix}
= 0
\]

Solution. So,

\[(x + 1)(x - 1) + 1 = 0 \quad \text{or} \quad x^2 = 0\]

So, \(x = 0\).
Example 3.1.11

Solve

\[
\begin{vmatrix}
  x^2 + 1 & 2 \\
  x    & 1
\end{vmatrix} = 0
\]

Solution. So,

\[
(x^2 + 1) \times 1 - 2x = 0 \quad \text{or} \quad (x - 1)^2 = 0
\]

So, \(x = 1\).