Determinant: §3.3 Properties of Determinants

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"*The most incomprehensible thing about the world is that it is comprehensible.*" - Albert Einstein

Goals

Learn some basic properties of determinant. Among them are:

- Determinant of the product of two matrices is the product of the determinant of the two matrices:

$$|AB| = |A||B|.$$

- For a $n \times n$ matrix A and a scalar c we have

$$|cA| = c^n|A|$$

$$Also, \ if \quad |A| \neq 0 \qquad \Longrightarrow \quad |A^{-1}| = \frac{1}{|A|}.$$

- A square matrix A is invertible $\Longleftrightarrow \quad |A| \neq 0.$

Theorem 3.3.1: The Product Formula

If A, B are two square matrix of order n then

$$|AB| = |A||B|.$$

Proof. (*It is too long, so will not be in the exams.*) However, suppose E is an elmentary metix.

- If E is obtained by switching two rows of I_n then $|E| = -1$. Then, EB is the matrix obtained by switching two rows of B. By the theorem is §3.2, $|EB| = -|B| = |E||B|$
- If E is obtained by multiplying a row of I_n by c, then $|E| = c$. Then, EB is the matrix obtained by multiplying the same row of B by c. By the same theorem (§3.2), $|EB| = c|B| = |E||B|$

Continued

- If E is obtained by adding a scalar multiple of a row of I_n to another row,then $|E| = 1$. Then, EB is the matrix obtained by doing the same with rows of B. By the same theorem (§3.2), $|EB| = |B| = |E||B|$

$$\text{So, if } E \textit{ is elementary} \quad |EB| = |E||B| \tag{1}$$

From this it follows that, by repeated application, for elementary matrices $E_1, \ldots, E_k$ we have

$$|E_1 E_2 \cdots E_k B| = |E_1||(E_2 \cdots E_k B)| = |E_1||E_2| \cdots |E_k||B|$$

Continued

If A is invertible, by theorem above, $A = E_1E_2 \cdots E_k$, for some elementary matrices E_i. So.

$$|AB| = |E_1E_2 \cdots E_kB| = |E_1||E_2| \cdots |E_k||B|$$

$$= |E_1E_2 \cdots E_k||B| = |A||B|.$$

Continued

If A is not invertible, then A is row equivalent to a matrix C with an entire row zero. That means $E_1E_2\cdots E_nA$ has an entire row zero, where E_i are elementary. Expanding by a zero row, we have $|E_k\cdots E_2E_1A| = 0$. By Equation 1, $|E_{k-1}\cdots E_2E_1A| = 0$. Inductively, it follows $|A| = 0$. Since $E_1E_2\cdots E_nA$ has an entire row zero, so does $E_1E_2\cdots E_nAB$. Therefore $|E_1E_2\cdots E_nAB| = 0$. Again, by repeated use of Equation 1, it follows $|AB| = 0$.

$$\text{So,} \qquad |AB| = 0 = 0 \times |B| = |A||B|$$

This completes the proof. ■

Product of more than 2 matrices

Corallary 3.3.2: If $A_1, A_2, \ldots A_k$ are k matrices then

$$|A_1 A_2 \cdots A_k| = |A_1||A_2| \cdots |A_k|$$

Proof. For 2 matrices this is true by theorem 3.5. For more than two matrices, we use induction:

$$|A_1(A_2 \cdots A_k)| = |A_1||(A_2 \cdots A_k)| = |A_1||A_2| \cdots |A_k|$$

The proof is complete. ■

Example 3.3.1

$$A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 3 & 1 \\ 7 & -3 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & 3 & 1 \\ 13 & 4 & 5 \\ 3 & 9 & -8 \end{bmatrix} \textit{ Verify } |AB| = |A||B|.$$

Solution: We have to compute $AB, |AB|, |A|, |B|$ verify $|AB| = |A||B|$.

$$AB = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 3 & 1 \\ 7 & -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 13 & 4 & 5 \\ 3 & 9 & -8 \end{bmatrix}$$

Continued

$$\begin{bmatrix} 4+13+9 & 6+4+27 & 2+5-24 \\ -2+39+3 & -3+12+9 & -1+15-8 \\ 14-39+6 & 21-12+18 & 7-15-16 \end{bmatrix}$$

$$= \begin{bmatrix} 26 & 37 & -17 \\ 40 & 18 & 6 \\ -19 & 27 & -24 \end{bmatrix}$$

Continued

We will compute the determinant of AB by writing the first two columns to the right:

$$\begin{bmatrix} 26 & 37 & -17 & 26 & 37 \\ 40 & 18 & 6 & 40 & 18 \\ -19 & 27 & -24 & -19 & 27 \end{bmatrix}$$

Recall the determiant is the $(sum\ of\ product\ of\ the\ entries\ in\ the\ left-to-right\ diagonals) - (sum\ of\ product\ of\ the\ entries\ in\ the\ right-to-left\ diagonals)$. So,

$$|AB| = -33810 - (-25494) = -8316$$

Continued

We will compute the determinant of A in the same way. So, write the first tow columns on the right side of A:

$$\begin{bmatrix} 2 & 1 & 3 & 2 & 1 \\ -1 & 3 & 1 & -1 & 3 \\ 7 & -3 & 2 & 7 & -3 \end{bmatrix}$$

So,

$$|A| = (12 + 7 + 9) - (63 - 6 - 2) = -27$$

Continued

For a change, we will compute $|B|$ be expanding by co-factors along the first row. The cofactors

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 4 & 5 \\ 9 & -8 \end{vmatrix} = -77, C_{12} = (-1)^{1+2} \begin{vmatrix} 13 & 5 \\ 3 & -8 \end{vmatrix} = 119$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 13 & 4 \\ 3 & 9 \end{vmatrix} = 105$$

$$|B| = 2 * (-77) + 3 * 119 + 1 * 105 = 308$$

Finally, $|A||B| = (-27) * 308 = -8316 = |AB|$ is varyfied.

Theorem 3.3.3

Let A be an $n \times n$ matrix and c be a scalar. Then,

$$|cA| = c^n|A|$$

Proof. Let

$$C = \begin{pmatrix} c & 0 & \cdots & 0 \\ 0 & c & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & c \end{pmatrix} \quad \text{be the diagonal matrix.}$$

of size $n \times n$. Then, by the product formula

$$|cA| = |CA| = |C||A| = c^n|A|.$$

The proof is complete.

Theorem 3.3.4: Determinant of A^{-1},

If A is invertible (of order n), then

$$|A^{-1}| = \frac{1}{|A|}$$

Proof. We have $AA^{-1} = I_n$. So,

$$|AA^{-1}| = |I_n| = 1. \quad \textit{By Product Formula} \quad |A||A^{-1}| = 1$$

So,

$$|A^{-1}| = \frac{1}{|A|}$$

The proof is complete. ∎

Determinant of Elementary Marrices

- First, note $|I_n| = 1$.
- **Theorem.** Let E be an elementary $n \times n$ matrix. Then
  - If E is obtained by adding a constant multiple of a row of I_n to another row of I_n, then $|E| = |I_n| = 1$.
  - If E is obtained from I_n by multiplying a row of I_n by a scalar c, then $|E| = c|I_n| = c$.
  - If E is obtained from I_n by switching two rows of I_n then $|E| = -|I_n| = -1$.

Theorem 3.3.5

Suppose A is a square matrix (of order n). Then,

$$A \text{ is invertible (nonsingualar)} \iff |A| \neq 0.$$

Proof. There are two statements to be proved. First, if A is invertible, we would prove $|A| \neq 0$. In this case, $AA^{-1} = I_n$. so $|A||A^{-1}| = |AA^{-1}| = |I_n| = 1$. So, $|A| \neq 0$.

Continued

Now we prove the converse. We assume $|A| \neq 0$, and prove that A is invertible. Notice $|A^T| = |A| \neq 0$. By using Gauss-Jordan elimination, for some matrix E, which is product of elementary matrices, EA is in reduced row-Echelon form. Write $B = EA$. Since, B is in reduced row-Echelon form, either $B = I_n$ or B has an entire row zero. If B has an entire row zero then $|B| = 0$. In that case $0 = |B| = |E||A| = 0$. Since $|E| \neq 0$, $|A| = 0$. which contradicts the hypothesis. So, $B = I_n$ So, $EA = I_n$. So, A has a left inverse. Likewise, there is a matrix F such that $FA^T = I$. Taking transpose $AF^T = I_n$. So, A has a right inverse. Now, if follows from the lemma in the next frame $E = F^T$ and A has an inverse. ■

Left and Right Inverses

Lemma: Suppose A is a square matrix of order n. Suppose A has a left inverse B, meaning $BA = I_n$. Also suppose A has right inverse C, meaning $AC = I_n$. Then, $B = C$ and $A^{-1} = B = C$.

Proof. We have

$$B = BI_n = B(AC) = (BA)C = I_nC = C.$$

Theorem 3.3.6: Nonsingularity

Let A be square matrix of order n. Then the following are equivalent:

- A is nonsingular
- The system $A\mathbf{a} = \mathbf{b}$ has a unique solution, for all $n \times 1$ matrix $\mathbf{b}$.
- The system $A\mathbf{a} = \mathbf{0}$ has only the trivial solution.
- A is row-equivalent to I_n.
- A can be written as product of elementary matrices.
- $|A| \neq 0$.

Continued

Proof. It follows by combining everything we proves above. We skip the details. ∎

Remark: This theorem summarizes as lot of things we did above. So, it is very important and useful.

Theorem 3.3.7: Determinant of transpose

Let A be a square matrix of order n. Then,

$$|A^T| = |A|.$$

Example 3.3.2.

$$|A| = \begin{vmatrix} 16 & 10 & 4 \\ 12 & 2 & 8 \\ 6 & 14 & 18 \end{vmatrix}$$

$$= 2^3 \begin{vmatrix} 8 & 5 & 2 \\ 6 & 1 & 4 \\ 3 & 7 & 9 \end{vmatrix} = -8 \begin{vmatrix} 6 & 1 & 4 \\ 8 & 5 & 2 \\ 3 & 7 & 9 \end{vmatrix} = +8 \begin{vmatrix} 1 & 6 & 4 \\ 5 & 8 & 2 \\ 7 & 3 & 9 \end{vmatrix}$$

(*Last two steps represent switching first and second rows, and then first and second column.*)

Now subtract 5 times first row from second, and then subtract 7 times first row from last.

$$= 8 \begin{vmatrix} 1 & 6 & 4 \\ 0 & -22 & -18 \\ 0 & -39 & -19 \end{vmatrix}$$

Expand by first column:

$$= 8 \left(1(-1)^2 \begin{vmatrix} -22 & -18 \\ -39 & -19 \end{vmatrix} + 0 * (-1)^3 C_{21} + 0 * (-1)^4 C_{31} \right)$$

$$= 8(22 * 19 - 18 * 39) = -2272$$

Example 3.3.3

Let

$$A = \begin{bmatrix} 1 & 5 & 4 \\ 0 & -6 & 2 \\ 0 & 0 & -3 \end{bmatrix}$$

- Compute $|A^T|$
- Compute $|A^2|$
- Compute $|AA^T|$
- Compute $|2A|$
- Compute $|A^{-1}|$

Solution.

First A is a triangular matrix and $|A| = 18$. So,

- $|A^T| = |A| = 18$
- Compute $|A^2| = |A|^2 = (18)^2 = 324$
- Compute $|AA^T| = |A||A^T| = |A||A| = 324$
- Compute $|2A| = 2^3|A| = 8 * 18 = 144$
- Compute $|A^{-1}| = \frac{1}{|A|} = \frac{1}{18}$

Example 3.3.4

$$A = \begin{bmatrix} 4 & -1 & 16 \\ -16 & 4 & -64 \\ -5 & 5 & 16 \end{bmatrix}$$

Is A non-singular (i. e. invertible)?

Solution: We know, A is non-singular if and only if $|A| \neq 0$.

$$|A| = \begin{vmatrix} 4 & -1 & 16 \\ -16 & 4 & -64 \\ -5 & 5 & 16 \end{vmatrix} = -4 \begin{vmatrix} 4 & -1 & 16 \\ 4 & -1 & 16 \\ -5 & 5 & 16 \end{vmatrix} = 0$$

because first and second rows are identical. Since $|A| = 0$ A is singular (i. e. not non-singular).

Example 3.3.5.

$$\begin{array}{cccc} 4x & -y & +16z & = 3 \\ -2x & +.5y & -4z & = 11 \\ -5x & +5y & +16z & = -1 \end{array}$$

Does this system have a unique solution? **Solution:** A sysmtem of 3 equiations in three variables, has unique solution if the coefficiant matrix A is non-singular. Which is, if $|A| \neq 0$. Here the coefficiant matrix

$$A = \begin{bmatrix} 4 & -1 & 16 \\ -2 & .5 & -4 \\ -5 & 5 & 16 \end{bmatrix} \quad and \quad |A| = 0,$$

because first row is 2 times the second row. So, the system does not have unique solution.