Determinant
§3.4 Application of Determinants

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Goals

We will discuss three applications of determinants.

- We we give formula to compute the inverse A^{-1} of an invertible matrix A.
- We will give the Cramer's Rule (formula) to solve systems of linear equations.
- Give formulas to compute area of a triangle and volume of a tetrahedron.

"*Science is a wonderful thing if one does not have to earn one's living at it.*" - Albert Einstein

Definition of Adjoint

Definition. Let A be a square matrix of order n. In section 3.1 we define cofactors C_{ij}. Define the **cofactor matrix** C as

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} & \cdots & C_{1n} \\ C_{21} & C_{22} & C_{13} & \cdots & C_{2n} \\ C_{31} & C_{32} & C_{33} & \cdots & C_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ C_{n1} & C_{n2} & C_{n3} & \cdots & C_{nn} \end{bmatrix}$$

Continued: Definition of Adjoint

The **Adjoint** matrix of A is defined to be the transpose C^T of the cofactor matrix C:

$$Adj(A) = C^T = \begin{bmatrix} C_{11} & C_{21} & C_{31} & \cdots & C_{n1} \\ C_{12} & C_{22} & C_{32} & \cdots & C_{n2} \\ C_{13} & C_{23} & C_{33} & \cdots & C_{n3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ C_{1n} & C_{2n} & C_{3n} & \cdots & C_{nn} \end{bmatrix}$$

Theorem 3.4.1: Inverse Formula

Theorem 3.4.1 Let A be a square matrix. Then,

$$A[Adj(A)] = [Adj(A)]A = \det(A)I_n$$

$$\text{So,} \qquad \det(A) \neq 0 \Longrightarrow A^{-1} = \frac{1}{\det(A)} Adj(A).$$

Proof. Expanding by first row, we have:

$$a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n} = \det(A)$$

which is the $(1, 1)$-entry in the following product:

Continued

$$A[Adj(A)] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} |A| & ? & \cdots & ? \\ ? & ? & \cdots & ? \\ \cdots & \cdots & \cdots & \cdots \\ ? & ? & \cdots & ? \end{bmatrix} = \begin{bmatrix} |A| & ? & \cdots & ? \\ ? & |A| & \cdots & ? \\ \cdots & \cdots & \cdots & \cdots \\ ? & ? & \cdots & |A| \end{bmatrix}$$

The i^{th}-diagonal entry is obtained by expanding $\det(A)$, along i^{th}-row.

Continued

Now I want to compute the element at the $(1,2)$ position of the product $A[Adj(A)]$.

$$It\ is = a_{11}C_{21} + a_{12}C_{22} + \cdots + a_{1n}C_{2n}$$

With

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \text{ let } B = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{11} & a_{12} & \cdots & a_{1n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Here B is obtained from A by replacing the second row of A by its first row.

Continued

So, the cofactors of entries of the second row of both matrices are same, which are that of A:

$$C_{21}, C_{22}, C_{23}, \ldots, C_{2n}.$$

So, the $(2,1)-$entry of the product $A[Adj(A)]$ is

$$= a_{11}C_{21} + a_{12}C_{22} + \cdots + a_{1n}C_{2n}$$

$$= \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{11} & a_{12} & \cdots & a_{1n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = 0 \quad (since\ 1^{st}\ two\ rows\ are\ equal).$$

Continued

$$A[Adj(A)] = \begin{bmatrix} |A| & 0 & \cdots & ? \\ ? & |A| & \cdots & ? \\ \cdots & \cdots & \cdots & \cdots \\ ? & ? & \cdots & |A| \end{bmatrix}$$

Similarly, all the other off-diagonal entries are zero. So

$$A[Adj(A)] = \begin{bmatrix} |A| & 0 & \cdots & 0 \\ 0 & |A| & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & |A| \end{bmatrix} = |A| I_n.$$

Continued

So,

$$A\left(\frac{1}{|A|}Adj(A)\right) = I_n$$

Similarly, using column expansion for determiant, we have

$$\left(\frac{1}{|A|}Adj(A)\right)A = I_n$$

So,

$$A^{-1} = \frac{1}{|A|}Adj(A)$$

The proof is complete. ■

Example 3.4.1

Compute the cofactor matrix, Adjoint and inverse of $A = \begin{bmatrix} 7 & 12 \\ 3 & 33 \end{bmatrix}$. The Cofactor Matrix

$$C = \begin{bmatrix} (-1)^{1+1}33 & (-1)^{1+2}3 \\ (-1)^{2+1}12 & (-1)^{2+2}7 \end{bmatrix} = \begin{bmatrix} 33 & -3 \\ -12 & 7 \end{bmatrix}.$$

$Adj(A) = C^T = \begin{bmatrix} 33 & -12 \\ -3 & 7 \end{bmatrix}$ Also the determinant $|A| = 7 * 33 - 12 * 3 = 195$. So $A^{-1} =$

$$\frac{1}{|A|} Adj(A) = \frac{1}{195} \begin{bmatrix} 33 & -12 \\ -3 & 7 \end{bmatrix} = \begin{bmatrix} \frac{33}{195} & -\frac{12}{195} \\ -\frac{3}{195} & \frac{7}{195} \end{bmatrix}$$

Example 3.4.2

Find the cofactor matrix, adjoint and inverse of

$$A = \begin{bmatrix} 3 & 3 & 5 \\ 3 & 5 & 9 \\ 5 & 9 & 17 \end{bmatrix}. \quad \textit{First, we compute all the 9 cofactors:}$$

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 5 & 9 \\ 9 & 17 \end{vmatrix} = 4, C_{12} = (-1)^{1+2} \begin{vmatrix} 3 & 9 \\ 5 & 17 \end{vmatrix} = -6,$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 3 & 5 \\ 5 & 9 \end{vmatrix} = 2.$$

Continued

$$C_{21} = (-1)^{2+1} \begin{vmatrix} 3 & 5 \\ 9 & 17 \end{vmatrix} = -6, \; C_{22} = (-1)^{2+2} \begin{vmatrix} 3 & 5 \\ 5 & 17 \end{vmatrix} = 26,$$

$$C_{23} = (-1)^{2+3} \begin{vmatrix} 3 & 3 \\ 5 & 9 \end{vmatrix} = -12, \; C_{31} = (-1)^{3+1} \begin{vmatrix} 3 & 5 \\ 5 & 9 \end{vmatrix} = 2$$

$$C_{32} = (-1)^{3+2} \begin{vmatrix} 3 & 5 \\ 3 & 9 \end{vmatrix} = -12, \; C_{33} = (-1)^{3+3} \begin{vmatrix} 3 & 3 \\ 3 & 5 \end{vmatrix} = 6.$$

Continued

So, cofactor matrix and adjoint of A is:

$$C = \begin{bmatrix} 4 & -6 & 2 \\ -6 & 26 & -12 \\ 2 & -12 & 6 \end{bmatrix}, Adj(A) = C^T = \begin{bmatrix} 4 & -6 & 2 \\ -6 & 26 & -12 \\ 2 & -12 & 6 \end{bmatrix}$$

So, $A^{-1} =$

$$\frac{1}{|A|} Adj(A) = \frac{1}{4} \begin{bmatrix} 4 & -6 & 2 \\ -6 & 26 & -12 \\ 2 & -12 & 6 \end{bmatrix} = \begin{bmatrix} 1 & -1.5 & .5 \\ -1.5 & 6.5 & -3 \\ .5 & -3 & 1.5 \end{bmatrix}$$

Foreword: Cramer's Rule

Cramer's rule gives a formula to solve systems of n linear equations in n variables when the system has exactly one solution. Recall that such a system has exactly one solution if and only if the determinant of the coefficient matrix $|A| \neq 0$.

Theorem 3.4.2: Cramer's Rule

Theorem: Suppose $A\mathbf{x} = \mathbf{b}$ is a system of n equations in n variables. Here A is a $n \times n$ matrix and

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \cdots \\ b_n \end{pmatrix}. \quad \text{Assume } |A| \neq 0. \text{ Then,}$$

Continued: Cramer's Rule

$$x_1 = \frac{|A_1|}{|A|}, x_2 = \frac{|A_2|}{|A|}, \cdots, x_n = \frac{|A_n|}{|A|}$$

where A_i is obtained from A by replacing the i^{th} row of A by the right hand side $\mathbf{b}$, For example,

$$A_1 = \begin{vmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_1 & a_{12} & \cdots & a_{1n} \\ b_3 & a_{32} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{vmatrix}, A_2 = \begin{vmatrix} a_{11} & b_1 & \cdots & a_{1n} \\ a_{11} & b_1 & \cdots & a_{1n} \\ a_{31} & b_3 & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & b_n & \cdots & a_{nn} \end{vmatrix}, \cdots$$

Proof.

Since $|A| \neq 0$, the coefficient matrix A is invertible and $A^{-1} = \frac{1}{|A|}Adj(A)$. We have $A\mathbf{x} = \mathbf{b}$ implies

$$A^{-1}A\mathbf{x} = A^{-1}\mathbf{b} \Longrightarrow \mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{|A|}Adj(A)\mathbf{b}$$

(*Read "$\Longrightarrow$" as "implies"*) So,

$$\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} = \frac{1}{|A|}Adj(A)\mathbf{b}$$

Continued

$$\begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} = \frac{1}{|A|} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_n \end{bmatrix} = \frac{1}{|A|} \begin{bmatrix} |A_1| \\ |A_3| \\ \cdots \\ |A_n| \end{bmatrix}$$

Last equality is obtained by column expansion of determinants. The proof is complete ■

Example 3.4.3

Use Cramer's rule to solve the following linear system

$$\begin{cases} 2x_1 & -3x_2 & +2x_3 & = 9 \\ x_1 & +x_2 & +x_3 & = -3 \\ 4x_1 & +x_2 & -x_3 & = 11 \end{cases}$$

The determinant of the coefficient matrix

$$|A| = \begin{vmatrix} 2 & -3 & 2 \\ 1 & 1 & 1 \\ 4 & 1 & -1 \end{vmatrix} = -25$$

Continued

$$x_1 = \frac{\begin{vmatrix} 9 & -3 & 2 \\ -3 & 1 & 1 \\ 11 & 1 & -1 \end{vmatrix}}{|A|} = \frac{-70}{-25} = 2.8$$

$$x_2 = \frac{\begin{vmatrix} 2 & 9 & 2 \\ 1 & -3 & 1 \\ 4 & 11 & -1 \end{vmatrix}}{|A|} = \frac{75}{-25} = -3$$

Continued

$$x_3 = \frac{\begin{vmatrix} 2 & -3 & 9 \\ 1 & 1 & -3 \\ 4 & 1 & 11 \end{vmatrix}}{|A|} = \frac{70}{-25} = -2.8$$

Example 3.4.4

Use Cramer's rule to solve the following linear system

$$\begin{cases} 2x & -3y & = 9 \\ x & +y & = -3 \end{cases}$$

The determinant of the coefficient matrix

$$|A| = \begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} = 5$$

Continued

$$x = \frac{\begin{vmatrix} 9 & -3 \\ -3 & 1 \end{vmatrix}}{|A|} = \frac{0}{5} = 0$$

$$y = \frac{\begin{vmatrix} 2 & 9 \\ 1 & -3 \end{vmatrix}}{|A|} = \frac{-15}{5} = -3$$

Example 3.4.5

Use Cramer's rule to solve the following linear system

$$\begin{cases} x_1 & +4x_2 & +6x_3 & = 3 \\ x_1 & +2x_2 & +2x_3 & = 1 \\ x_1 & +2x_2 & +4x_3 & = 2 \end{cases}$$

The determiant of the coefficient matrix

$$|A| = \begin{vmatrix} 1 & 4 & 6 \\ 1 & 2 & 2 \\ 1 & 2 & 4 \end{vmatrix} = -4$$

Continued

$$x_1 = \frac{\begin{vmatrix} 3 & 4 & 6 \\ 1 & 2 & 2 \\ 2 & 2 & 4 \end{vmatrix}}{|A|} = \frac{0}{-4} = 0$$

$$x_2 = \frac{\begin{vmatrix} 1 & 3 & 6 \\ 1 & 1 & 2 \\ 1 & 2 & 4 \end{vmatrix}}{|A|} = \frac{0}{-4} = 0$$

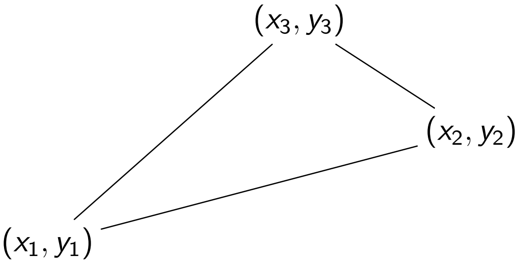
Continued

$$x_3 = \frac{\begin{vmatrix} 1 & 4 & 3 \\ 1 & 2 & 1 \\ 1 & 2 & 2 \end{vmatrix}}{|A|} = \frac{-2}{-4} = .5$$

■

Area of Δ

Determiant can be used to compute area of triangles and volume of tetrahedorns.



Formula for Area of Δ

Formula: Area of a triangle whose vertices are $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ is give by

$$Area = \pm\frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

Final answer is obviously non-negative, and we adjust $\pm$ accordingly.

Remark. These three points are colinear, if the above determinant is zero.

Exercise 3.4.6

Find the area of the triangle with vertices at $(1,-9),(2,3)(5,-7)$.
Solution: $Area =$

$$\pm\frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = \pm\frac{1}{2}\begin{vmatrix} 1 & 1 & -9 \\ 1 & 2 & 3 \\ 1 & 5 & -7 \end{vmatrix} = \pm\frac{1}{2}(-46) = 23\,(unit)^2$$

Formula for volume of tetrahedron

Formula:

- Recall a tetrahedron had four sides and four vertices.
- The volume of a tetrahedron whose vertices are $(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4)$ is give by

$$Volume = \pm\frac{1}{6}\begin{vmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{vmatrix}$$

Final answer is obviously non-negative, and we adjust $\pm$ accordingly.

Continued

- **Remark.** These four points are coplanar, if the above determinant is zero.

Example 3.4.7

Find the volume of the tetrahedron with vertices at $(1,-7,2),(2,2,3)(-4,-7,1),(1,-1,2)$.
Solution: $Vo; ume =$

$$\pm\frac{1}{6}\begin{vmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{vmatrix} = \pm\frac{1}{6}\begin{vmatrix} 1 & 1 & -7 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & -4 & -7 & 1 \\ 1 & 1 & -1 & 2 \end{vmatrix}$$

$$= \pm\frac{1}{6}(-24) = 4\ (unit)^3$$