Vector Spaces
§4.4 Spanning and Independence

Satya Mandal, KU

Summer 2017

Goals

Discuss two important basic concepts:

- Define linear combination of vectors.
- Define *Span*(S) of a set S of vectors.
- Define linear Independence of a set of vectors.

Set theory and set theoretic Notations

Borrow (re-introduce) some Set theoretic lingo and notations.

- A collection S of objects is called a set.
- Objects in S are called elements of S.
- We write "$x \in S$" to mean "*x is in S*" or "*x is an element of S.*"
- Given two sets, T, S we say T is a subset of S, if each element of T is in S. We write

$$T \subseteq S \quad \textit{to mean} \quad T \quad \textit{is a subset of } S.$$

- Read the notation $\Longrightarrow$ as "implies".

Linear Combination

Definition. Let V be a vector space and $\mathbf{v}$ be a vector in V. Then, $\mathbf{v}$ is said to be a linear combination of vectors $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_k}$ in V, if

$$\mathbf{v} = c_1\mathbf{u_1} + c_2\mathbf{u_2} + \cdots + c_k\mathbf{u_k} \quad \textit{for some scalars} \; c_1, c_2, \ldots, c_k \in \mathbb{R}.$$

Example 4.4.1a: Linear Combination

Let $S = \{(-1,-2,2),(-2,1,-1)\}$ be a set of two vectors in $\mathbb{R}^3$. Write $\mathbf{u} = (-8,-1,1)$ as a linear combination of the vectors in S, if possible.

Solution:

- Write $(-8,-1,1) = a(-1,-2,2) + b(-2,1,-1)$.
- So, $(-8,-1,1) = (-a-2b, -2a+b, 2a-b)$.
- So,

$$\begin{cases} -a & -2b & = -8 \\ -2a & +b & = -1 \\ 2a & -b & = 1 \end{cases}$$

- The augmented matrix

$$\begin{pmatrix} -1 & -2 & -8 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{pmatrix}$$

- Its row echelon form (use TI-84 "ref")

$$\begin{pmatrix} 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}. \qquad So, \quad b = 3, a = 2.$$

- So,

$$(-8, -1, 1) = 2(-1, -2, 2) + 3(-2, 1, -1)$$

Example 4.4.1b: Linear Combination

Let $S = \{(-1,-2,2),(-2,1,-1)\}$ be a set of two vectors in $\mathbb{R}^3$. Write $\mathbf{v} = (-3,-1,3)$ as a linear combination of the vectors in S, if possible.

Solution:

- Write $(-3,-1,3) = a(-1,-2,2) + b(-2,1,-1)$.
- So, $(-3,-1,3) = (-a-2b, -2a+b, 2a-b)$.
- So,

$$\begin{cases} -a & -2b & = -3 \\ -2a & +b & = -1 \\ 2a & -b & = 3 \end{cases}$$

- The augmented matrix

$$\begin{pmatrix} -1 & -2 & -3 \\ -2 & 1 & -1 \\ 2 & -1 & 3 \end{pmatrix}$$

- Its row echelon form (use TI-84 "ref")

$$\begin{pmatrix} 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}. \qquad \text{Last rwo gives} \quad 0 = 1$$

  So, they system has no solution.
- $\mathbf{v} = (-3, -1, 3)$ is not a linear combination of $(-1, -2, 2), (-2, 1, -1)$.

Example 4.4.1c: Linear Combination

Let $S = \{(-1, -2, 2), (-2, 1, -1)\}$ be a set of two vectors in $\mathbb{R}^3$. Write $\mathbf{v} = (-3, -1, 1)$ as a linear combination of the vectors in S, if possible.

Solution:

- Write $(-3, -1, 1) = a(-1, -2, 2) + b(-2, 1, -1)$.
- So, $(-3, -1, 1) = (-a - 2b, -2a + b, 2a - b)$.
- So,

$$\begin{cases} -a & -2b & = -3 \\ -2a & +b & = -1 \\ 2a & -b & = 1 \end{cases}$$

- The augmented matrix

$$\begin{pmatrix} -1 & -2 & -3 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{pmatrix}$$

- Its row echelon form (use TI-84 "ref")

$$\begin{pmatrix} 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \qquad \text{So} \quad b = 1, \ a = 1$$

  So, they system has no solution.

- So, $(-3, -1, 1) = (-1, -2, 2) + (-2, 1, -1)$

Example 4.4.1d

Let $S = \{(-1, -2, 2), (-2, 1, -1)\}$ be a set of two vectors in $\mathbb{R}^3$. Write $\mathbf{w} = (-9, -13, 13)$ as a linear combination of the vectors in S, if possible.

Solution:

- Write $(-9, -13, 13) = a(-1, -2, 2) + b(-2, 1, -1)$.
- So, $(-9, -13, 13) = (-a - 2b, -2a + b, 2a - b)$.
- So,

$$\begin{cases} -a & -2b & = -9 \\ -2a & +b & = -13 \\ 2a & -b & = 13 \end{cases}$$

- The augmented matrix

$$\begin{pmatrix} -1 & -2 & -9 \\ -2 & 1 & -13 \\ 2 & -1 & 13 \end{pmatrix}$$

- Its row echelon form (use TI-84 "ref")

$$\begin{pmatrix} 1 & -\frac{1}{2} & \frac{13}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \qquad so \quad b = 1, \ a = 7$$

- So,

$$\mathbf{w} = (-9, -13, 13) = 7(-1, -2, 2) + (-2, 1, -1)$$

Example 4.4.1e

Let $S = \{(-1, -2, 2), (-2, 1, -1)\}$ be a set of two vectors in $\mathbb{R}^3$. Write $\mathbf{z} = (-4, -3, 3)$ as a linear combination of the vectors in S, if possible.

Solution:

- Write $(-4, -3, 3) = a(-1, -2, 2) + b(-2, 1, -1)$.
- So, $(-4, -3, 3) = (-a - 2b, -2a + b, 2a - b)$.
- So,

$$\begin{cases} -a & -2b & = -4 \\ -2a & +b & = -3 \\ 2a & -b & = 3 \end{cases}$$

- The augmented matrix

$$\begin{bmatrix} -1 & -2 & -4 \\ -2 & 1 & -3 \\ 2 & -1 & 3 \end{bmatrix}$$

- Its row echelon form (use TI-84 "ref")

$$\begin{bmatrix} 1 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}. \qquad So, \quad a = 2, \quad b = 1.$$

- So,

$$\mathbf{z} = (-4, -3, 3) = 2(-1, -2, 2) + (-2, 1, -1).$$

Span of a Sets

Definition. Let $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$ be a subset of a vector space V.

- The span of S is the set of all linear combinations of vectors in S. So,

$$span(S) = \{c_1\mathbf{v_1} + c_2\mathbf{v_2} \cdots + c_k\mathbf{v_k} : c_1, c_2, \cdots, c_k \text{ are scalars}\}$$

  The $span(S)$ is also denoted by $span(\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k})$.
- If $V = span(S)$, we say V is spanned by S.

$Span(S)$ is a subspace of V

Theorem 4.4.1 Let $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$ be a subset of a vector space V.

- Then, $span(S)$ is a subspace of V.
- In fact, $Span(S)$ is the smallest subspace of V that contains S. That means, if W is a subspace of V then,

$$S \subseteq W \qquad \Longrightarrow \qquad span(S) \subseteq W.$$

Proof. First, we show $span(S)$ is a subspace of V.

- First, $\mathbf{0} = 0\mathbf{v_1} + 0\mathbf{v_2} + \cdots + 0\mathbf{v_k} \in spanS$. So, $span(S)$ is nonempty.
- Let $\mathbf{u}, \mathbf{v} \in Span(S)$ and c be a scalar. Then

$$\mathbf{u} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_k\mathbf{v_k}, \quad \mathbf{v} = d_1\mathbf{v_1} + d_2\mathbf{v_2} + \cdots + d_k\mathbf{v_k}$$

where $c_1, c_2, \ldots, c_k, d_1, d_2, \ldots, d_k$ are scalars. Then

$$\mathbf{u} + \mathbf{v} = (c_1 + d_1)\mathbf{v_1} + (c_2 + d_2)\mathbf{v_2} + \cdots + (c_k + d_k)\mathbf{v_k}$$

$$c\mathbf{u} = (cc_1)\mathbf{v_1} + (cc_2)\mathbf{v_2} + \cdots + (cc_k)\mathbf{v_k}$$

So $\mathbf{u} + \mathbf{v}, c\mathbf{u} \in span(S)$. So $span(S)$ is a subspace of V.

- So, we have shown that $span(S)$ is nonempty and closed under both addition and scalar multiplication. So, by Theorem 4.3.1, $span(S)$ is a subspace of V.
- Now, we prove that $span(S)$ is the smallest subspace W, of V, that contains S. Suppose W is a subspace of V and $S \subseteq W$. Let $\mathbf{u} \in span(S)$. we will have to show $\mathbf{u} \in W$. Then,

$$\mathbf{u} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_k\mathbf{v_k},$$

where $c_1, c_2, \ldots, c_k$ are scalars. Now, $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k} \in W$. Since W is closed under scalar multiplication $c_i\mathbf{v_i} \in W$. Since W is closed under addition $\mathbf{u} \in W$. The proof is complete. ∎

Examples 4.4.2: of Spanning Sets

- Most obvious and natural spanning set of the 3−space $\mathbb{R}^3$ is $S = \{(1,0,0),(0,1,0),(0,0,1)\}$ Because for any vector $\mathbf{u} = (a,b,c) \in \mathbb{R}^3$ w e have

$$\mathbf{u} = (a,b,c) = a(1,0,0) + b(0,1,0) + c(0,0,1)$$

- Similarly, most obvious and natural spanning set of the real plane $\mathbb{R}^2$ is $S = \{(1,0),(0,1)\}$.

Continued

- More generally, we give the natural spanning set of $\mathbb{R}^n$.

$$\text{Let} \qquad \begin{cases} \mathbf{e}_1 = (1, 0, 0, \ldots, 0) \\ \mathbf{e}_2 = (0, 1, 0, \ldots, 0) \\ \mathbf{e}_3 = (0, 0, 1, \ldots, 0) \\ \qquad \cdots \\ \mathbf{e}_n = (0, 0, 0, \ldots, 1) \end{cases} \qquad (1)$$

  Then, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \ldots, \mathbf{e}_n\}$ is a spanning set of $\mathbb{R}^n$.
- **Remark.** If S is spanning set of V and T is a bigger set (i.e. $S \subseteq T$) than T is also a spanning set of V.

Example 4.4.3

Let $S = \{(1,1), (-1,1)\}$. Is S a spanning set of $\mathbb{R}^2$?

Solution.

- Yes, it is a spanning set of $\mathbb{R}^2$. We need to show that any vector $(x, y) \in \mathbb{R}^2$ is a linear combination of elements is S.

$$So, \quad a(1,1) + b(-1,1) = (x,y) \quad OR \quad \left\{ \begin{array}{lll} a & -b & = x \\ a & +b & = y \end{array} \right.$$

must have at least one solution, for any (x, y). In the matrix form

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

- Use TI-84

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

- Since the systems have solution for all (x, y), S is a spanning set $\mathbb{R}^2$. Therefore, S is a spanning set of $\mathbb{R}^2$.
- We have could just argued $\det \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = 2 \neq 0$. Hence the system has a solution. That would suffice.

Example 4.4.4

Let $S = \{(1,1,1)\}$. Is S a spanning set of $\mathbb{R}^3$?
Solution. No. Because

$$span(S) = \{c(1,1,1) : c \in \mathbb{R}\} = \{(c,c,c) : c \in \mathbb{R}\}.$$

is only the line through the origin and $(1,1,1)$. It is strictly smaller than $\mathbb{R}^3$. For example, $(1,0,0) \notin span(S)$.

Example 4.4.5

Let $S = \{(1,0,0),(0,1,0)\}$. Is S a spanning set of $\mathbb{R}^3$?
Solution. No. Because

$$span(S) = \{a(1,0,0) + b(0,1,0) : a, b \in \mathbb{R}\}$$

$$= \{(a,b,0) : a, b \in \mathbb{R}\}.$$

is only the xy-plane, which is strictly smaller than $\mathbb{R}^3$. For example, $(0,0,1) \notin span(S)$.

Example 4.4.5a

Let $S = \{(1,0,1),(1,1,0),(0,1,1)\}$. Is S a spanning set of $\mathbb{R}^3$?

Solution.

- Yes, it is a spanning set of $\mathbb{R}^3$. We need to show that any vector $(x,y,z) \in \mathbb{R}^3$ is a linear combination of elements is S.

$$So, \quad a(1,0,1) + b(1,1,0) + c(0,1,1) = (x,y,z)$$

$$OR \quad \begin{cases} a & +b & & = x \\ & b & +c & = y \\ a & & +c & = z \end{cases}$$

must have at least one solution, for any (x,y,z).

- In the matrix form

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

- Use TI-84, we have $\det \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} = 2 \neq 0$ So, the system has a solution for all $(x, y, z) \in \mathbb{R}^3$. Therefore, S is a spanning set of $\mathbb{R}^3$.
- **Remark.** Note, we did not have to solve the system explicitly.

Example 4.4.6

Let $S = \{(1,1,1),(1,-1,1),(1,1,-1),(7,13,17)\}$. Is S a spanning set of $\mathbb{R}^3$?

Solution. To check, Write

$$a(1,1,1)+b(1,-1,1)+c(1,1,-1)+d(7,13,17)=(x,y,z)$$

$$OR \quad \left\{\begin{array}{lllll} a & +b & +c & +7d & =x \\ a & -b & +c & +13d & =y \\ a & +b & -c & +17d & =z \end{array}\right.$$

Question is, whether the system has one or more solutions, for any (x,y,z).

Continued

Write the equation in matrix form

$$\begin{pmatrix} 1 & 1 & 1 & 7 \\ 1 & -1 & 1 & 13 \\ 1 & 1 & -1 & 17 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Remark. Since the coefficient matrix is not a square matrix, we cannot use the determinant trick we used before.

Continued

The augmented matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 7 & x \\ 1 & -1 & 1 & 13 & y \\ 1 & 1 & -1 & 17 & z \end{pmatrix}$$

Since the matrix has variables, we have to do it by hand.
Subtract first row from second and third:

$$\begin{pmatrix} 1 & 1 & 1 & 7 & x \\ 0 & -2 & 0 & 6 & y-x \\ 0 & 0 & -2 & 10 & z-x \end{pmatrix}, \text{ This is (nearly) in Echelon form.}$$

Continued

So, the equivalent system:

$$\begin{cases} a & +b & +c & +7d & = x \\ & -2b & & +6d & = y - x \\ & & -2c & +10d & = z - x \end{cases}$$

This system has a solution. Any value of d leads to a solution. For convenience, we take $d = 0$. So, the system becomes

$$\begin{cases} a & +b & +c & = x \\ & -2b & & = y - x \\ & & -2c & = z - x \end{cases}$$

Continued

Given any $(x, y, z) \in \mathbb{R}^3$, we can take

$$\begin{cases} c & = -\frac{z-x}{2} \\ b & = -\frac{y-x}{2} \\ a & = x - b - c \\ d & = 0 \end{cases}$$

Therefore, S is a spanning set of $\mathbb{R}^3$.

Liniear Independence

- **Definition.** Let $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$ be a subset of a vector space V. The set S is said to be linearly independent, if

$$c_1\mathbf{v_1} + c_2\mathbf{v_2}, \cdots + c_k\mathbf{v_k} = \mathbf{0} \implies \quad c_1 = c_2 = \cdots = c_k = 0.$$

  That means, the equation on the left has only the trivial solution.
- If S is not linearly independent, we say that S is linearly dependent.

Comments

- (1) Let $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$ be a subset of a vector space V. If $\mathbf{0} \in S$ then, S is linearly dependent.
  Proof. For simplicity, assume $\mathbf{v_1} = \mathbf{0}$. Then,
  $$1\mathbf{v_1} + 0\mathbf{v_2} + \cdots + 0\mathbf{v_k} = \mathbf{0}$$
  So, S is not linearly indpendent. ■
- (2)**Methods to test Independence:** We will mostly be working with vector in $\mathbb{R}^2, \mathbb{R}^3$, or $n-$spaces $\mathbb{R}^n$. Gauss-Jordan elimination (with TI-84) will be used to check if a set is independent.

A Property of Linearly Dependent Sets

Theorem 4.4.2 Let $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$ be a subset of a vector space V. Assume S has at least 2 elements ($k \geq 2$). Then, S is linearly dependent if and only if one of the vectors $\mathbf{v_j}$ can be written as a linear combination of rest of the vectors in S.

Proof. Again, we have to prove two statments.

- First, we prove "if" part. We assume that one of the vectors $\mathbf{v_j}$ can be written as a linear combination of rest of the vectors in S.
- For simplicity, we assume that $\mathbf{v_1}$ is a linear combination of rest of the vectors in S. So,

$$\mathbf{v_1} = c_2\mathbf{v_2} + c_3\mathbf{v_3} + \cdots + c_k\mathbf{v_k}$$

$$So, \quad (-1)\mathbf{v_1} + c_2\mathbf{v_2} + c_3\mathbf{v_3} + \cdots + c_k\mathbf{v_k} = \mathbf{0}$$

This has at least one coefficient -1 that is nonzero. This establishes that S is a linearly dependent set.

- Now, we prove the "only if" part. We assume that S is linearly dependent. So, there are scalars $c_1, \cdots, c_k$, at least one non-zero, such that

$$c_1\mathbf{v_1} + c_2\mathbf{v_2} + c_3\mathbf{v_3} + \cdots + c_k\mathbf{v_k} = \mathbf{0}$$

- Without loss of generality (i.e. for simplicity) assume $c_1 \neq 0$.

$$So, \quad \mathbf{v_1} = \left(-\frac{c_2}{c_1}\right)\mathbf{v_2} + \left(-\frac{c_3}{c_1}\right)\mathbf{v_3} + \cdots + \left(\frac{c_k}{c_1}\right)\mathbf{v_k}$$

- Therefore, $\mathbf{v_1}$ is a linear combination of the rest. The proof is complete. ∎

Examples 4.4.7

- Again, most natural example of linearly independent set in 3−space $\mathbb{R}^3$ is $S = \{(1,0,0),(0,1,0),(0,0,1)\}$ Because

  $$a(1,0,0)+b(0,1,0)+c(0,0,1) = (0,0,0) \implies a = b = c = 0.$$

- Similarly, most natural example of linearly independent set in the real plane $\mathbb{R}^2$ is $S = \{(1,0),(0,1)\}$.

Continued

- More generally, the natural linearly independent set of $\mathbb{R}^n$:

$$\text{Let} \qquad \begin{cases} \mathbf{e}_1 = (1, 0, 0, \ldots, 0) \\ \mathbf{e}_2 = (0, 1, 0, \ldots, 0) \\ \mathbf{e}_3 = (0, 0, 1, \ldots, 0) \\ \qquad \cdots \\ \mathbf{e}_n = (0, 0, 0, \ldots, 1) \end{cases} \qquad (2)$$

  Then, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \ldots, \mathbf{e}_n\}$ is a linearly independent subset of $\mathbb{R}^n$. (*Compare this with Example 4.2.2, that this set is also a spanning set of* $\mathbb{R}^n$)
- **Remark.** If S is a linearly independent subset of V and if $R \subseteq S$, then R is also a linearly independent subset of V.

Example 4.4.8

Is the set $S = \{(-2, 4), (1, -2)\}$. linearly independent?

Solution.

- We can see a non-trivial linear combination $1 * (-2, 4) + 2 * (1, -2) = (0, 0)$. So, S is not linearly independent.
- Alternately, to prove it methodically, let $a(-2, 4) + b(1, -2) = (0, 0)$.
- Then,

$$\begin{cases} -2a & +b & = 0 \\ 4a & -2b & = 0 \end{cases} \quad Or \quad \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- The question is, if this system has only the trivial solution or not.?
- Add 2 times the first equation to the second:
$$\left\{\begin{array}{rrl} -2a & +b & = 0 \\ & 0 & = 0 \end{array}\right. \quad So, \quad a = t, b = 2t$$
- So, there are lots of non-zero (non trivial) a, b. Hence, $S = \{(-2, 4), (1, -2)\}$ is not linearly independent.
- Alternately, $\begin{vmatrix} -2 & 1 \\ 4 & -2 \end{vmatrix} = 0$. So, this homogeneous system has nontrivial solutions. So, S is not linearly independent.

■

Example 4.4.9

Let $S = \{(1,1,1), (1,-1,3), (1,0,0)\}$. Is it linearly independent or dependent?

Solution. Let $a(1,1,1) + b(1,-1,3) + c(1,0,0) = (0,0,0)$
So,

$$\left\{\begin{array}{llll} a & +b & +c & =0 \\ a & -b & & =0 \\ a & +3b & & =0 \end{array}\right. \quad \text{Or,} \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 3 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The question is, if the it has only the trivial solution or not.?

- Short Method: $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 3 & 0 \end{vmatrix} = 4 \neq 0$. So, the system has only the zero solution. So, S is linearly independent.
- Explicit Method: Write the augmented matrix

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 3 & 0 & 0 \end{pmatrix}.$$

$$\text{Its Echelon reduction}: \quad \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

- So, the only solution is the zero-solution: $a = 0$, $b = 0$, $c = 0$. We conclude that S is linearly independent.

Example 4.4.10

Let $S = \{x^2 - x + 1, 2x^2 + x\}$ be a set of polynomials. Is it linearly independent or dependent?

Solution.

- Write $a(x^2 - x + 1) + b(2x^2 + x) = 0.$
- So, $(a + 2b)x^2 + (-a + b)x + a = 0$
- , Equating coefficients of x^2, x and the constant terms:

$$a + 2b = 0, -a + b = 0, a = 0 \qquad or \qquad a = b = 0.$$

- We conclude, S is linearly independent.

■