Vector Spaces
§4.5 Basis and Dimension

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Goals

Discuss two related important concepts:

- Define Basis of a Vectors Space V.
- Define Dimension $\dim(V)$ of a Vectors Space V.

Definition:Linear Independence of infinite sets

In fact, we defined linear independence of finite sets S, only. Before we proceed, we define the same for infinite sets.
Definition. Suppose V is a vector space and $S \subseteq V$ is a subset (possibly infinite). We say S is Linearly Independent, if any finite subset $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \subseteq S$ is linearly independent.
That means, for any finite subset $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \subseteq S$ and scalars $c_1, \ldots, c_n$,

$$c_1\mathbf{v}_1, + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0} \Longrightarrow c_1 = c_2 = \cdots = c_n = 0.$$

Basis

Let V be a vector space (over $\mathbb{R}$). A set S of vectors in V is called a basis of V if

1. $V = Span(S)$ and
2. S is linearly independent.

- In words, *we say that S is a basis of V if S spans V* and if S is linearly independent.
- First note, it would need a proof (i.e. it is a theorem) that any vector space has a basis.

Continued

- The definition of basis does not require that S is a finite set.
  - However, we will only deal with situations when $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ is a finite set.
  - If V has a finite basis $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$, then we say that V is finite dimensional. Otherwise, we say that V is infinite dimensional.

Example 4.5.1a

The set $S = \{(1,0,0),(0,1,0),(0,0,1)\}$ is a basis of the 3−space $\mathbb{R}^3$.

Proof. We have seen, in § 4.4 that S is spans $\mathbb{R}^3$ and it is linearly independent. We repeat the proof.

- Given any $(x,y,z) \in \mathbb{R}^3$ we have

$$(x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1).$$

  So, for any $(x,y,z) \in \mathbb{R}^3$, $(x,y,z) \in span(S)$. So, $\mathbb{R}^3 = Span(S)$.

- Also, S us linearly independent; because

$$a(1,0,0)+b(0,1,0)+c(0,0,1) = (0,0,0) \implies a = b = c = 0.$$

So, S is a basis of $\mathbb{R}^3$.

Example 4.5.1b

Similarly, a basis of the $n-$space $\mathbb{R}^n$ is given by the set

$$S = \{\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}\}$$

$$\text{where,} \quad \begin{cases} \mathbf{e}_1 = (1, 0, 0, \ldots, 0) \\ \mathbf{e}_2 = (0, 1, 0, \ldots, 0) \\ \mathbf{e}_3 = (0, 0, 1, \ldots, 0) \\ \quad \cdots \\ \mathbf{e}_n = (0, 0, 0, \ldots, 1) \end{cases} \tag{1}$$

This one is called the standard basis of $\mathbb{R}^n$.

Example 4.5.2

The set $S = \{(1,-1,0),(1,1,0),(1,1,1)\}$ *is a basis of* $\mathbb{R}^3$.

Proof.

- First we prove $Span(S) = \mathbb{R}^3$. Let $(x,y,z) \in \mathbb{R}^3$. We need to find a, b, c such that

$$(x,y,z) = a(1,-1,0) + b(1,1,0) + c(1,1,1)$$

So,

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \quad \textit{Notationally} \quad A\mathbf{a} = \mathbf{v}$$

Continued

$$\text{Using} \quad \text{TI} - 84, \qquad \begin{vmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 2 \neq 0$$

So, the above system has a solution.
Therefore $(x, y, z) \in span(S)$. So, $span(S) = \mathbb{R}^3$.
Remark. We could so the same, by long calculation.

- Now, we prove S is linearly independent. Let
$$a(1,-1,0)+b(1,1,0)+c(1,1,1)=(0,0,0).$$
In the matrix from, this equation is
$$A\begin{pmatrix} a \\ b \\ c \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{where } A \text{ is as above.}$$
where A is as above. Since, $|A| = 2 \neq 0$,
$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$
So, S is linearly independent.
- Since, $span(S) = \mathbb{R}^3$ and S is linearly independent, S forms a bais of $\mathbb{R}^3$.

Examples 4.5.3

- Let P_3 be a vector space of all polynomials of degree less of equal to 3. Then $S = \{1, x, x^2, x^3\}$ is a basis of P_3.
  Proof. Clearly $span(S) = P_3$. Also S is linearly independent, because

$$a1 + bx + cx^2 + dx^3 = 0 \quad \Longrightarrow a = b = c = d = 0.$$

(Why?) ■

Example 4.5.4

- Let $\mathbb{M}_{3,2}$ be the vector space of all 3×2 matrices. Let

$$A_{1,1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, A_{1,2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, A_{2,1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$A_{2,2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, A_{3,1} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}, A_{3,2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Then,

$$A = \{A_{11}, A_{12}, A_{2,1}, A_{2,2}, A_{3,1}, A_{3,2}\}$$

is a basis of $\mathbb{M}_{3,2}$.

Theorem 4.5.1

Theorem 4.5.1(Uniqueness of basis representation): Let V be a vector space and $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ be a basis of V. Then, any vector $\mathbf{v} \in V$ can be written in one and only one way as linear combination of vectors in S.

Proof. Suppose $\mathbf{v} \in V$. Since $Span(S) = V$

$$\mathbf{v} = a_1\mathbf{v_1} + a_2\mathbf{v_2} + \cdots + a_n\mathbf{v_n} \quad where \quad a_i \in \mathbb{R}.$$

Now suppose there are two ways:

$$\mathbf{v} = a_1\mathbf{v_1} + a_2\mathbf{v_2} + \cdots + a_n\mathbf{v_n} \quad \textit{and} \quad \mathbf{v} = b_1\mathbf{v_1} + b_2\mathbf{v_2} + \cdots + b_n\mathbf{v_n}$$

We will prove $a_1 = b_1, a_2 = b_2, \ldots, a_n = b_n$.

Subtracting $\mathbf{0} = (a_1 - b_1)\mathbf{v_1} + (a_2 - b_2)\mathbf{v_2} + \cdots + (a_n - b_n)\mathbf{v_n}$

Since, S is linearly independent,
$a_1 - b_1 = 0, a_2 - b_2 = 0, \ldots, a_n - b_n = 0$ or
$a_1 = b_1, a_2 = b_2, \ldots, a_n = b_n$. The proof is complete. ∎

Theorem 4.5.2

Theorem 4.5.2 (Bases and cardinalities) Let V be a vector space and $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ be a basis of V, containing n vectors. Then any set containing more than n vectors in V is linearly dependent.

Proof. Let $T = \{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_m}\}$ be set of m vectors in V with $m > n$. For simplicity, assume $n = 3$ and $m = 4$. So, $S = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ and $T = \{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, \mathbf{u_4}\}$. To prove that T is dependent, we will have to find scalars a_1, a_2, a_3, a_4, not all zeros, such that not all zero,

$$a_1\mathbf{u_1} + a_2\mathbf{u_2} + a_3\mathbf{u_3} + a_4\mathbf{u_4} = \mathbf{0} \qquad Equation - I$$

Subsequently, we will show that Equation-I has non-trivial solution.

Continued

Since S is a basis we can write

$$\begin{array}{rlll} \mathbf{u_1} = & c_{11}\mathbf{v_1} & +c_{12}\mathbf{v_2} & +c_{13}\mathbf{v_3} \\ \mathbf{u_2} = & c_{21}\mathbf{v_1} & +c_{22}\mathbf{v_2} & +c_{23}\mathbf{v_3} \\ \mathbf{u_3} = & c_{31}\mathbf{v_1} & +c_{32}\mathbf{v_2} & +c_{33}\mathbf{v_3} \\ \mathbf{u_4} = & c_{41}\mathbf{v_1} & +c_{42}\mathbf{v_2} & +c_{43}\mathbf{v_3} \end{array}$$

We substitute these in Equation-I and re-group:

$$\begin{array}{llll} (c_{11}a_1 & +c_{21}a_2 & +c_{31}a_3 & +c_{41}a_4)\mathbf{v_1} \\ +(c_{12}a_1 & +c_{22}a_2 & +c_{32}a_3 & +c_{42}a_4)\mathbf{v_2} \\ +(c_{13}a_1 & +c_{23}a_2 & +c_{33}a_3 & +c_{43}a_4)\mathbf{v_3} & = \mathbf{0} \end{array}$$

Since $S = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is linearly independent, the coeffients of $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ are zero. So, we have (in the next frame):

Continued

$$\begin{array}{lllll} c_{11}a_1 & +c_{21}a_2 & +c_{31}a_3 & +c_{41}a_4 & =0 \\ c_{12}a_1 & +c_{22}a_2 & +c_{32}a_3 & +c_{42}a_4 & =0 \\ c_{13}a_1 & +c_{23}a_2 & +c_{33}a_3 & +c_{43}a_4 & =0 \end{array}$$

In matrix notation:

$$\begin{pmatrix} c_{11} & c_{21} & c_{31} & c_{41} \\ c_{12} & c_{22} & c_{32} & c_{42} \\ c_{13} & c_{23} & c_{33} & c_{43} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This is a system of three homogeneous linear equations in four variables. (less equations than number of variable. So, the system has non-trivial (infinitely many) solutions. So, there are a_1, a_2, a_3, a_4, not all zeros, so that Equation-I is valid. So, $T = \{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, \mathbf{u_4}\}$ is linearly dependent. The proof is complete. ■

Theorem 4.5.3

Suppose V is a vector space. If V has a basis with n elements then all bases have n elements.

Proof. Suppose $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ and $T = \{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_m}\}$ are two bases of V.
Since, the basis S has n elements, and T is linealry independent, by the theorem above m cannot be bigger than n. So, $m \leq n$.
By switching the roles of S and T, we have $n \leq m$. So, $m = n$. The proof is complete. ■

Dimension of Vector Spaces

Definition. Let V be a vector space. Suppose V has a basis $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ consisting of n vectors. Then, we say n is the dimension of V and write

$$\dim(V) = n.$$

If V consists of the zero vector only, then the dimension of V is defined to be zero.

Examples 4.5.5

We have

- From above example $\dim(\mathbb{R}^n) = n$.
- From above example $\dim(P_3) = 4$. Similalry, $\dim(P_n) = n + 1$.
- From above example $\dim(\mathbb{M}_{3,2}) = 6$. Similarly, $\dim(\mathbb{M}_{n,m}) = mn$.

Corollary 4.5.4: Dimensions of Subspaces

Corollary 4.5.4: Let V be a vector space and W be a subspace of V. Then

$$\dim(W) \leq \dim(V).$$

Proof. For simplicity, assume $\dim V = n < \infty$. We give a proof by contrapositive argument.
Suppose $\dim W > n = \dim V$. Then, there is a basis $\mathbf{w}_1, \ldots, \mathbf{w}_n.\mathbf{w}_{n+1}, \cdots$ of W. In particular, $\mathbf{w}_1, \ldots, \mathbf{w}_n.\mathbf{w}_{n+1}$ is linearly independent. Since $\dim V = n$, by Theorem 4.5.2, $\mathbf{w}_1, \ldots, \mathbf{w}_n.\mathbf{w}_{n+1}$ is linearly dependent. This is a contradiction. So, $\dim W \leq \dim V$. This completes the proof. ∎

Example 4.5.6

$$Let \quad W = \{(x, y, 2x + 3y) : x, y \in \mathbb{R}\}$$

Then, W is a subspace of $\mathbb{R}^3$ and $\dim(W) = 2$.
Proof. Note $\mathbf{0} = (0, 0, 0) \in W$, and W is closed under addition and scalar multiplication. So, W is a subspace of $\mathbb{R}^3$. Given $(x, y, 2x + 3y) \in W$, we have

$$(x, y, 2x + 3y) = x(1, 0, 2) + y(0, 1, 3)$$

This shows $span(\{(1, 0, 2), (0, 1, 3)\}) = W$. Also $\{(1, 0, 2), (0, 1, 3)\}$ is linearly independent. So, $\{(1, 0, 2), (0, 1, 3)\}$ is a basis of W and $\dim(W) = 2$. ■

Example 4.5.7

Let

$$S = \{(1,3,-2,13),(-1,2,-3,12),(2,1,1,1)\}$$

and $W = span(S)$. Prove $\dim(W) = 2$.

- **Proof.** Denote the three vectors in S by $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$.
- Then $\mathbf{v_3} = \mathbf{v_1} - \mathbf{v_2}$. Write $T = \{\mathbf{v_1}, \mathbf{v_2}\}$.
- It follows, any linear combination of vectors in S is also a linear combination of vectors in T.
- $$So, \quad W = span(S) = span(T).$$
- Also T is linearly independent. So, T is a basis and $\dim(W) = 2$.

Theorem 4.5.5

(Basis Tests): Let V be a vector space and $\dim(V) = n$.

- If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a linearly independent set in V (consisting of n vectors), then S is a basis of V.
- If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ spans V, then S is a basis of V

Proof. To prove the first one, we need to prove $spanS = W$. We use contrapositive argument. Assume $V \neq span(S)$. Then, there is a vector $\mathbf{v}_{n+1} \in V$, such that $\mathbf{v}_{n+1} \notin span(S)$. Then, it follows $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}, \mathbf{v}_{n+1}\}$ is linearly independent. On the other hand, by Theorem 4.5.2, $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}, \mathbf{v}_{n+1}\}$ is linearly dependent. This is a contradiction. So, $span(S) = V$ and S is a basis of V.

Continued

Now we prove the second statement. We again use contrapositive argument. So, assume S is not linearly independent. By Theorem 4.4.2, at least one of the vectors in S is linear combination of the rest. Without loss of generality, we can assume $\mathbf{v}_n$ is linear combination of $S_1 := \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n-1}\}$. So, $\mathbf{v}_n \in span(S_1)$. From this it follows, $V = span(S) = span(S_1)$. Now, if S_1 is not linearly independent, this process can continue and we can find a subset $T \subseteq S$, $S \neq T$, such that $span(T) = V$. So, T would be a basis of V. Since number of elements in T is less than n, this would contradict that $\dim V = n$.
This completes the proof. ■

Corollary 4.5.6

Let V be a vector space and $\dim(V) = n < \infty$

- Suppose $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\} \subseteq S$ is a linearly independent set in V (consisting of m vectors). Then, $m \leq n$ and S extends to a basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m, \mathbf{v}_{m+1}, \cdots, \mathbf{v}_n\}$ of V.
- Suppose a set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\} \subseteq S$ (consisting of m vectors), spans V. Then, $m \geq n$ and there is a subset $T \subseteq S$, such that T is a basis of V

Proof. Similar to the proof of Theorem 4.5.5. ■

Corollary 4.5.7

Let V be a vector space and Suppose $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\} \subseteq S$ is a subset of V. Then,

$$\dim(S) \leq m$$

Proof. Corollary 4.5.6, there is a subset $T \subseteq S$ that is a basis of $span(S)$. Since, So,

$$\dim(span(S)) = (\text{number of elements in T}) \leq m$$

■

Example 4.5.8

- (**Example**) Let $S = \{(13, 7), (-26, -14)\}$. Give a reason, why S is not a basis for $\mathbb{R}^2$?
  Answer: S is linearly dependent. This is immediate because the first vector is a multiple of the second.
- (**Example**)

$$\text{Let} \quad S = \{(5, 3, 1), (-2, 3, 1), (7, -8, 11), (\sqrt{2}, 2, \sqrt{2})\}$$

  Give a reason, why S is not a basis for $\mathbb{R}^3$ where
  Answer: Here $\dim(\mathbb{R}^3) = 3$. So, any basis would have 3 vectors, while S has four.

Examples 4.5.8: Continues

- **Example**. Let $S = \{1 - x, 1 - x^2, 3x^2 - 2x - 1\}$. Give a reason, why S is not a basis for P_2?
  Answer: $\dim P_2 = 3$ and S has 3 elements. So, we have to give different reason. In fact, S is linearly dependent:
  $$3x^2 - 2x - 1 = 2(1 - x) - 3(1 - x^2)$$

Examples 4.5.8: Continues

- **Example.**

$$\text{Let} \quad S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\}$$

Give a reason, why S is not a basis for $\mathbb{M}_{22}$, where
Answer: $\dim(\mathbb{M}_{22}) = 4$ and S has 3 elements.

Example 4.5.9

$$\text{Let} \quad S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \right\}$$

Does S form a basis for $\mathbb{M}_{22}$, where
Answer: $\dim(\mathbb{M}_{22}) = 4$ and S has 4 elements. Further, S is linearly independent. So, S is a basis of $\mathbb{M}_{22}$. To see they are linearly independent: Let

$$a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + c \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + d \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} a+b+c+d & c+d \\ b+d & a+b+c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow a = b = c = d = 0$$

Basis of subspaces

Suppose V is subspace of $\mathbb{R}^n$, spanned by a few given vectors. To find a basis of V do the following:

- Form a matrix A with these vectors, as rows.
- Then, row space of A is V.
- A basis of the row space would be a basis of V, which also gives the dimension.

Example 4.5.10

Let $S = \{(3,2,2),(6,5,-1),(1,1,-1)\}$. Find a basis of $span(S)$, and $\dim(span(S))$.

Solution. Form the matrix A, with these rows.

$$A = \begin{pmatrix} 3 & 2 & 2 \\ 6 & 5 & -1 \\ 1 & 1 & -1 \end{pmatrix}$$

Solution: We try to reduce the matrix, to a matrix essentially in Echelon form.

Continued

Switch first and third rows:

$$\begin{pmatrix} 1 & 1 & -1 \\ 6 & 5 & -1 \\ 3 & 2 & 2 \end{pmatrix}$$

Subtract 6 times 1^{st} row, from 2^{nd} and 3 times 1^{st} row, from 3^{rd}:

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 5 \\ 0 & -1 & 5 \end{pmatrix}$$

Continued

Subtract 2^{nd} row from 3^{rd}:

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 5 \\ 0 & 0 & 0 \end{pmatrix}$$

The matrix is essentially in row Echelon form. So,

$$\begin{cases} Basis\ of\ \ span(S) = \{(1,1,-1),(0,-1,5)\} \\ \dim(span(S)) = 2 \end{cases}$$