Inner Product Spaces
§6.2 Inner product spaces

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Goals

- Concept of length, distance, and angle in $\mathbb{R}^2$ or $\mathbb{R}^n$ is extended to abstract vector spaces V. Such a vector space will be called an Inner Product Space.
- An Inner Product Space V comes with an inner product that is like dot product in $\mathbb{R}^n$.
- The Euclidean space $\mathbb{R}^n$ is only one example of such Inner Product Spaces.

Inner Product

Definition Suppose V is a vector space.

- An inner product on V is a function

$$\langle *, * \rangle : V \times V \to \mathbb{R} \quad \textit{that associates}$$

to each ordered pair $(\mathbf{u}, \mathbf{v})$ of vectors a real number $\langle \mathbf{u}, \mathbf{v} \rangle$, such that for all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V and scalar c, we have

1. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$.
2. $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$.
3. $c\langle \mathbf{u}, \mathbf{v} \rangle = \langle c\mathbf{u}, \mathbf{v} \rangle$.
4. $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ and $v = 0 \iff \langle \mathbf{v}, \mathbf{v} \rangle = 0$.

- The vector space V with such an inner product is called an inner product space.

Theorem 6.2.1: Properties

Let V be an inner product space. Let $\mathbf{u}, \mathbf{v} \in V$ be two vectors and c be a scalar, Then,

1. $\langle \mathbf{0}, \mathbf{v} \rangle = 0$
2. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$
3. $\langle \mathbf{u}, c\mathbf{v} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$

Proof. We would have to use the properties in the definition.

1. Use (3): $\langle \mathbf{0}, \mathbf{v} \rangle = \langle 0\mathbf{0}, \mathbf{v} \rangle = 0\langle \mathbf{0}, \mathbf{v} \rangle = 0$.
2. Use commutativity (1) and (2):
   $\langle \mathbf{u}+\mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{w}, \mathbf{u}+\mathbf{v} \rangle = \langle \mathbf{w}, \mathbf{u} \rangle + \langle \mathbf{w}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$
3. Use (1) and (3): $\langle \mathbf{u}, c\mathbf{v} \rangle = \langle c\mathbf{v}, \mathbf{u} \rangle = c\langle \mathbf{v}, \mathbf{u} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$

The proofs are complete. ∎

Definitions

Definitions Let V be an inner product space and $\mathbf{u}, \mathbf{v} \in V$.

1. The length or norm of $\mathbf{v}$ is defined as
$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}.$$
2. The distance between $\mathbf{u}, \mathbf{v} \in V$ is defined as
$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$
3. The angle θ vectors $\mathbf{u}, \mathbf{v} \in V$ is defined by the formula:
$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \, \|\mathbf{v}\|} \qquad 0 \leq \theta \leq \pi.$$

   A version of Cauchy-Swartz inequality, to be given later, would assert that right side is between -1 and 1.

Theorem(s) 6.2.2

Let V be an inner product space and $\mathbf{u}, \mathbf{v} \in V$. Then,

1. Cauchy-Schwartz Inequality: $|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \, \|\mathbf{v}\|$.
2. Triangle Inequality: $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$.
3. (Definition) We say that $\mathbf{u}, \mathbf{v}$ are (mutually) orthogonal or perpendicular, if

$$\langle \mathbf{u}, \mathbf{v} \rangle = 0. \quad \textit{We write} \quad \mathbf{u} \perp \mathbf{v}.$$

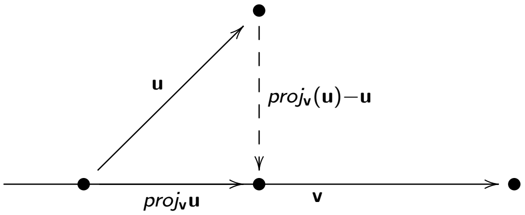
4. Pythagorean Theorem. If $\mathbf{u}, \mathbf{v}$ are orthogonal, then

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2.$$

Proof. Exactly similar to the corresponding theorems in §6.1 for $\mathbb{R}^n$.

Orthogonal Projection

Definition. Let V be an inner product space. Suppose $\mathbf{v} \in V$ is a non-zero vector. Then, for $\mathbf{u} \in V$ define Orthogonal Projection of $\mathbf{u}$ on to $\mathbf{v}$: $proj_{\mathbf{v}}(\mathbf{u}) = \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\|\mathbf{v}\|^2}\mathbf{v}$



Theorem 6.2.3

Let V be an inner product space. Suppose $\mathbf{v} \in V$ is a non-zero vector. Then, $(\mathbf{u} - proj_{\mathbf{v}}(\mathbf{u})) \perp proj_{\mathbf{v}}(\mathbf{u})$.

Proof.

$$\langle \mathbf{u} - proj_{\mathbf{v}}(\mathbf{u}), proj_{\mathbf{v}}(\mathbf{u}) \rangle = \left\langle \mathbf{u} - \left(\frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\|\mathbf{v}\|^2} \mathbf{v} \right), \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\|\mathbf{v}\|^2} \mathbf{v} \right\rangle$$

$$= \left\langle \mathbf{u}, \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\|\mathbf{v}\|^2} \mathbf{v} \right\rangle - \left\langle \left(\frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\|\mathbf{v}\|^2} \mathbf{v} \right), \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\|\mathbf{v}\|^2} \mathbf{v} \right\rangle$$

$$= \frac{\langle \mathbf{v}, \mathbf{u} \rangle^2}{\|\mathbf{v}\|^2} - \frac{\langle \mathbf{v}, \mathbf{u} \rangle^2}{\|\mathbf{v}\|^4} \langle \mathbf{v}, \mathbf{v} \rangle = 0$$

The proof is complete. ∎

Examples 6.2.1

- **Remark.** If $\mathbf{v} = (1,0)$ (or on $x-$axis) and $\mathbf{u} = (x,y)$, then $proj_{\mathbf{v}}\mathbf{u} = (x,0)$.
- (1) **The Obvious Example**: With dot product as the inner product, the Euclidean $n-$space $\mathbb{R}^n$ is an inner product space.

Examples 6.2.2: Integration

Integration is a great way to define inner product.
Let $V = C[a, b]$ be the vector space of all continuous functions $f : [a, b] \to \mathbb{R}$. For $f, g \in C[a, b]$, define inner product

$$\langle f, g \rangle = \int_a^b f(x)g(x)dx.$$

It is easy to check that $\langle f, g \rangle$ satisfies the properties of inner product spaces. Namely,

1. $\langle f, g \rangle = \langle g, f \rangle$, for all $f, g \in C[a, b]$.
2. $\langle f, g + h \rangle = \langle f, g \rangle + \langle f, h \rangle$, for all $f, g, h \in C[a, b]$.
3. $c\langle f, g \rangle = \langle cf, g \rangle$, for all $f, g \in C[a, b]$ and $c \in \mathbb{R}$.
4. $\langle f, f \rangle \geq 0$ for all $f \in C[a, b]$ and $f = 0 \Leftrightarrow \langle f, f \rangle = 0$.

Continued

Accordingly, for $f \in C[a, b]$, we can define length (or norm)

$$\|f\| = \sqrt{\langle f, f\rangle} = \sqrt{\int_a^b f(x)^2 dx}.$$

This 'length' of continuous functions would have all the properties that you expect "length" or "magnitude" to have.

Examples 6.2.2A: Double Integration

Let $D \subseteq \mathbb{R}^2$ be any connected region. Let $V = C(D)$ be the vector space of all bounded continuous functions $f(x, y) : D \to \mathbb{R}$. For $f, g \in V$ define inner product

$$\langle f, g \rangle = \int \int_D f(x, y) g(x, y) dxdy.$$

As in Example 6.2.2, it is easy to check that $\langle f, g \rangle$ satisfies the properties of inner product spaces.
In this case, length or norm of $f \in V$ is given by

$$\|f\| = \sqrt{\langle f, f \rangle} = \sqrt{\int \int_D f(x, y)^2 dxdy}.$$

Continued

In particular:

- **Example a:** If $D = [a, b] \times [c, d]$, then
$$\langle f, g\rangle = \int_c^d \int_a^b f(x, y)g(x, y)dxdy.$$
- **Example b:** If D is the unit disc: $D = \{(x, y) : x^2 + y^2 \leq 1\}$, then for $f, g \in C(D)$ is:
$$\langle f, g\rangle = \int\int_D f(x, y)g(x, y)dxdy.$$
$$= \int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x, y)g(x, y)dxdy$$

Example 6.2.3

In $\mathbb{R}^2$, define an inner product (as above): for $\mathbf{u} = (u_1, u_2)$, $\mathbf{v} = (v_1, v_2)$ define $\langle \mathbf{u}, \mathbf{v} \rangle = 2(u_1 v_1 + u_2 v_2)$. It is easy to check that this is an Inner Product on $\mathbb{R}^2$ (we skip the proof.)

Let $\mathbf{u} = (1, 3), \quad \mathbf{v} = (2, -2)$.

- (1) Compute $\langle \mathbf{u}, \mathbf{v} \rangle$. **Solution:**

$$\langle \mathbf{u}, \mathbf{v} \rangle = 2(u_1 v_1 + u_2 v_2) = 2(2 - 6) = -8$$

- (2) Compute $\|\mathbf{u}\|$. **Solution:**

$$\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} = \sqrt{2(u_1 u_1 + u_2 u_2)} = \sqrt{2(1 + 9)} = \sqrt{20}.$$

Continued

- (3) Compute $\|\mathbf{v}\|$. **Solution:**

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = \sqrt{2(4+4)} = 4$$

- (4) Compute $d(\mathbf{u}, \mathbf{v})$. **Solution:**

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \|(-1, 5)\|$$

$$= \sqrt{2(1+25)} = \sqrt{52}.$$

Example 6.2.4

Let $V = C[0,1]$ with inner product

$$\langle f, g\rangle = \int_0^1 f(x)g(x)dx \qquad for \qquad f, g, \in V.$$

Let $f(x) = 2x$ and $g(x) = x^2 + x + 1$.

- (1) Compute $\langle f, g\rangle$. **Solution:** We have

$$\langle f, g\rangle = \int_0^1 f(x)g(x)dx = \int_0^1 2\left(x^3 + x^2 + x\right) dx$$

$$= 2\left[\frac{x^4}{4} + \frac{x^3}{3} + \frac{x^2}{2}\right]_{x=0}^1 = 2\left[\frac{1}{4} + \frac{1}{3} + \frac{1}{2}\right] - 0 = \frac{13}{6}$$

Continued

- (2) Compute norm $\|f\|$.
  Solution: We have

$$\|f\| = \sqrt{\langle f, f\rangle} = \sqrt{\int_0^1 f(x)^2 dx} = \sqrt{\int_0^1 4x^2 dx}$$

$$= \sqrt{4\left[\frac{x^3}{3}\right]_{x=0}^{1}} = \sqrt{\frac{4}{3} - 0} = 2\sqrt{\frac{1}{3}}$$

Continued

- (3) Compute norm $\|g\|$. **Solution:** We have

$$\|g\| = \sqrt{\langle g, g\rangle} = \sqrt{\int_0^1 g(x)^2 dx} = \sqrt{\int_{-1}^1 (x^2 + x + 1)^2 dx}$$

$$= \sqrt{\int_0^1 \left(x^4 + 2x^3 + 3x^2 + 2x + 1\right) dx}$$

$$= \sqrt{\left[\frac{x^5}{5} + 2\frac{x^4}{4} + 3\frac{x^3}{3} + 2\frac{x^2}{2} + x\right]_0^1}$$

Continued

$$= \sqrt{\left[\frac{1}{5} + 2\frac{1}{4} + 3\frac{1}{3} + 2\frac{1}{2} + 1\right] - 0} = \sqrt{\frac{37}{10}}$$

Continued

- (4) Compute $d(f, g)$.
  Solution: We have $d(f, g) = \|f - g\| =$

$$\sqrt{\langle f - g, f - g\rangle} = \sqrt{\int_0^1 (-x^2 + x - 1)^2 dx}$$

$$= \sqrt{\int_0^1 \left(x^4 - 2x^3 + 3x^2 - 2x + 1\right) dx}$$

$$= \sqrt{\left[\frac{x^5}{5} - 2\frac{x^4}{4} + 3\frac{x^3}{3} - 2\frac{x^2}{2} + x\right]_0^1}$$

Continued

$$= \sqrt{\left[\frac{1}{5} - 2\frac{1}{4} + 3\frac{1}{3} - 2\frac{1}{2} + 1\right] - 0} = \sqrt{\frac{7}{10}}$$

Example 6.2.4

Let $V = C[-\pi, \pi]$ with inner product $\langle f, g\rangle$ as in Example 6.2.2 (by definite integral). Let $f(x) = x^3$ and $g(x) = x^2 - 3$. Show that f and g are orthogonal.

Solution: We have to show that $\langle f, g\rangle = 0$. We have $\langle f, g\rangle =$

$$\int_{-\pi}^{\pi} f(x)g(x)dx = \int_{-\pi}^{\pi} x^3(x^2 - 3)dx = \int_{-\pi}^{\pi} (x^5 - 3x^3dx$$

$$= \left[\frac{x^6}{6} - \frac{x^4}{4}\right]_{-\pi}^{\pi} = 0.$$

So, $f \perp g$.

Example 6.2.5

Exercise Let $\mathbf{u} = (\sqrt{2}, \sqrt{2})$ and $\mathbf{v} = (3, -4)$.

- Compute $proj_{\mathbf{v}}(\mathbf{u})$ and $proj_{\mathbf{u}}(\mathbf{v})$
- **Solution**. First $\langle \mathbf{u}, \mathbf{v} \rangle = \sqrt{2} * 3 - \sqrt{2} * 4 = -\sqrt{2}$,

$$\|\mathbf{u}\| = \sqrt{\sqrt{2}^2 + \sqrt{2}^2} = 4, \quad \|\mathbf{v}\| = \sqrt{3^2 + (-4)^2} = 5$$

- $$proj_{\mathbf{v}}(\mathbf{u}) = \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\|\mathbf{v}\|^2}\mathbf{v} = -\sqrt{2}(3, -4) = \left(-3\sqrt{2}, 4\sqrt{2}\right)$$

- $$proj_{\mathbf{u}}(\mathbf{v}) = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\|^2}\mathbf{u} = -\sqrt{2}(\sqrt{2}, \sqrt{2}) = (-2, -2)$$

Example 6.2.6

Let $V = C[0,1]$ with inner product

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx \qquad \textit{for} \qquad f, g, \in V.$$

Let $f(x) = 2x$ and $g(x) = x^2 + x + 1$. Compute the orthogonal projection of f onto g, and the orthogonal projection of g onto f.

Solution From Example 6.2.4, where we worked these two functions f, g, we have

$$\langle f, g \rangle = \frac{13}{6}, \quad \|f\| = 2\sqrt{\frac{1}{2}}, \|g\| = \sqrt{\frac{37}{10}}$$

Continued

$$proj_f(g) = \frac{\langle g, f\rangle}{\|f\|^2} f = \frac{\frac{13}{6}}{2}(2x) = \frac{13}{6}x$$

Also,

$$proj_g(f) = \frac{\langle g, f\rangle}{\|g\|^2} g = \frac{\frac{13}{6}}{\frac{37}{10}}(x^2 + x + 1) = \frac{130}{222}(x^2 + x + 1)$$

Example 6.2.7

Let $V = C[0,1]$ with inner product $\langle f, g\rangle$ as in Example 6.2.2 (by definite integral). Let $f(x) = x^3 + x$ and $g(x) = 2x + 1$. Compute the orthogonal projection of f onto g.
Solution Recall the definition: $proj_{\mathbf{v}}(\mathbf{u}) = \frac{\langle \mathbf{v},\mathbf{u}\rangle}{\|\mathbf{v}\|^2}\mathbf{v}$ So,

$$proj_g(f) = \frac{\langle g, f\rangle}{\|g\|^2} g$$

- First compute $\langle g, f\rangle =$

$$\int_0^1 (x^3 + x)(2x + 1)dx \int_0^1 (2x^4 + x^3 + 2x^2 + x)dx$$

Continued

- $$= \left[2\frac{x^5}{5} + \frac{x^4}{4} + 2\frac{x^3}{3} + \frac{x^2}{2} \right]_0^1 = \frac{109}{60}$$
- So $\langle g, f \rangle = \frac{109}{60}$

Continued

- Now compute $\|g\|^2 =$

$$\int_0^1 (2x+1)^2 dx = \int_{-1}^1 (4x^2+4x+1) dx = \left[4\frac{x^3}{3} + 4\frac{x^2}{2} + x\right]_0^1$$

$$= \left(4\frac{1}{3} + 4\frac{1}{2} + 1\right) - 0 = \frac{13}{3}$$

Continued

- So,

$$proj_g(f) = \frac{\langle g, f \rangle}{\|g\|^2} g = \frac{\frac{109}{60}}{\frac{13}{3}}(2x+1) = \frac{109}{260}(2x+1)$$