Inner Product Spaces
§6.3 Orthonormal Bases

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Goals

- Define Orthonormal Basis of an Inner Product Spaces
- Discuss Gram-Schmidt Method of finding an Orthonormal Basis

Definition: Orthonormal Basis

Definition Suppose $(V, \langle -, - \rangle)$ is an Inner product space.

- A subset $S \subseteq V$ is said to be an Orthogonal subset, if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$, for all $\mathbf{u}, \mathbf{v} \in S$, with $\mathbf{u} \neq \mathbf{v}$. That means, if elements in S are pairwise orthogonal.
- An Orthogonal subset $S \subseteq V$ is said to be an Orthonormal subset if, in addition, $\|u\| = 1$, for all $\mathbf{u} \in S$.
- If an Orthonormal set S is also a basis of V, then it is called an Orthonormal Basis. That means, if

$$\begin{cases} \langle \mathbf{u}, \mathbf{v} \rangle = 0 & \forall\ \mathbf{u}, \mathbf{v} \in S,\ \mathbf{u} \neq \mathbf{v} \\ \|\mathbf{u}\| = 1 & \mathbf{u} \in S \\ S \text{ is a basis of } V & \end{cases}$$

Continued

- We remark that, we did not require that S is a finite set. However, we would mainly be considering finite such subsets $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\} \subseteq V$.
- So, a finite subset $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\} \subseteq V$ is an orthonormal basis, if

$$\begin{cases} \langle \mathbf{u}_i, \mathbf{u}_j \rangle = 0 & \forall\ i, j = 1, 2, \ldots, n\ i \neq j \\ \|\mathbf{u}_i\| = 1 & \forall\ i, = 1, 2, \ldots, n \\ S \text{ is a basis of } V & \end{cases}$$

Example 6.3.1

Most obvious example of an orthonormal basis is standard basis $S = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\} \subseteq \mathbb{R}^n$, where

$$\begin{cases} \mathbf{e}_1 = (1, 0, 0, \ldots, 0) \\ \mathbf{e}_2 = (0, 1, 0, \ldots, 0) \\ \mathbf{e}_3 = (0, 0, 1, \ldots, 0) \\ \quad \cdots \\ \mathbf{e}_n = (0, 0, 0, \ldots, 1) \end{cases} \tag{1}$$

Lemma 6.3.1

Suppose $(V, \langle -, - \rangle)$ is an inner product space. Suppose $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\} \subseteq V$ is an orthogonal set, consisting of nonzero vectors. Then, S is linearly independent.

Proof. Suppose

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_n\mathbf{u}_n = \mathbf{0} \qquad \text{where} \quad c_1, \ldots, c_n \in \mathbb{R}.$$

$$\Longrightarrow \langle \mathbf{u}_1, c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_n\mathbf{u}_n \rangle = \langle \mathbf{u}_1, \mathbf{0} \rangle = 0$$

$$\Longrightarrow c_1\langle \mathbf{u}_1, \mathbf{u}_1 \rangle + c_2\langle \mathbf{u}_1, \mathbf{u}_2 \rangle + \cdots + c_n\langle \mathbf{u}_1, \mathbf{u}_n \rangle = 0$$

$$\Longrightarrow c_1\langle \mathbf{u}_1, \mathbf{u}_1 \rangle + c_2 0 + \cdots + c_n 0 = 0. \quad \Longrightarrow c_1\langle \mathbf{u}_1, \mathbf{u}_1 \rangle = 0.$$

$$\Longrightarrow c_1 = 0. \quad \text{Likewise,} \quad c_2 = \cdots = c_n = 0$$

This completes the proof. ∎

Recall: Orthogonal Projections

Suppose $(V, \langle -, - \rangle)$ is an inner product space and $\mathbf{u}, \mathbf{v} \in V$. Then the orthogonal projective of $\mathbf{u}$ along $\mathbf{v}$ is

$$Proj_{\mathbf{v}}(\mathbf{u}) = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{v}\|^2}\mathbf{v}$$

In the next frame, we would discuss Gram-Schmidt Orthogonalization Process.

Theorem 6.3.2: Gram-Schmidt Orthogonalization Process

Let $(V, \langle -, - \rangle)$ be an inner product space and $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be a basis of V. Construct the following sequence of vectors:

$$\begin{cases} \mathbf{v}_1 = \mathbf{u}_1 \\ \mathbf{v}_2 = \mathbf{u}_2 - Proj_{\mathbf{v}_1}(\mathbf{u}_2) \\ \mathbf{v}_3 = \mathbf{u}_3 - Proj_{\mathbf{v}_1}(\mathbf{u}_3) - Proj_{\mathbf{v}_2}(\mathbf{u}_3) \\ \cdots \quad \cdots \\ \mathbf{v}_n = \mathbf{u}_n - Proj_{\mathbf{v}_1}(\mathbf{u}_n) - Proj_{\mathbf{v}_2}(\mathbf{u}_n) - \cdots - Proj_{\mathbf{v}_{n-1}}(\mathbf{u}_n) \end{cases} \tag{2}$$

Then, $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is an Orthogonal Basis of V.

Proof.

Proof. It is easy to see, inductively,

$$span\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} = span\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\} = V.$$

Since dim $V = n$, the theorem follows from Theorem 4.5.5. ■

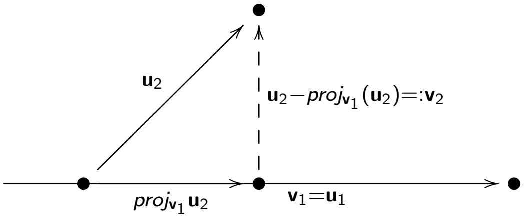
Continued

In this frame, we write Equation 2, more explicitly.

$$\begin{cases} \mathbf{v}_1 = \mathbf{u}_1 \\ \mathbf{v}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 \\ \mathbf{v}_3 = \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 \\ \cdots \quad \cdots \\ \mathbf{v}_n = \mathbf{u}_n - \frac{\langle \mathbf{u}_n, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_n, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 - \cdots - \frac{\langle \mathbf{u}_n, \mathbf{v}_2 \rangle}{\|\mathbf{v}_{n-1}\|^2}\mathbf{v}_{n-1} \end{cases} \tag{3}$$

Continued

In this frame, we show first two terms geometrically:



Corollary 6.3.3

Let $(V, \langle -, - \rangle)$ be an inner product space and $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be a basis of V. Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be as in Equation 2, which is an orthogonal basis of V. Then,

$$\left\{ \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}, \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|}, \ldots, \frac{\mathbf{v}_n}{\|\mathbf{v}_n\|} \right\}$$

is an Orthonormal Basis of V.

Example 6.3.2

Let $\mathbf{u}_1 = (1, 1, 1)$, $\mathbf{u}_2 = (1, -1, 1)$, $\mathbf{u}_3 = (1, 1, -1)$. Use Gram-Schmidt Orthogonalization process, to compute an orthonormal basis of $\mathbb{R}^3$, in two steps, using Theorem 6.3.2 and Corollary 6.3.3.

Solution: We have

$$\|\mathbf{u}_1\|^2 = \|\mathbf{u}_2\|^2 = \|\mathbf{u}_3\|^2 = 3,$$

$$\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = \langle \mathbf{u}_1, \mathbf{u}_3 \rangle = 1, \quad \langle \mathbf{u}_2, \mathbf{u}_3 \rangle = -1$$

Continued

$$
\begin{cases}
\mathbf{v}_1 = \mathbf{u}_1 = (1,1,1) \\
\mathbf{v}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = (1,-1,1) - \frac{1}{3}(1,1,1) = \left(\frac{2}{3}, -\frac{4}{3}, \frac{2}{3}\right) \\
\|\mathbf{v}_2\|^2 = \frac{24}{9}, \quad \langle \mathbf{u}_3, \mathbf{v}_2 \rangle = -\frac{4}{3} \\
\mathbf{v}_3 = \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 \\
= (1,1,-1) - \frac{1}{3}(1,1,1) - \frac{-4/3}{24/9}\left(\frac{2}{3}, -\frac{4}{3}, \frac{2}{3}\right) \\
= (1,1,-1) - \frac{1}{3}(1,1,1) + \left(\frac{1}{3}, -\frac{2}{3}, \frac{1}{3}\right) = (1,1,-1) + (0,-1,0) \\
= (1,0,-1)
\end{cases}
$$

Continued

So, an Orthogonal Basis of $\mathbb{R}^3$ is

$$\left\{(1,1,1), \left(\frac{2}{3}, -\frac{4}{3}, \frac{2}{3}\right), (1,0,-1)\right\}$$

$$\text{Also,} \quad \|\mathbf{v}_1\| = \sqrt{3}, \quad \|\mathbf{v}_2\| = \frac{2\sqrt{2}}{\sqrt{3}} \quad \|\mathbf{v}_3\| = \sqrt{2}$$

Continued

So, an Orthonormal basis of $\mathbb{R}^3$ is

$$\left\{\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right),\frac{\sqrt{3}}{2\sqrt{2}}\left(\frac{2}{3},-\frac{4}{3},\frac{2}{3}\right),\left(\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}}\right)\right\}$$

$$=\left\{\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right),\left(\frac{1}{\sqrt{6}},-\frac{2}{\sqrt{6}},\frac{1}{\sqrt{6}}\right),\left(\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}}\right)\right\}$$