Chapter 2

Matrices

2.1 Operations with Matrices

Homework: §2.1 (page 56): 7, 9, 13, 15, 17, 25, 27, 35, 37, 41, 46, 49, 67

Main points in this section:

1. We define a few concept regarding matrices. This would include addition of matrices, scalar multiplication and multiplication of matrices.

2. We also represent a system of linear equation as equation with matrices.
In this section, we will do some algebra of matrices. That means, we will add, subtract, multiply matrices. Matrices will usually be denoted by upper case letters, \( A, B, C, \ldots \). Such a matrix

\[
A = \begin{bmatrix}
  a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
  a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\
  a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\
  \cdots & \cdots & \cdots & \cdots & \cdots \\
  a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn}
\end{bmatrix}
\]

is also denoted by \([a_{ij}]\).

**Remark.** I used parentheses in the last chapter, not a square brackets. The textbook uses square brackets.

**Definition 2.1.1** Two matrices \( A = [a_{ij}] \) and \( B = [b_{ij}] \) are equal if both \( A \) and \( B \) have same size \( m \times n \) and the entries

\[ a_{ij} = b_{ij} \quad \text{for all} \quad 1 \leq i \leq m \quad \text{and} \quad 1 \leq j \leq n. \]

Read [Textbook, Example 1, p. 47] for examples of unequal matrices.

**Definition 2.1.2** Following are some standard terminologies:

1. A matrix with only one column is called a **column matrix** or **column vector**. For example,

\[
a = \begin{bmatrix} 13 \\ 19 \\ 23 \end{bmatrix}
\]

is a column matrix.

2. A matrix with only one row is called a **row matrix** or **row vector**. For example

\[
b = \begin{bmatrix} 4 & 11 & 13 & 19 & 23 \end{bmatrix}
\]

is a row matrix.
3. Bold face lower case letters, as above, will often be used to denote row or column matrices.

### 2.1.1 Matrix Addition

**Definition 2.1.3** We define addition of two matrices of *same size*. Suppose $A = [a_{ij}]$ and $B = [b_{ij}]$ be two matrices of same size $m \times n$. Then the *sum* $A + B$ is defined to be the matrix of size $m \times n$ given by

$$A + B = [a_{ij} + b_{ij}].$$

1. For example, with

$$A = \begin{bmatrix} 4 & -5 \\ 3 & -8 \\ 10 & 14 \end{bmatrix}, \quad B = \begin{bmatrix} 9 & 15 \\ -5 & 18 \\ 11 & 1 \end{bmatrix} \quad \text{we have } A + B = \begin{bmatrix} 13 & 10 \\ -2 & 10 \\ 21 & 15 \end{bmatrix}.$$  

2. Also, for example, with

$$C = \begin{bmatrix} -3.5 & -2 \\ 3 & -2.2 \end{bmatrix}, \quad D = \begin{bmatrix} 0.5 & 2.7 \\ -3 & -5 \end{bmatrix} \quad \text{we have } C + D = \begin{bmatrix} -3 & 0.7 \\ 0 & -7.2 \end{bmatrix}.$$  

3. While, the sum $A + C$ is not defined because $A$ and $C$ do not have same size.

4. Read [Textbook, Example 2, p. 48] for more such examples.

### 2.1.2 Scalar Multiplication

Recall, in some contexts, real numbers are referred to as *scalars* (*in contrast with vectors.*) We define, multiplication of a matrix $A$ by a scalar $c$. 

Definition 2.1.4 Let \( A = [a_{ij}] \) be an \( m \times n \) matrix and \( c \) be a scalar. We define **Scalar multiple** \( cA \) of \( A \) by \( c \) as the matrix of same size given by

\[
cA = [ca_{ij}].
\]

1. For a matrix \( A \), the negative of \(-A\) denotes \((1)A\). Also \( A - B := A + (-1)B \).

2. Let \( c = 11 \) and

\[
A = \begin{bmatrix}
4 & -5 \\
3 & -8 \\
10 & 14
\end{bmatrix}, \quad B = \begin{bmatrix}
9 & 15 \\
-5 & 18 \\
11 & 1
\end{bmatrix}
\]

Then, with \( c = 11 \) we have

\[
cA = 11 \begin{bmatrix}
4 & -5 \\
3 & -8 \\
10 & 14
\end{bmatrix} = \begin{bmatrix}
44 & -55 \\
33 & -88 \\
110 & 154
\end{bmatrix}.
\]

Likewise, \( A - B = A + (-1)B = \)

\[
\begin{bmatrix}
4 & -5 \\
3 & -8 \\
10 & 14
\end{bmatrix} + (-1) \begin{bmatrix}
9 & 15 \\
-5 & 18 \\
11 & 1
\end{bmatrix} = \begin{bmatrix}
4 & -5 \\
3 & -8 \\
10 & 14
\end{bmatrix} + \begin{bmatrix}
-9 & -15 \\
5 & -18 \\
-11 & -1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
4 - 9 & -5 - 15 \\
3 + 5 & -8 - 18 \\
10 - 11 & 14 - 1
\end{bmatrix} = \begin{bmatrix}
-5 & -20 \\
8 & -26 \\
-1 & 13
\end{bmatrix}
\]

3. **Read** [Textbook, Example 3, p. 49] for more such computations.
2.1. OPERATIONS WITH MATRICES

2.1.3 Matrix Multiplication

The textbook gives a helpful motivation for defining matrix multiplication in page 49. Read it, if you are not already motivated.

**Definition 2.1.5** Suppose $A = [a_{ij}]$ is a matrix of size $m \times n$ and $B = [b_{ij}]$ is a matrix of size $n \times p$. Then the product $AB = [c_{ij}]$ is a matrix size $m \times p$ where

$$c_{ij} := a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}.$$  

1. Note that number of columns ($n$) of $A$ must be equal to number of rows ($n$) of $B$, for the product $AB$ to be defined.

2. Note the number of rows of $AB$ is equal to is same as that ($m$) of $A$ and number of columns of $AB$ is equal to is same as that ($n$) of $B$.

3. Read [Textbook, Example 4-5, p.51-52]. I will workout a few below.

**Exercise 2.1.6 (Ex. 12, p. 57)** Let

$$A = \begin{bmatrix} 1 & -1 & 7 \\ 2 & -1 & 8 \\ 3 & 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 2 \end{bmatrix}$$

Compute $AB$ abd $BA$. We have

$$AB = \begin{bmatrix} 1 \cdot 1 + (-1) \cdot 2 + 7 \cdot 1 & 1 \cdot 1 + (-1) \cdot 1 + 7 \cdot (-3) & 1 \cdot 2 + (-1) \cdot 1 + 7 \cdot 2 \\
2 \cdot 1 + (-1) \cdot 2 + 8 \cdot 1 & 2 \cdot 1 + (-1) \cdot 1 + 8 \cdot (-3) & 2 \cdot 2 + (-1) \cdot 1 + 8 \cdot 2 \\
3 \cdot 1 + 1 \cdot 2 + (-1) \cdot 1 & 3 \cdot 1 + 1 \cdot 1 + (-1) \cdot (-3) & 3 \cdot 2 + 1 \cdot 1 + (-1) \cdot 2 \end{bmatrix}$$
Now, we compute $BA$. We have
\[
BA = \begin{bmatrix}
1 \cdot 1 + 1 \cdot 2 + 2 \cdot 3 & 1 \cdot (-1) + 1 \cdot (-1) + 2 \cdot 1 & 1 \cdot 7 + 1 \cdot 8 + 2 \cdot (-1) \\
2 \cdot 1 + 1 \cdot 2 + 1 \cdot 3 & 2 \cdot (-1) + 1 \cdot (-1) + 1 \cdot 1 & 2 \cdot 7 + 1 \cdot 8 + 1 \cdot (-1) \\
1 \cdot 1 + (-3) \cdot 2 + 2 \cdot 3 & 1 \cdot (-1) + (-3) \cdot (-1) + 2 \cdot 1 & 1 \cdot 7 + (-3) \cdot 8 + 2 \cdot (-1)
\end{bmatrix}
\]
\[
= \begin{bmatrix}
9 & 0 & 13 \\
7 & -2 & 21 \\
1 & 4 & -19
\end{bmatrix}.
\]

Also note that $AB \neq BA$.

Exercise 2.1.7 (Ex. 16, p. 57) Let
\[
A = \begin{bmatrix}
0 & -1 & 3 \\
4 & 0 & 2 \\
8 & -1 & 7
\end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix}
2 \\
-3 \\
1
\end{bmatrix}.
\]

Compute $AB$ and $BA$, if defined.

Solution: Note $BA$ is not defined, because size of $B$ is $3 \times 1$ and size of $A$ is $3 \times 3$. But $AB$ is defined and is a $3 \times 1$ matrix (or a column matrix). We have
\[
AB = \begin{bmatrix}
0 \cdot 2 + (-1) \cdot (-3) + 3 \cdot 1 \\
4 \cdot 2 + 0 \cdot (-3) + 2 \cdot 1 \\
8 \cdot 2 + (-1) \cdot (-3) + 7 \cdot 1
\end{bmatrix}
= \begin{bmatrix}
6 \\
10 \\
26
\end{bmatrix}.
\]

Exercise 2.1.8 (Ex. 18, p. 57) Let
\[
A = \begin{bmatrix}
1 & 0 & 3 & -2 & 4 \\
6 & 13 & 8 & -17 & 20
\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}
1 & 6 \\
4 & 2
\end{bmatrix}
\]
Compute $AB$ and $BA$, if defined.

**Solution** Note $AB$ is not defined, because size of $A$ is $2 \times 5$ and size of $B$ is $2 \times 2$. But $BA$ is defined and we have 

$$BA = 
\begin{bmatrix}
1 \cdot 1 + 6 \cdot 6 & 1 \cdot 0 + 6 \cdot 13 & 1 \cdot 3 + 6 \cdot 8 & 1 \cdot (-2) + 6 \cdot (-17) & 1 \cdot 4 + 6 \cdot 20 \\
4 \cdot 1 + 2 \cdot 6 & 4 \cdot 0 + 2 \cdot 13 & 4 \cdot 3 + 2 \cdot 8 & 4 \cdot (-2) + 2 \cdot (-17) & 4 \cdot 4 + 2 \cdot 20
\end{bmatrix} =
\begin{bmatrix}
37 & 78 & 51 & -104 & 124 \\
16 & 26 & 28 & -42 & 56
\end{bmatrix}.$$ 

### 2.1.4 Systems of linear equations

Systems of linear equations can be represented in a matrix form. Given a system of linear equations

\[
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3 \\
\vdots \\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m
\]

we can write this system in the matrix form as

$$Ax = b$$

where 

$$A = 
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\
a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn}
\end{bmatrix}, \quad x = 
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\vdots \\
x_n
\end{bmatrix}, \quad b = 
\begin{bmatrix}
b_1 \\
b_2 \\
b_3 \\
\vdots \\
b_n
\end{bmatrix}.$$ 

Clearly, $A$ is the coefficient matrix, $x$ would be called the unknown (or variable) matrix (or vector) and $b$ would be called the constant vector (or matrix).
1. With

\[
\begin{bmatrix}
a_{11} \\
a_{21} \\
a_{31} \\
\vdots \\
a_{m1}
\end{bmatrix},
\begin{bmatrix}
a_{12} \\
a_{22} \\
a_{32} \\
\vdots \\
a_{m2}
\end{bmatrix}, \ldots
\begin{bmatrix}
a_{1j} \\
a_{2j} \\
a_{3j} \\
\vdots \\
a_{mj}
\end{bmatrix}, \ldots
\begin{bmatrix}
a_{1n} \\
a_{2n} \\
a_{3n} \\
\vdots \\
a_{mn}
\end{bmatrix}
\]

we can write (or think)

\[
A = \begin{bmatrix}
a_1 & a_2 & \cdots & a_j & \cdots & a_n
\end{bmatrix}
\]

as a matrix of matrices.

2. The above system of linear equations can be (re)written as

\[
x_1a_1 + x_2a_2 + \cdots + x_ja_j + \cdots + x_na_n = b.
\]

We say, \( b \) is a \textit{linear combination} of the column matrices (or vectors) \( a_1, a_2, \ldots, a_n \) with \textit{coefficients} \( x_1, x_2, \ldots, x_n \).

3. So, the solutions of the above system are precisely those \( c_1, \ldots, c_n \) such that \( b \) is a \textit{linear combination} of the vectors \( a_1, a_2, \ldots, a_n \) with \textit{coefficients} \( c_1, c_2, \ldots, c_n \).

\textbf{Exercise 2.1.9 (Ex. 34 (changed), p. 58)} Consider the system of equation:

\[
\begin{align*}
x_1 + x_2 - 3x_3 &= -1 \\
-x_1 + 2x_2 &= 1 \\
x_1 - 2x_2 + x_3 &= 2
\end{align*}
\]

Write this system in the matrix-form \( Ax = b \) and solve matrix equation for \( x \).

\textbf{Solution:} The equation reduces to

\[
\begin{bmatrix}
1 & 1 & -3 \\
-1 & 2 & 0 \\
1 & -2 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
= \begin{bmatrix}
-1 \\
1 \\
2
\end{bmatrix}
\]
This answers the first part. To solve, the augmented matrix of the system is
\[
\begin{bmatrix}
1 & 1 & -3 & -1 \\
-1 & 2 & 0 & 1 \\
1 & -2 & 1 & 2
\end{bmatrix}
\]
Using TI-84, we can reduce it to Gauss-Jordan form:
\[
\begin{bmatrix}
1 & 0 & 0 & 5 \\
0 & 1 & 0 & 3 \\
0 & 0 & 1 & 3
\end{bmatrix}
\]
The corresponding linear system is:
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} =
\begin{bmatrix}
5 \\
3 \\
3
\end{bmatrix}
\]
Multiplying, this give the solution:
\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} =
\begin{bmatrix}
5 \\
3 \\
3
\end{bmatrix}
\]

Exercise 2.1.10 (Ex. 40, .58)
\[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}
\begin{bmatrix}
2 & 1 \\
3 & 1
\end{bmatrix} =
\begin{bmatrix}
3 & 17 \\
4 & -1
\end{bmatrix}
\]

Solution: Here we have four unknowns a, b, c, d. In any case, multiplying:
\[
\begin{bmatrix}
2a + 3b & a + b \\
2c + 3d & c + d
\end{bmatrix} =
\begin{bmatrix}
3 & 17 \\
4 & -1
\end{bmatrix}
\]
Equating respective entries:

\[
\begin{align*}
2a + 3b &= 3 \\
a + b &= 17 \\
2c + 3d &= 4 \\
c + d &= -1
\end{align*}
\]

The augmented matrix of the system:

\[
\begin{bmatrix}
2 & 3 & 0 & 0 & 3 \\
1 & 1 & 0 & 0 & 17 \\
0 & 0 & 2 & 3 & 4 \\
0 & 0 & 1 & 1 & -1
\end{bmatrix}
\]

Use TI-84 to reduce it to the Gauss-Jordan form:

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 48 \\
0 & 1 & 0 & 0 & -31 \\
0 & 0 & 1 & 0 & -7 \\
0 & 0 & 0 & 1 & 6
\end{bmatrix}
\]

Looking at the corresponding to this matrix, we get

\[
a = 48, b = -31, c = -7, d = 6 \quad \text{and} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 48 & -31 \\ -7 & 6 \end{bmatrix}.
\]
2.2 Properties of Matrix operations

Homework: §2.2 (page 70): 1, 3, 7, 9, 15, 17, 21, 23, 29, 31, 35, 37, 39, 43, 57, 59

Main points in this section:

1. We write down some of the properties of matrix addition and multiplication.

2. We define transpose of a matrix.

3. We start writing some proofs.
We start developing some algebra of matrices.

**Theorem 2.2.1 (Properties)** Let $A, B, C$ be three $m \times n$ matrices and $c, d$ be scalars. Then

1. $A + B = B + A$ \quad Commutativity of matrix addition
2. $(A + B) + C = A + (B + C)$ \quad Associativity of matrix addition
3. $(cd)A = c(dA)$ \quad Associativity of scalar multiplication
4. $1A = A$ \quad identity of scalar multiplication
5. $c(A + B) = cA + cB$ \quad Distributivity of scalar multiplication
6. $(c + d)A = cA + dA$ \quad Distributivity of scalar multiplication

**Proof.** One needs to prove all these statements using definitions of addition and scalar multiplication. To prove the commutativity of matrix addition (1), Let $A = [a_{ij}], B = [b_{ij}]$. Both $A$ and $B$ have same size $m \times n$, so $A + B, B + A$ are defined. From definition

$$A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}] \quad \text{and} \quad B + A = [b_{ij}] + [a_{ij}] = [b_{ij} + a_{ij}].$$

From commutative property of addition of real numbers, we have $a_{ij} + b_{ij} = b_{ij} + a_{ij}$. Therefore, from definition of equality of matrices, $A + B = B + A$. So, (1) is proved. Other properties are proved similarly. $\blacksquare$

**Remark.** For matrices $A, B, C$ as in the theorem, by the expression $A + B + C$ we mean $(A + B) + C$ or $A + (B + C)$. It is well defined, because $(A + B) + C = A + (B + C)$ by associative property(2) of matrix addition.

**Theorem 2.2.2** Let $O_{mn}$ denote the $m \times n$ matrix whose entries are all zero. Let $A$ be a $m \times n$ matrix. Then

1. Then $A + O_{mn} = A$. 

---

We start developing some **algebra of matrices**.

**Theorem 2.2.1 (Properties)** Let $A, B, C$ be three $m \times n$ matrices and $c, d$ be scalars. Then

1. $A + B = B + A$ \quad Commutativity of matrix addition
2. $(A + B) + C = A + (B + C)$ \quad Associativity of matrix addition
3. $(cd)A = c(dA)$ \quad Associativity of scalar multiplication
4. $1A = A$ \quad identity of scalar multiplication
5. $c(A + B) = cA + cB$ \quad Distributivity of scalar multiplication
6. $(c + d)A = cA + dA$ \quad Distributivity of scalar multiplication

**Proof.** One needs to prove all these statements using definitions of addition and scalar multiplication. To prove the commutativity of matrix addition (1), Let $A = [a_{ij}], B = [b_{ij}]$. Both $A$ and $B$ have same size $m \times n$, so $A + B, B + A$ are defined. From definition

$$A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}] \quad \text{and} \quad B + A = [b_{ij}] + [a_{ij}] = [b_{ij} + a_{ij}].$$

From commutative property of addition of real numbers, we have $a_{ij} + b_{ij} = b_{ij} + a_{ij}$. Therefore, from definition of equality of matrices, $A + B = B + A$. So, (1) is proved. Other properties are proved similarly. $\blacksquare$

**Remark.** For matrices $A, B, C$ as in the theorem, by the expression $A + B + C$ we mean $(A + B) + C$ or $A + (B + C)$. It is well defined, because $(A + B) + C = A + (B + C)$ by associative property(2) of matrix addition.

**Theorem 2.2.2** Let $O_{mn}$ denote the $m \times n$ matrix whose entries are all zero. Let $A$ be a $m \times n$ matrix. Then

1. Then $A + O_{mn} = A$. 

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2.2. PROPERTIES OF MATRIX OPERATIONS

2. We have \( A + (-A) = O_{mn} \)

3. If \( cA = O_{mn} \), then either \( c = 0 \) or \( A = 0 \).

*We say, on the set of all \( m \times n \) matrices, \( O_{mn} \) is an additive identity (property (1)), and \((-A)\) is the additive inverse of \( A \) (property (2))*

2.2.1 Properties of matrix multiplication

Theorem 2.2.3 (Mult-Properties) Let \( A, B, C \) be three matrices (of varying sizes) so that all the products below are defined and \( c \) be a scalars. Then

1. \( AB \neq BA \)  
   **Failure of Commutativity of multiplication**

2. \((AB)C = A(BC)\)  
   **Associativity of multiplication**

3. \((A + B)C = AC + BC\)  
   **Left – Distributivity of multiplication**

4. \(A(B + C) = AB + AC\)  
   **Right – Distributivity of multiplication**

5. \(c(AB) = (cA)B\)  
   **are Associativity**

In (1), by \( AB \neq BA \), we mean \( AB \) is not always equal to \( BA \).

**Proof.** We will only prove (4). In this case, let \( A \) be a matrix of size \( m \times n \) and then \( B, C \) would have to be of same size \( n \times p \). Write

\[
A = \begin{bmatrix}
  a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
  a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\
  a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn}
\end{bmatrix},
\]
and

\[
B = \begin{bmatrix}
  b_{11} & b_{12} & b_{13} & \cdots & b_{1p} \\
  b_{21} & b_{22} & b_{23} & \cdots & b_{2p} \\
  b_{31} & b_{32} & b_{33} & \cdots & b_{3p} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  b_{n1} & b_{n2} & b_{n3} & \cdots & b_{np}
\end{bmatrix}, \quad C = \begin{bmatrix}
  c_{11} & c_{12} & c_{13} & \cdots & c_{1p} \\
  c_{21} & c_{22} & c_{23} & \cdots & c_{2p} \\
  c_{31} & c_{32} & c_{33} & \cdots & c_{3p} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  c_{n1} & c_{n2} & c_{n3} & \cdots & c_{np}
\end{bmatrix}
\]

Therefore, \(A(B + C) =\)

\[
\begin{bmatrix}
  a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
  a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\
  a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn}
\end{bmatrix}
\begin{bmatrix}
  b_{11} + c_{11} & b_{12} + c_{12} & b_{13} + c_{13} & \cdots & b_{1p} + c_{1p} \\
  b_{21} + c_{21} & b_{22} + c_{22} & b_{23} + c_{23} & \cdots & b_{2p} + c_{2p} \\
  b_{31} + c_{31} & b_{32} + c_{32} & b_{33} + c_{33} & \cdots & b_{3p} + c_{3p} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  b_{n1} + c_{n1} & b_{n2} + c_{n2} & b_{n3} + c_{n3} & \cdots & b_{np} + c_{np}
\end{bmatrix}
\]

which is

\[
\begin{bmatrix}
  \sum_{k=1}^{n} a_{1k}(b_{k1} + c_{k1}) & \sum_{k=1}^{n} a_{1k}(b_{k2} + c_{k2}) & \cdots & \sum_{k=1}^{n} a_{1k}(b_{kp} + c_{kp}) \\
  \sum_{k=1}^{n} a_{2k}(b_{k1} + c_{k1}) & \sum_{k=1}^{n} a_{2k}(b_{k2} + c_{k2}) & \cdots & \sum_{k=1}^{n} a_{2k}(b_{kp} + c_{kp}) \\
  \sum_{k=1}^{n} a_{3k}(b_{k1} + c_{k1}) & \sum_{k=1}^{n} a_{3k}(b_{k2} + c_{k2}) & \cdots & \sum_{k=1}^{n} a_{3k}(b_{kp} + c_{kp}) \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  \sum_{k=1}^{n} a_{mk}(b_{k1} + c_{k1}) & \sum_{k=1}^{n} a_{mk}(b_{k2} + c_{k2}) & \cdots & \sum_{k=1}^{n} a_{mk}(b_{kp} + c_{kp})
\end{bmatrix}
\]
which is

\[
\begin{bmatrix}
\sum_{k=1}^{n} a_{1k} b_{k1} & \sum_{k=1}^{n} a_{1k} b_{k2} & \cdots & \sum_{k=1}^{n} a_{1k} b_{kp} \\
\sum_{k=1}^{n} a_{2k} b_{k1} & \sum_{k=1}^{n} a_{2k} b_{k2} & \cdots & \sum_{k=1}^{n} a_{2k} b_{kp} \\
\sum_{k=1}^{n} a_{3k} b_{k1} & \sum_{k=1}^{n} a_{3k} b_{k2} & \cdots & \sum_{k=1}^{n} a_{3k} b_{kp} \\
\vdots & \vdots & \ddots & \vdots \\
\sum_{k=1}^{n} a_{mk} b_{k1} & \sum_{k=1}^{n} a_{mk} b_{k2} & \cdots & \sum_{k=1}^{n} a_{mk} b_{kp}
\end{bmatrix} = \sum_{k=1}^{n} a_{1k} b_{k1} + \sum_{k=1}^{n} a_{1k} b_{k2} + \cdots + \sum_{k=1}^{n} a_{1k} b_{kp} = AB + AC
\]

So, \( A(B + C) = AB + AC \) and the proof is complete.

Alternate way to write the same proof: Let \( A \) be a matrix of size \( m \times n \) and then \( B, C \) would have to be of same size \( n \times p \). Write

\[ A = [a_{ik}], \quad B = [b_{kj}], \quad C = [c_{kj}] \]

Then \( B + C = [b_{kj} + c_{kj}] \). So,

\[ A(B + C) = [a_{ik}][b_{kj} + c_{kj}] = [\alpha_{ij}] \text{ (say)} \]

Then the \((ij)\)th entry \( \alpha_{ij} \) of \( A(B + C) \) is given by

\[ \alpha_{ij} = \sum_{k=1}^{n} a_{ik}(b_{kj} + c_{kj}) = \sum_{k=1}^{n} a_{ik}b_{kj} + \sum_{k=1}^{n} a_{ik}c_{kj} \]

But

\[ AB = \left[ \sum_{k=1}^{n} a_{ik}b_{kj} \right], \quad \text{and} \quad AC = \left[ \sum_{k=1}^{n} a_{ik}c_{kj} \right] \]

So, the \((ij)\)th entry of \( AB + AC \) is also equal to \( \alpha_{ij} \). Therefore \( A(B + C) = AB + AC \) and the proof is complete.
**Remark.** For matrices $A, B, C$ as in the theorem so that all the multiplications in the associative law (2) are defined. Then the expression $ABC$ would mean $(AB)C$ or $A(BC)$. It is well defined, because $(AB)C = A(BC)$ by associative property (2) of matrix multiplication.

**Reading assignment**

1. [Textbook, Example 3, p. 64] to 'experience' that associativity (property 2) works.

2. [Textbook, Example 4, p. 64] to see examples that commutativity for multiplication fails (property 1).

3. [Textbook, Example 5, p. 65] to see examples that cancellation fails. That means, examples $AB = AC$ for nonzero $A, B, C$ with $B \neq C$. This makes solving matrix equation like $AX = AC$, different from solving equation in real numbers, like $ax = c$.

**Theorem 2.2.4** Let $I_n$ denote the square matrix of order $n$ whose main diagonal entries are 1 and all the entries are zero. So,

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}, I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Let $A$ be an $m \times n$ matrix. Then

$$I_m A = A \quad \text{and} \quad AI_n = A.$$

Because of these two properties, $I_n$ is called the **identity matrix** of order $n$.

**Proof.** Easy.
2.2. PROPERTIES OF MATRIX OPERATIONS

Reading assignment

1. [Textbook, Example 6, p. 66]
2. [Textbook, Example 7, p. 66], just to get used to computing $A^r$.

Following the definition 1.1.2 in item (4) we gave a classification of system of equation, which we state as a theorem and prove as follows.

**Theorem 2.2.5** Let $Ax = b$ be a linear system of $m$ equations in $n$ unknowns. Then exactly one of the following is true:

1. The system has no solution.
2. The system has exactly one solution.
3. The system has infinitely many solutions.

**Proof.** If the system has no solution or have exactly one solution, then there is nothing to prove. So, assume, it has at least two distinct solutions $x_1, x_2$. So, $Ax_1 = b$ and $Ax_2 = b$. Subtracting the second from the first, we have $A(x_1 - x_2) = 0$. Write $y = x_1 - x_2$.

Then $y \neq 0$ and $Ay = 0$. So, for any scalar $c$, we have

$$A(x_1 + cy) = Ax_1 + A(cy) = b + (Ac)y = b + c(Ay) = b + c0 = b.$$ 

So, $x_1 + cy$ is a solution, for each scalar $c$ (and they are all distinct). So, the system has infinitely many solutions. This completes the proof. ■

2.2.2 Transpose of a matrix

**Definition 2.2.6** The transpose of a matrix $A$ is obtained by writing the rows as columns (and/or writing the columns as rows). The transpose of $A$ is denoted by $A^T$. So, for the matrix
A square matrix $A$ is said to be symmetric, if $A = A^T$.

**Theorem 2.2.8** Let $A, B$ be matrices (of varying sizes so that the sum or product is defined) and $c$ be a scalar. Then,

1. $(A^T)^T = A$. (That means transpose of transpose is the same.)
2. $(A + B)^T = A^T + B^T$.
3. $(cA)^T = cA^T$.
4. $(AB)^T = B^T A^T$. (This is the most important one in this list.)

*In fact, properties 2, 4 extends to sum of higher number of matrices.*

**Reading assignment**

1. [Textbook, Example 8, p. 68] to compute transpose of a matrix.
2. [Textbook, Example 9, p. 66], about transpose and product of matrices.
Exercise 2.2.9 (Ex. 8, p. 70) Let

\[ A = \begin{bmatrix} -2 & -1 \\ 1 & 0 \\ 3 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 3 \\ 2 & 0 \\ -4 & -1 \end{bmatrix}. \]

Then

1. Solve \( X = 3A - 2B \)

   **Solution:** In this case

   \[
   X = 3A - 2B = 3 \begin{bmatrix} -2 & -1 \\ 1 & 0 \\ 3 & -4 \end{bmatrix} - 2 \begin{bmatrix} 0 & 3 \\ 2 & 0 \\ -4 & -1 \end{bmatrix}
   \]

   which is

   \[
   \begin{bmatrix} -6 & -3 \\ 3 & 0 \\ 9 & -12 \end{bmatrix} - \begin{bmatrix} 0 & 6 \\ 4 & 0 \\ -8 & -2 \end{bmatrix} = \begin{bmatrix} -6 & -9 \\ -1 & 0 \\ 17 & -10 \end{bmatrix}.
   \]

2. Solve \( 2X = 2A - B \).

   **Solution:** We have

   \[
   2X = 2A - B = 2 \begin{bmatrix} -2 & -1 \\ 1 & 0 \\ 3 & -4 \end{bmatrix} - \begin{bmatrix} 0 & 3 \\ 2 & 0 \\ -4 & -1 \end{bmatrix} = \begin{bmatrix} -4 & -2 \\ 2 & 0 \\ 6 & -8 \end{bmatrix} - \begin{bmatrix} 0 & 3 \\ 2 & 0 \\ -4 & -1 \end{bmatrix}
   \]

   OR

   \[
   2X = \begin{bmatrix} -4 & -5 \\ 0 & 0 \\ 10 & -7 \end{bmatrix}
   \]

   Divide by 2 (i.e. multiply by \( \frac{1}{2} \)), we have

   \[
   X = \frac{1}{2} \begin{bmatrix} -4 & -5 \\ 0 & 0 \\ 10 & -7 \end{bmatrix} = \begin{bmatrix} -2 & -2.5 \\ 0 & 0 \\ 5 & -3.5 \end{bmatrix}.
   \]
3. Solve $2X + 3A = B$.

**Solution:** We have $2X + 3A = B$. Subtracting $3A$ from both sides, $2X = B - 3A$. So,

$$2X = B - 3A = \begin{bmatrix} 0 & 3 \\ 2 & 0 \\ -4 & -1 \end{bmatrix} - 3 \begin{bmatrix} -2 & -1 \\ 1 & 0 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 6 & 6 \\ -1 & 0 \\ -13 & 11 \end{bmatrix}.$$

Divide by 2 (i.e. multiply by $\frac{1}{2}$), we have

$$X = \frac{1}{2} \begin{bmatrix} 6 & 6 \\ -1 & 0 \\ -13 & 11 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ -0.5 & 0 \\ -6.5 & 5.5 \end{bmatrix}.$$

**Exercise 2.2.10 (Ex. 10-changed, p. 70)** Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 3 \\ -1 \\ 2 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Compute $C(BA)$ and $A(BC)$, if defined.

**Solution:** Note $A(BC)$ is not defined because $A$ has 3 columns and $BC$ has two rows. We compute $C(BA)$. We have $C(BA) = (CB)A$ and

$$CB = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -1 & -3 \end{bmatrix}.$$

So, $C(BA) = (CB)A$

$$= \begin{bmatrix} -1 & 2 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & -5 \\ -1 & -5 & 0 \end{bmatrix}.$$

**Exercise 2.2.11 (Ex. 18, p. 70)** Let

$$A = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 \\ -0.5 & 1 \end{bmatrix}.$$
2.2. PROPERTIES OF MATRIX OPERATIONS

Then $AB = 0$ but $A \neq 0$ nor $B \neq 0$. *(Note, such a phenomenon will not occur with real numbers. That is why with real numbers, $ax = 0, a \neq 0 \Rightarrow x = 0$. We cannot do similar algebra with matrices. With $A \neq 0$ the solution to the equation $AX = 0$ is not necessarily $X = 0$.)

Solution: Obvious.

Exercise 2.2.12 (Ex. 22, p. 70) Let

$$A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}, \quad \text{and} \quad I = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. $$

Compute $A + IA$.

Solution: We have

$$A + IA = A + A = 2A = 2 \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 0 & -2 \end{bmatrix}. $$

Exercise 2.2.13 (Ex. 24. p. 70) Let

$$A = \begin{bmatrix} 1 & -1 \\ 3 & 4 \\ 0 & -2 \end{bmatrix}. $$

Compute $A^T, A^T A$ and $AA^T$.

Solution: We have

$$A^T = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 4 & -2 \end{bmatrix}. $$

So, $A^T A =$

$$\begin{bmatrix} 1 & 3 & 0 \\ -1 & 4 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 3 & 4 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 + 9 + 0 & -1 + 12 + 0 \\ -1 + 12 + 0 & 1 + 16 + 4 \end{bmatrix} = \begin{bmatrix} 10 & 11 \\ 11 & 21 \end{bmatrix}. $$
Also $AA^T =$
\[
\begin{bmatrix}
1 & -1 \\
3 & 4 \\
0 & -2 \\
\end{bmatrix}
\begin{bmatrix}
1 & 3 & 0 \\
-1 & 4 & -2 \\
\end{bmatrix}
= \begin{bmatrix}
1+1 & 3-4 & 0+2 \\
3-4 & 9+16 & 0-8 \\
0+2 & 0-8 & 0+4 \\
\end{bmatrix}
\]
which is
\[
\begin{bmatrix}
2 & -1 & 2 \\
-1 & 25 & -8 \\
2 & -8 & 4 \\
\end{bmatrix}
\]

Exercise and comment: Notice in this exercise 24 above, both $AA^T$ and $A^T A$ are symmetric matrices. This is not a surprise.

In fact, for any matrix $A$, both $AA^T$ and $A^T A$ are symmetric matrices.

Proof.
\[(AA^T)^T = (A^T)^T A^T = AA^T.
\]
So, $AA^T$ is symmetric. Similarly, $A^T A$ is symmetric.

Exercise 2.2.14 (Ex. 38. p. 71) Let
\[
X = \begin{bmatrix}
1 \\
2 \\
3 \\
\end{bmatrix}, Y = \begin{bmatrix}
1 \\
0 \\
2 \\
\end{bmatrix}, Z = \begin{bmatrix}
1 \\
4 \\
\end{bmatrix}, W = \begin{bmatrix}
0 \\
0 \\
1 \\
\end{bmatrix}, O = \begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix}
\]

1. Find scalars $a, b$ such that $Z = aX + bY$.

Solution: This is, in fact, a problem of solving a system of linear equations. Suppose $Z = aX + bY$. Then,
\[
\begin{bmatrix}
X & Y \\
\end{bmatrix}
\begin{bmatrix}
a \\
b \\
\end{bmatrix}
= Z. \quad OR \quad \begin{bmatrix}
1 & 1 \\
2 & 0 \\
3 & 2 \\
\end{bmatrix}
\begin{bmatrix}
a \\
b \\
\end{bmatrix}
= \begin{bmatrix}
1 \\
4 \\
\end{bmatrix}
\]
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Here \(a, b\) are unknown to be solved for. The augmented matrix of this system is:

\[
\begin{bmatrix}
1 & 1 & 1 \\
2 & 0 & 4 \\
3 & 2 & 4 \\
\end{bmatrix}
\]

By TI-84, the matrix reduced to the Gauss-Jordan form:

\[
\begin{bmatrix}
1 & 0 & 2 \\
0 & 1 & -1 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

This gives \(a = 2, b = -1\). (We can check \(2X - Y = Z\).)

2. Show that there do not exist scalar \(a, b\) such that \(W = aX + bY\).

**Solution:** This is, in fact, a problem of solving a system of linear equations. Suppose \(W = aX + bY\). Then,

\[
\begin{bmatrix}
X & Y \\
\end{bmatrix}
\begin{bmatrix}
a \\
b \\
\end{bmatrix}
= W. \text{ OR } \begin{bmatrix}
1 & 1 & 1 \\
2 & 0 & 0 \\
3 & 2 & 1 \\
\end{bmatrix}
\begin{bmatrix}
a \\
b \\
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
1 \\
\end{bmatrix}
\]

Here \(a, b\) are unknown to be solved for. The augmented matrix of this system is:

\[
\begin{bmatrix}
1 & 1 & 0 \\
2 & 0 & 0 \\
3 & 2 & 1 \\
\end{bmatrix}
\]

By TI-84, the matrix reduced to the Gauss-Jordan form:

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]
The system corresponding to this matrix is \( a = 0, b = 0, 0 = 1 \), which is inconsistent.

3. Show that \( aX + bY + cW = O \) then \( a = b = c = 0 \).

**Solution:** This is, again, a problem of solving a system of linear equations. Suppose \( O = aX + bY + cW \). Then,

\[
\begin{bmatrix}
X & Y & W
\end{bmatrix}
\begin{bmatrix}
a \\
b \\
c
\end{bmatrix}
= O
\]

OR

\[
\begin{bmatrix}
a \\
b \\
c
\end{bmatrix}
= \begin{bmatrix} 0 \\
0 \\
0 \end{bmatrix}
\]

The augmented matrix of this system is given by

\[
\begin{bmatrix}
1 & 1 & 0 & 0 \\
2 & 0 & 0 & 0 \\
3 & 2 & 1 & 0
\end{bmatrix}
\]

From TI-83/84, the matrix reduces to the following Gauss-Jordan form:

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]

So, \( a = 0, b = 0, c = 0 \), as was required to show.

4. Find the scalars \( a, b, c \), not all zero (nontrivial solution), such that \( aX + bY + cZ = O \).

**Solution:** This is, again, a problem of solving a system of linear equations. Suppose \( O = aX + bY + cZ \). Then,

\[
\begin{bmatrix}
X & Y & Z
\end{bmatrix}
\begin{bmatrix}
a \\
b \\
c
\end{bmatrix}
= O
\]

OR

\[
\begin{bmatrix}
a \\
b \\
c
\end{bmatrix}
= \begin{bmatrix} 0 \\
0 \\
0 \end{bmatrix}
\]

\[
\begin{bmatrix}
a \\
b \\
c
\end{bmatrix}
= \begin{bmatrix} 0 \\
0 \\
0 \end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 1 & 1 \\
2 & 0 & 4 \\
3 & 2 & 4
\end{bmatrix}
\begin{bmatrix}
a \\
b \\
c
\end{bmatrix}
= \begin{bmatrix} 0 \\
0 \\
0 \end{bmatrix}
\]
The augmented matrix of this system is given by
\[
\begin{bmatrix}
1 & 1 & 1 & 0 \\
2 & 0 & 4 & 0 \\
3 & 2 & 4 & 0
\end{bmatrix}
\]
From TI-83/84, the matrix reduces to the following Gauss-Jordan form:
\[
\begin{bmatrix}
1 & 0 & 2 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
The linear system corresponding to this matrix is
\[
\begin{align*}
a + 2c &= 0 \\
b - c &= 0 \\
0 &= 0
\end{align*}
\]
Using \(c = t\) as parameter, we have \(a = -2t, b = t, c = t\). So, for any scalar \(t\), we have \(-2tx + ty + tz = O\).

**Exercise 2.2.15 (Ex. 44, p. 71)** Let
\[
f(x) = x^2 - 7x + 6 \quad \text{and} \quad A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}.
\]
Compute \(f(A)\).

**Solution:** We have
\[
A^2 = AA = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 29 & 28 \\ 7 & 8 \end{bmatrix}.
\]
So, \(f(A) = A^2 - 7A + 6I_2 =
\[
\begin{bmatrix} 29 & 28 \\ 7 & 8 \end{bmatrix} - 7 \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} + 6 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]
\[
\begin{bmatrix}
29 & 28 \\
7 & 8
\end{bmatrix}
- 
\begin{bmatrix}
35 & 28 \\
7 & 14
\end{bmatrix}
+ 
\begin{bmatrix}
6 & 0 \\
0 & 6
\end{bmatrix}
= 
\begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}.
\]

So, \(f(A) = O\). (\textit{One can say that }A\textit{ is a (matrix) root of }f(x).\)
2.3. The Inverse of a matrix

Homework: [Textbook, Ex. 9, 13, 25, 27, 33, 35, 37, 41, 45; page 84-]

Main point in the section is to define and compute inverse of matrices:

1. We give a formula to compute inverse of a $2 \times 2$ matrices.

2. We describe the method of row reduction to compute inverse of a matrix of any size.

3. We use inverse of matrices to solve system of linear equations.
Definition 2.3.1 For a square matrix $A$ of size $n \times n$, we say that $A$ is invertible (or nonsingular) if there is an $n \times n$ matrix $B$ such that

$$AB = BA = I_n$$

where $I_n$ is the identity matrix of order $n$. The matrix $B$ is called the (multiplicative) inverse of $A$. If a matrix does not have an inverse, it is called a noninvertible (or singular) matrix.

A non-square matrix does have an inverse. Because if $A$ is an $m \times n$ matrix with $m \neq n$, then for $AB$ and $BA$ to be defined, the size of $B$ has to be $n \times m$. The size of $AB$ would be $m \times m$ and size of $BA$ would be $n \times n$. So, $AB \neq BA$.

Theorem 2.3.2 If $A$ is invertible, then the inverse is unique. The inverse is denoted by $A^{-1}$.

Proof. Suppose $B$ and $C$ are two inverses of $A$. We need to prove $B = C$. We have

$$AB = BA = I \quad \text{and} \quad AC = CA = I.$$ 

So,

$$B = BI = B(AC) = (BA)C = IC = C.$$ 

So, the proof is complete.

Remark. (1) Note that the proof works if $C$ was only a right-inverse and $B$ was a left-inverse of $A$. (2) Also note that the proof did not use any property of matrices, except associativity.

2.3.1 Finding Inverse of a matrix

There are many methods of finding inverse of a matrix. We will discuss at least two of them. The easy one, for a $2 \times 2$ matrices is given by the theorem:
Theorem 2.3.3 Suppose

\[ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \]

is a non-zero 2 \times 2 matrix. Then

1. If \( ad - bc = 0 \), then \( A \) has no inverse (i.e. \( A \) is a singular matrix).

2. If \( ad - bc \neq 0 \), then

\[ A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \]

Proof. Write

\[ B = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \]

First,

\[ AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} = (ad - bc)I_2. \]

1. (Case 1:) Assume \( ad - bc = 0 \). Then, we have \( AB = O \). So, \( A \) cannot have an inverse (otherwise we will get \( B = A^{-1}(AB) = O \), which is not the case).

2. (Case 2:) Assume \( ad - bc \neq 0 \). We need to prove \( A(\frac{1}{ad - bc} B) = I_2 = (\frac{1}{ad - bc} B)A \). Multiply the above equation by \( \frac{1}{ad - bc} \), we get

\[ A \left( \frac{1}{ad - bc} B \right) = I_2. \]

Similarly, \( (\frac{1}{ad - bc} B) A = I_2 \). So, the proof is complete. \( \square \)

Exercise 2.3.4 (Ex. 6, p.84) Find the inverse of the matrix

\[ A = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}, \text{ if exists.} \]
Solution: We have \(ad - bc = 1(-3) - (-2)2 = 1\). So, by theorem 2.3.3,

\[
A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}
\]

Reading assignment: Read [Textbook, Example 1 and 2, p. 74-75] and [Textbook, Example 5, p. 79].

2.3.2 2nd method of finding inverse of a matrix

The second method of finding the inverse evolves out of the spirit of solving equation. Suppose \(A\) is an \(n \times n\) matrix and it has an inverse \(X\), where \(X\) is an \(n \times n\) matrix. To find \(X\), we have to solve the equation \(AX = I_n\). The following method is suggested to solve this equation:

1. Write the augmented matrix of this Equation \(AX = I_n\) as 

\[
[A \mid I_n]
\]

2. If possible, row reduce \(A\) to the identity matrix \(I\) using elementary row operations on the entire augmented matrix \([A \mid I]\). If the result is \([I \mid C]\), then \(A^{-1} = C\). If this is not possible, the matrix is not invertible (or singular).

3. Check your work by multiplying \(AA^{-1}\) and \(A^{-1}A\) to see \(AA^{-1} = A^{-1}A = I\).

Justification or Proof. Suppose \(B\) is an \(m \times n\) matrix. Then, each elementary row operation on a matrix \(B\) is same as left-multiplication of \(B\) by a matrix:

1. For example, multiplying the \(i^{th}\)—row \(B\) by a constant is same as left-multiplication of \(B\) by the diagonal matrix of order \(n\), whose \((i, i)\)—entry is \(c\) and other diagonal entries are 1 (and rest of the entries are zero).
2.3. THE INVERSE OF A MATRIX

2. Interchanging of (for example) first and second row is left-multiplication of $B$ by the (partitioned) matrix:

$$S = \begin{bmatrix} 0 & 1 & O \\ 1 & 0 & O \\ O & I_{m-2} \end{bmatrix}.$$ 

We use the notation $O$ to denote the zero matrix of appropriate size. So, we mean $SB$ is the matrix obtained by switching first and second row of $B$.

3. Adding $c$-times first row to the second is same as left-multiplication of $B$ by the (partitioned) matrix:

$$E = \begin{bmatrix} 1 & 0 \\ c & 1 & O \\ O & I_{m-2} \end{bmatrix}.$$ 

So, we mean $EB$ is the matrix obtained from $B$ by adding $c$-times the first row to second. We can write down a similar fact for adding $c$-times the $i^{th}$ row to $j^{th}$ row.

Our method says, there is sequence of matrices $E_1, E_2, \ldots, E_r$ such that the left-multiplication(s) gives

$$(E_1E_2 \cdots E_r)[A \mid I] = [I \mid C].$$

With $E = E_1E_2 \cdots E_r$, we have

$$E[A \mid I] = [I \mid C] \implies [EA \mid E] = [I \mid C].$$

This means, $EA = I$ and $E = C$. So, $E = C$ is the (left) inverse of $A$. This completes the justification/proof.

**Reading assignment:** Read [Textbook, Example 3 and 4, p. 76-75].

**Exercise 2.3.5 (Ex. 10, p. 84)** Compute the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix}.$$
Solution: Augment this matrix with the identity matrix $I_3$ and we have

$$
\begin{bmatrix}
1 & 2 & 3 & 1 & 0 & 0 \\
3 & 7 & 9 & 0 & 1 & 0 \\
-1 & -4 & -7 & 0 & 0 & 1 \\
\end{bmatrix}.
$$

According to the above method, we will try to reduce first $3 \times 3$ part to $I_3$, using row operations. Subtract 3 times first row from second and add first row to third:

$$
\begin{bmatrix}
1 & 2 & 3 & 1 & 0 & 0 \\
0 & 1 & 0 & -3 & 1 & 0 \\
0 & -2 & -4 & 1 & 0 & 1 \\
\end{bmatrix}.
$$

Now subtract 2 times second row from first and add 2 times second row to third:

$$
\begin{bmatrix}
1 & 0 & 3 & 7 & -2 & 0 \\
0 & 1 & 0 & -3 & 1 & 0 \\
0 & 0 & -4 & -5 & 2 & 1 \\
\end{bmatrix}.
$$

Divide third row by $-4$, we get

$$
\begin{bmatrix}
1 & 0 & 3 & 7 & -2 & 0 \\
0 & 1 & 0 & -3 & 1 & 0 \\
0 & 0 & 1 & 1.25 & -5 & -25 \\
\end{bmatrix}.
$$

Subtract 3 times third row from first:

$$
\begin{bmatrix}
1 & 0 & 0 & 3.25 & -5 & .75 \\
0 & 1 & 0 & -3 & 1 & 0 \\
0 & 0 & 1 & 1.25 & -5 & -25 \\
\end{bmatrix}.
$$

So,

$$
A^{-1} = \begin{bmatrix}
3.25 & -5 & .75 \\
-3 & 1 & 0 \\
1.25 & -5 & -25 \\
\end{bmatrix}.
$$
Exercise 2.3.6 (Ex. 12, p. 84) Compute the inverse of the matrix

\[
A = \begin{bmatrix}
10 & 5 & -7 \\
-5 & 1 & 4 \\
3 & 2 & -2
\end{bmatrix}.
\]

Solution: Augment this matrix with the identity matrix \( I_3 \) and we have

\[
\begin{bmatrix}
10 & 5 & -7 & 1 & 0 & 0 \\
-5 & 1 & 4 & 0 & 1 & 0 \\
3 & 2 & -2 & 0 & 0 & 1
\end{bmatrix}.
\]

According to the above method, we will try to reduce the left 3\( \times \)3 part to \( I_3 \), using row operations. Divide first row by 10:

\[
\begin{bmatrix}
1 & 0.5 & -0.7 & 0.1 & 0 & 0 \\
-5 & 1 & 4 & 0 & 1 & 0 \\
3 & 2 & -2 & 0 & 0 & 1
\end{bmatrix}.
\]

Add 5 times first row to second and subtract 3 times first row from third:

\[
\begin{bmatrix}
1 & 0.5 & -0.7 & 0.1 & 0 & 0 \\
0 & 3.5 & 0.5 & 0.5 & 1 & 0 \\
0 & 0.5 & 1 & -0.3 & 0 & 1
\end{bmatrix}.
\]

Divide second row by 3.5:

\[
\begin{bmatrix}
1 & 0.5 & -0.7 & 0.1 & 0 & 0 \\
0 & 1 & 0.1 & 0.1 & 0.2 & 0 \\
0 & 0.5 & 1 & -0.3 & 0 & 1
\end{bmatrix}.
\]

Subtract 0.5 times second row from both the first and third:

\[
\begin{bmatrix}
1 & 0 & -\frac{54}{70} & \frac{1}{35} & -\frac{1}{7} & 0 \\
0 & 1 & \frac{1}{7} & \frac{1}{7} & \frac{2}{7} & 0 \\
0 & 0 & \frac{1}{35} & -\frac{13}{35} & -\frac{1}{7} & 1
\end{bmatrix}.
\]
Multiply last row by 35:

$$
\begin{bmatrix}
1 & 0 & -\frac{54}{70} & \frac{1}{35} & -\frac{1}{7} & 0 \\
0 & 1 & \frac{1}{7} & \frac{1}{7} & \frac{2}{7} & 0 \\
0 & 0 & 1 & -13 & -5 & 35
\end{bmatrix}.
$$

Subtract $\frac{1}{7}$ times third row from second and add $\frac{54}{70}$ times third row from the first:

$$
\begin{bmatrix}
1 & 0 & 0 & -10 & -4 & 27 \\
0 & 1 & 0 & 2 & 1 & -5 \\
0 & 0 & 1 & -13 & -5 & 35
\end{bmatrix}.
$$

So,

$$
A^{-1} = \begin{bmatrix}
-10 & -4 & 27 \\
2 & 1 & -5 \\
-13 & -5 & 35
\end{bmatrix}.
$$

### 2.3.3 Properties of Inverses

Following is a list properties of inverses:

**Theorem 2.3.7** Suppose $A$ is an invertible matrix and $c \neq 0$ is a scalar. Also, let $k$ be a positive integer. Then

1. $(A^{-1})^{-1} = A$.
2. $(A^k)^{-1} = (A^{-1})^k$.
3. $(cA)^{-1} = \frac{1}{c}A^{-1}$.
4. $(A^T)^{-1} = (A^{-1})^T$. 
2.3. \textsc{The Inverse of a Matrix} \hfill 75

\textbf{Proof.} From definitions, \( B \) is an inverse of \( A \), if
\[ AB = BA = I. \]
So, (1) is obvious. To prove (2), we need to show \((A^k)(A^{-1})^k = (A^{-1})^k A^k = I\). For \( k = 2 \), we need to prove \((A^2)(A^{-1})^2 = (A^{-1})^2 A^2 = I\). But
\[(A^2)(A^{-1})^2) = (AA)(A^{-1}A^{-1}) = A((AA^{-1})A^{-1}) = A(IA^{-1}) = AA^{-1} = I\]
Similarly, \((A^{-1})^2 A^2 = I\). For any integer \( k > 2 \), we prove it similarly or we can prove it by \textbf{method of induction}: To do this we assume that \((A^{-1})^k \) is the inverse of \( A^k \) and use it to prove that \((A^{-1})^{k+1} \) is the inverse of \( A^{k+1} \), as follows:
\[(A^{k+1})(A^{-1})^{k+1} = A^k(AA^{-1})(A^{-1})^k = A^k I(A^{-1})^k = A^k(A^{-1})^k = I\]
So, (2) is proved.

Also
\[ (cA) \left( \frac{1}{c} A^{-1} \right) = (c\frac{1}{c})(AA^{-1}) = 1I = I. \]
Similarly, \((A^{-1})A^{-1} c = I\). So, (3) is proved. Finally,
\[ A^T (A^{-1})^T = (A^{-1}A)^T = I^T = I. \]
Similarly, \((A^{-1})^T A^T = I\). So, all the proofs are complete. \( \blacksquare \)

\textbf{Reading Assignment:} Read [Textbook, Example 6, 7 p. 80-81]

\textbf{Theorem 2.3.8} Let \( A, B \) be two invertible matrices of order \( n \). Then \( AB \) is invertible and
\[ (AB)^{-1} = B^{-1} A^{-1}. \]

\textbf{Proof.} We need to prove,
\[ (AB)(B^{-1}A^{-1}) = I = (B^{-1}A^{-1})(AB). \]
But
\[ (AB)(B^{-1}A^{-1}) = A ((BB^{-1}) A^{-1}) = A (IA^{-1}) = AA^{-1} = I. \]
Similarly, we prove \((B^{-1}A^{-1})(AB) = I\). The proof is complete. \( \blacksquare \)
Theorem 2.3.9 (Cancellation Property) Let $A, B$ be two invertible matrices of order $n$ and $C$ be an invertible matrix of the same order. Then

$$AC = BC \implies A = B \quad \text{and} \quad CA = CB \implies A = B.$$ 

**Proof.** Suppose $AC = BC$. Multiply this equation by $C^{-1}$ from the right side (we can do this because it is given that $C$ has an inverse), we get

$$(AC)C^{-1} = (BC)C^{-1}. \quad \text{So} \quad A(CC^{-1}) = B(CC^{-1}) \quad \text{So} \quad AI = BI.$$ 

So $A = B$. Similarly, we prove the other one. The proof is complete $\blacksquare$

Theorem 2.3.10 Let $A$ be an invertible matrices of order $n$. Then the system of linear equations $Ax = b$ has a unique solution given by

$$x = A^{-1}b.$$ 

**Proof.** (Proof is exactly same as that of theorem 2.3.9) Suppose $Ax = b$. Multiply this equation by $A^{-1}$, and we get $(A^{-1}A)x = A^{-1}b$. Therefore $Ix = A^{-1}b$. Hence $x = A^{-1}b$. The proof is complete $\blacksquare$

Exercise 2.3.11 (Ex. 26, p. 85) Solve the following three linear systems.

1. Solve the linear system of equations:

$$
2x - y = -3 \\
2x + y = 7
$$

**Solution:** With

$$A = \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix} \quad x = \begin{bmatrix} x \\ y \end{bmatrix} \quad b = \begin{bmatrix} -3 \\ 7 \end{bmatrix}$$
the system can be written as \( A\mathbf{x} = \mathbf{b} \).

By theorem 2.3.3,

\[
A^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix}.
\]

The solution is

\[
\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} -3 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}.
\]

2. Solve the linear system of equations:

\[
\begin{align*}
2x - y &= -1 \\
2x + y &= -3
\end{align*}
\]

**Solution:** The system can be written as \( A\mathbf{x} = \mathbf{b} \) where \( A, \mathbf{x} \) are same as in (1) and

\[
\mathbf{b} = \begin{bmatrix} -1 \\ -3 \end{bmatrix}. \quad We\ already\ computed\ \ A^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix}.
\]

The solution is

\[
\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ -3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}.
\]

3. Solve the linear system of equations:

\[
\begin{align*}
2x - y &= 6 \\
2x + y &= 10
\end{align*}
\]

**Solution:** The system can be written as \( A\mathbf{x} = \mathbf{b} \) where \( A, \mathbf{x} \) are same as in (1) and

\[
\mathbf{b} = \begin{bmatrix} 6 \\ 10 \end{bmatrix}. \quad We\ already\ computed\ \ A^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix}.
\]
The solution is
\[
x = A^{-1}b = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 \\ -2 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 10 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.
\]

Exercise 2.3.12 (Ex. 28, p. 85) Solve the following two linear systems.

1. Solve the linear system of equations:
   \[
   \begin{align*}
   x_1 + x_2 - 3x_3 &= 0 \\
   x_1 - 2x_2 + x_3 &= 0 \\
   x_1 - x_2 - x_3 &= -1
   \end{align*}
   \]

Solution: With
\[
A = \begin{bmatrix} 1 & 1 & -3 \\ 1 & -2 & 1 \\ 1 & -1 & -1 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}
\]
the system can be written as \(Ax = b\). We need to find the inverse of \(A\). To do this augment the identity matrix \(I_3\), to \(A\) and we get
\[
\begin{bmatrix} 1 & 1 & -3 & 1 & 0 & 0 \\ 1 & -2 & 1 & 1 & 0 & 0 \\ 1 & -1 & -1 & 0 & 0 & 1 \end{bmatrix}
\]

Subtract first row from second and third:
\[
\begin{bmatrix} 1 & 1 & -3 & 1 & 0 & 0 \\ 0 & -3 & 4 & -1 & 1 & 0 \\ 0 & -2 & 2 & -1 & 0 & 1 \end{bmatrix}
\]

Divide second row by \(-3\), we get
\[
\begin{bmatrix} 1 & 1 & -3 & 1 & 0 & 0 \\ 0 & 1 & \frac{4}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & -2 & 2 & -1 & 0 & 1 \end{bmatrix}
\]
Subtract second row from first and add 2 times second to third row:

\[
\begin{bmatrix}
1 & 0 & -\frac{5}{3} & \frac{2}{3} & \frac{1}{3} & 0 \\
0 & 1 & -\frac{4}{3} & \frac{1}{3} & -\frac{1}{3} & 0 \\
0 & 0 & -\frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} & 1
\end{bmatrix}
\]

Multiply third row by $-\frac{3}{2}$, we get

\[
\begin{bmatrix}
1 & 0 & -\frac{5}{4} & \frac{2}{3} & \frac{1}{3} & 0 \\
0 & 1 & -\frac{4}{3} & \frac{1}{3} & -\frac{1}{3} & 0 \\
0 & 0 & 1 & \frac{1}{2} & 1 & -\frac{3}{2}
\end{bmatrix}
\]

Add $\frac{5}{3}$ times third row to first and add $\frac{4}{3}$ times third row to second:

\[
\begin{bmatrix}
1 & 0 & 0 & \frac{3}{2} & 2 & -\frac{5}{2} \\
0 & 1 & 0 & 1 & 1 & -2 \\
0 & 0 & 1 & \frac{1}{2} & 1 & -\frac{3}{2}
\end{bmatrix}
\]

By theorem 2.3.3,

\[
A^{-1} = \begin{bmatrix}
\frac{3}{2} & 2 & -\frac{5}{2} \\
1 & 1 & -2 \\
\frac{1}{2} & 1 & -\frac{3}{2}
\end{bmatrix}
\]

The solution is

\[
x = A^{-1}b = \begin{bmatrix}
\frac{3}{2} & 2 & -\frac{5}{2} \\
1 & 1 & -2 \\
\frac{1}{2} & 1 & -\frac{3}{2}
\end{bmatrix} \begin{bmatrix} 0 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \end{bmatrix}
\]

2. Solve the linear system of equations:

\[
\begin{align*}
x_1 + x_2 - 3x_3 &= -1 \\
x_1 - 2x_2 + x_3 &= 2 \\
x_1 - x_2 - x_3 &= 0
\end{align*}
\]
CHAPTER 2. MATRICES

Solution: With $A$ and $x$ as in (1) and with

$$b = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

the system can be written as $Ax = b$. We already computed $A^{-1}$. So, the solution is

$$x = A^{-1}b = \begin{bmatrix} \frac{3}{2} & 2 & -\frac{5}{2} \\ 1 & 1 & -2 \\ \frac{1}{2} & 1 & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 1 \\ 1.5 \end{bmatrix}.$$ 

Exercise 2.3.13 (Ex. 34, p. 85) Suppose $A, B$ are two $2 \times 2$ matrices and

$$A^{-1} = \begin{bmatrix} -\frac{2}{7} & \frac{1}{7} \\ \frac{3}{7} & \frac{2}{7} \end{bmatrix} \text{ and } B^{-1} = \begin{bmatrix} \frac{5}{11} & \frac{2}{11} \\ \frac{3}{11} & -\frac{1}{11} \end{bmatrix}.$$ 

1. Compute $(AB)^{-1}$.

   Solution: By theorem 2.3.8, we have

   $$(AB)^{-1} = B^{-1}A^{-1} = \begin{bmatrix} \frac{5}{11} & \frac{2}{11} \\ \frac{3}{11} & -\frac{1}{11} \end{bmatrix} \begin{bmatrix} -\frac{2}{7} & \frac{1}{7} \\ \frac{3}{7} & \frac{2}{7} \end{bmatrix}$$

   $$= \frac{1}{11} \begin{bmatrix} 5 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} -4 & 9 \\ -9 & 1 \end{bmatrix}.$$ 

2. Compute $(A^T)^{-1}$.

   Solution: By theorem 2.3.7, we have

   $$(A^T)^{-1} = (A^{-1})^T = \left( \begin{bmatrix} -\frac{2}{7} & \frac{1}{7} \\ \frac{3}{7} & \frac{2}{7} \end{bmatrix} \right)^T = \begin{bmatrix} -\frac{2}{7} & \frac{3}{7} \\ \frac{1}{7} & \frac{2}{7} \end{bmatrix}.$$
3. Compute $A^{-2}$. (In page 80, for a positive integer $k$ it was defined $A^{-k} := (A^{-1})^k$.)

We have $A^{-2} = A^{-1}A^{-1} =
\begin{bmatrix}
-\frac{2}{7} & \frac{1}{7} \\
\frac{3}{7} & \frac{2}{7}
\end{bmatrix}
\begin{bmatrix}
-\frac{2}{7} & \frac{1}{7} \\
\frac{3}{7} & \frac{2}{7}
\end{bmatrix}
= \frac{1}{49}
\begin{bmatrix}
-2 & 1 \\
3 & 2
\end{bmatrix}
\begin{bmatrix}
-2 & 1 \\
3 & 2
\end{bmatrix}
= \frac{1}{49}
\begin{bmatrix}
-1 & 0 \\
0 & 5
\end{bmatrix}.

4. Compute $(2A)^{-1}$.

Solution: By theorem 2.3.7, we have

$$(2A)^{-1} = \frac{1}{2} A^{-1} = \frac{1}{2}
\begin{bmatrix}
-\frac{2}{7} & \frac{1}{7} \\
\frac{3}{7} & \frac{2}{7}
\end{bmatrix}
= \begin{bmatrix}
-\frac{1}{7} & \frac{1}{14} \\
\frac{3}{14} & \frac{2}{14}
\end{bmatrix}.$$

Exercise 2.3.14 (Ex. 38, p. 85) Let

$$A = \begin{bmatrix}
2 & x \\
-1 & -2
\end{bmatrix}.$$

Find $x$, so that $A$ is its own inverse.

Solution: If $A$ is its own inverse then $A^2 = I_2$. So, we have

$$\begin{bmatrix}
2 & x \\
-1 & -2
\end{bmatrix}
\begin{bmatrix}
2 & x \\
-1 & -2
\end{bmatrix}
= \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}.$$

Multiplying, we have

$$\begin{bmatrix}
4 - x & 0 \\
0 & 4 - x
\end{bmatrix}
= \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}.$$

so $4 - x = 1$; so $x = 3$.

2.4 Elementary Matrices

We skip this section. I included some of it in subsection 2.3.2.
2.5 Application of matrix operations

Homework: [Textbook, §2.5 Ex. 15, 17, 19, 21, 23; p. 113].

Main point in this section is to do some applications of matrices.

1. We discuss stochastic matrices. But we will not go deeper into it.

2. We discuss application of matrices in Cryptography.
2.5. APPLICATION OF MATRIX OPERATIONS

2.5.1 Stochastic matrices

**Definition 2.5.1** A square matrix

\[
P = \begin{bmatrix}
p_{11} & p_{12} & p_{13} & \cdots & p_{1n} \\
p_{21} & p_{22} & p_{23} & \cdots & p_{2n} \\
p_{31} & p_{32} & p_{33} & \cdots & p_{3n} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
p_{n1} & p_{n2} & p_{n3} & \cdots & p_{nn}
\end{bmatrix}
\]
of size $n \times n$

is called a **stochastic** matrix if

1. we have $0 \leq p_{ij} \leq 1$ and

2. sum of all the entries in a column is 1. For example,

\[p_{11} + p_{21} + p_{31} + \cdots + p_{n1} = 1.\]

**Remark.** We will not emphasize on this topic of stochastic matrices very much in this course. We encounter such matrices in probability theory and statistics.

**Reading Assignment:** Read [Textbook, Example 1,2 p. 99-100] and the discussion preceding this.

2.5.2 Cryptography

A **cryptogram** is a message written according to a secret code. We describle a method of using matrix multiplication to **encode** and **decode**.
First, we assign a number to each letter in the letter as follows:

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<tr>
<td>0</td>
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<td>26</td>
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</tr>
</tbody>
</table>

Example. Write uncoded matrices of size $1 \times 3$ for the message LINEAR ALGEBRA.

Solution:

\[
\begin{bmatrix}
L & I & N \\
E & A & R \\
A & L & G \\
E & B & R \\
A &   &   
\end{bmatrix}
\]

\[
\begin{bmatrix}
12 & 9 & 14 \\
5 & 1 & 18 \\
0 & 1 & 12 \\
7 & 5 & 2 \\
18 & 1 & 0 
\end{bmatrix}
\]

2.5.3 Encoding

Following is the method of encoding and decoding a message:

1. Given a message, first it is written as sequence of row matrices of a fixed size. This was done in the above example, by writing the message in a sequence of uncoded matrices of size $1 \times 3$. These will be called uncoded matrices.

2. The message is encoded by using a square matrix $A$ of appropriate size and simply multiplying the uncoded matrices by $A$.

3. Decoding of the encoded matrices is done by multiplying the encoded matrices by the inverse $A^{-1}$ of $A$.

Exercise 2.5.2 (Ex. 16, p.113) Use the ending matrix

\[
A = \begin{bmatrix}
14 & 2 & 1 \\
-3 & -3 & -1 \\
3 & 2 & 1 
\end{bmatrix}
\]
to encode the message: PLEASE SEND MONEY

**Solution:** We first write the message in uncoded row matrices of size $1 \times 3$ as follows:

$$
\begin{array}{cccccc}
P & L & E & A & S & E \\
16 & 12 & 5 & 1 & 19 & 5 \\
S & E & N & D \\
0 & 19 & 5 & 14 & 4 & 0 \\
M & O & N & E & Y \\
13 & 15 & 14 & 5 & 25 & 0
\end{array}
$$

So, the encoded matrices are given by:

$$
\begin{array}{c|c}
\text{uncoded} \times A & \text{encoded} \\
\hline
16 & 12 & 5 & A & = & 203 & 6 & 9 \\
1 & 19 & 5 & A & = & -28 & -45 & -13 \\
0 & 19 & 5 & A & = & -42 & -47 & -14 \\
14 & 4 & 0 & A & = & 184 & 16 & 10 \\
13 & 15 & 14 & A & = & 179 & 9 & 12 \\
5 & 25 & 0 & A & = & -5 & -65 & -20 \\
\end{array}
$$

**Exercise 2.5.3 (Ex. 22. p. 113)** Let

$$
A = \begin{bmatrix} 3 & -4 & 2 \\ 0 & 2 & 1 \\ 4 & -5 & 3 \end{bmatrix}
$$

be the encoding matrix. Decode the cryptogram:

$$
$$

**Solution:** We use TI to find:

$$
A^{-1} = \begin{bmatrix} 11 & 2 & -8 \\ 4 & 1 & -3 \\ -8 & -1 & 6 \end{bmatrix}.
$$
So, the uncoded matrices (and corresponding letters) are given by:

<table>
<thead>
<tr>
<th>encoded $\times A^{-1}$</th>
<th>unencoded</th>
<th>word</th>
</tr>
</thead>
<tbody>
<tr>
<td>112 -140 83 $A^{-1}$</td>
<td>8 1 22</td>
<td>HAV</td>
</tr>
<tr>
<td>19 -25 13 $A^{-1}$</td>
<td>5 0 1</td>
<td>E A</td>
</tr>
<tr>
<td>72 -76 61 $A^{-1}$</td>
<td>0 7 18</td>
<td>GR</td>
</tr>
<tr>
<td>95 -118 71 $A^{-1}$</td>
<td>5 1 20</td>
<td>EAT</td>
</tr>
<tr>
<td>20 21 38 $A^{-1}$</td>
<td>0 23 5</td>
<td>WE</td>
</tr>
<tr>
<td>35 -23 36 $A^{-1}$</td>
<td>5 11 5</td>
<td>EKE</td>
</tr>
<tr>
<td>42 -48 32 $A^{-1}$</td>
<td>14 4 0</td>
<td>ND</td>
</tr>
</tbody>
</table>

So the message is HAVE A GREAT WEEKEND.