Chern Classes by Induction - Last Lecture

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Suppose X is an algebraic Scheme over a field with $\dim X = n$ and $\mathcal{E}$ be a locally free sheaf of rank r. We will try to define all the chern classes of $\mathcal{E}$.

Notation 0.1

1. *$A^r(X)$ will denote the Chow group of codimension r cycles and*
2. *$A_r(X)$ will denote the Chow group of dimension r cycles.*
3. $$A(X) = \oplus_{i=0}^{n} A^r(X) = \oplus_{i=0}^{\infty} A^r(X)$$
   will denote the total chow group.

1 Nonsingular Case

1. We will assume X is smooth and hence the total Chow group
   $$A(X) = \oplus_{k=0}^{\dim X} A^k(X)$$
   is a GRADED ring.
2. **We want the Chern classes to have the following properties:**
   (a) the $k-th$ Chern class $C^k(\mathcal{E}) \in A^k(X)$.
   (b) So, $C^k(\mathcal{E}) = 0$ for $k > dimX$.
   (c) Also $C^0(\mathcal{E}) = 1$.
   (d) The total Chern class of $\mathcal{E}$ will be denoted by $C(\mathcal{E}) = 1 + C^1(\mathcal{E}) + C^2(\mathcal{E}) + \cdots$

(e) So, the total Chern class is an UNIT in $A(X)$.

(f) Given any exact sequence

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$$

We must have the

$$C(\mathcal{E}) = C(\mathcal{E}')C(\mathcal{E}'').$$

(g) For a rank one locally free sheaf $\mathcal{L}$ on X we have defined the first chern class $C^1(\mathcal{L})$, as in the book of Fulton.

In fact, if $\mathcal{L}$ is isomorphic to the invertible sheaf of ideals $\mathcal{I}$ (which is not alaways the case) then

$$C^1(\mathcal{L}) = cycle(\frac{\mathcal{O}_X}{\mathcal{I}}),$$

The total Chern class of $\mathcal{L}$ is

$$C(\mathcal{L}) = 1 + C^1(\mathcal{L}).$$

(h) **(Pullback)** Let us write $X_1 = Proj(Symm(\mathcal{E}))$ and $p : X_1 \to X$ be the projection map. Then the pullback must commute with chern classes. That means

$$C(p^*\mathcal{E}) = p^*(C(\mathcal{E})) \qquad OR \qquad C^k(p^*\mathcal{E}) = p^*(C^k(\mathcal{E}))$$

(i) Let $p : X_1 \to X$ be as above. Then the pullback map

$$A(X) \to A(X_1)$$

is an injective map of GRADED rings. (See page 15 of Mohan Kumar's Note.)

3. **(Splitting Principle)** Let $X_1 = Proj(Symm(\mathcal{E}))$ and $p : X_1 \to X$ be the projection map. Then there is an exact sequence

$$0 \to \mathcal{E}' \to p^*\mathcal{E} \to \mathcal{O}(1) \to 0$$

where the kernel $\mathcal{E}'$ is, clearly, a locally free sheaf of rank $r - 1$.

4. **Inductive Definition:** Use the injectivity of

$$A(X) \to A(X_1)$$

and define the total Chern class

$$C(\mathcal{E}) = C(p^*(\mathcal{E})) = C(\mathcal{E}')C(\mathcal{O}(1)) = C(\mathcal{E}')(1+\eta).$$

Here

$$\eta = C^1(\mathcal{O}(1)) = Cycle(\mathcal{O}/\mathcal{I})$$

where

$$\mathcal{I} = (\oplus_{i>0} Symm^i(\mathcal{E}))^{\sim}.$$

5. **(Exercise)** Let $\mathcal{F}$ be a FREE sheaf of rank r over X. Prove that the total Chern class

$$C(\mathcal{F}) = 1.$$

6. **(Exercise)** Let $\mathcal{E}$ be a locally free sheaf of rank r. Prove that $C^k(\mathcal{E}) = 0$ for all $k > r$.

7. **(Exercise)** Let $\mathcal{E}$ be a locally free sheaf of rank r over X. It needs a proof that $C^k(\mathcal{E}) \in A^k(X)$.

2 First and the Top Chern Class

As above, suppose X is an algebraic Scheme over a field with $\dim X = n$ and $\mathcal{E}$ be a locally free sheaf of rank r.

The r^{th} Chern class $C^r(\mathcal{E})$ of $\mathcal{E}$ will be called the TOP Chern class of $\mathcal{E}$.

1. Description of the first Chern class is given by

$$C^1(\mathcal{E}) = C^1(det(\mathcal{E})).$$

For the right hand side, we have to look at an invertible subsheaf of $K(X)$ that is isomorphic of $det(\mathcal{E})$ OR the Cartier divisor corresponding to $det(\mathcal{E})$.

2. For simplicity, assume that $X = Spec(A)$ and $\dim A = n$. Now let P be a projective $A-$module of rank r.

   To describe the top Chern class of P we do the following: Let

   $$\lambda : P \to I \subseteq A$$

   be surjective linear map, where I is a locally complete intersection ideal of height r. (Such maps and ideals exist.) The

   $$C^r(P) = (-1)^r Cycle(A/I) \qquad AND \qquad C^r(P^*) = Cycle(A/I)$$

   where $P^* = Hom(P, A)$.

3. Same can be done for non-affine schemes. Let $\mathcal{E}$ be a locally free sheaf on a scheme X. Let $s \in \Gamma(\mathcal{E}, X)$ be a global section, such that $Y = \{x \in X : s(x) = 0\}$ is a locally complete intersection subscheme, of codimension r, of X. (Such sections may not exist.) Then the top Chern class of $\mathcal{E}$ is given by

   $$C^r(\mathcal{E}) = cycle(Y).$$

3 The Singular Case

Now we assume that X is not necessarily nonsingular.

1. So, the total Chow group $A(X) = \oplus A^r(X)$ does not have a ring structure.

2. **(Definition)**A group homomorphism $\varphi : A(X) \to A(X)$ is said to be a graded homomorphism of **degree d**, if $\varphi(A^r(X)) \subseteq A^{r+d}(X)$. for all $r = 0, 1, 2, \ldots$.

3. For a cartier divisor or a line bundle D, intersection was define

   $$D \cap \_ : A(X) \to A(X)$$

   as a homomorphism of degree one (see Section 2.3 of Fulton).

4. Let

   $$GrHomA(X) = \oplus_{i=0}^{n} Hom^r A(X)$$

   denote the group of all graded homomorphisms, where $Hom^r A(X)$ is the group of homomorhisms of degree r.

5. Note that $GrHomA(X)$ has a graded ring structure under composition.
6. also note that $D \cap \_ \in Hom^1 A(X)$.
7. define total Chern class of a line bundle L as

$$C(L) = 1 + C^1(L) = 1 + D \cap \_.$$

   This is an element in $1 + Hom^1 A(X) \subseteq GrHomA(X)$.
8. For a locally free sheaf $\mathcal{E}$ of rank r total Chern class is defined

$$C(\mathcal{E}) = 1 + C^1(\mathcal{E}) + \cdots + C^r(\mathcal{E})$$

   where $C^k(\mathcal{E}) \in Hom^k A(X)$ is a homomorphism of degree k.
9. The rest is using induction as above in the nonsingular case.
10. For our purpose, $GrHomA(X)$ behaves quite like the Chow group $A(X)$ in nonsingular case.
11. For locally free sheaf $\mathcal{E}$ of rank r, we will use the above exact sequence and define the total chern class

$$C(\mathcal{E}) = C(p^*(\mathcal{E})) = C(\mathcal{E}')C(\mathcal{O}(1)) = C(\mathcal{E}')(1 + \eta),$$

   where $\eta : A(X_1) \to A(X_1)$ is the first chern class of $\mathcal{O}(1)$.
12. It needs a proof to show that $C^k(\mathcal{E}) : A(X) \to A(X)$.

I did not have chance to proof read. Thanks you all!