# Chapter IV
# The Brauer Group

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August 15 2013

## 1 The Brauer Group

1. As always, $F$ would denote a field.
2. All $F-$algebras considered are assumed to be finite dimensional.
3. Let $A$ be an $F-$algebra and $S$ be a subset of $A$. Denote
$$C_A(S) := \{a \in A : as = sa \ \forall \ s \in S\},$$
which is called the centralizer of $S$ in $A$.
   (a) It follows $C_A(S)$ is a subalgebra of $A$.
   (b) By defintion of $F-$algebras, $F \subseteq C_A(S)$.
   (c) In particular, denote $Z(A) := C_A(A)$, to be called the center of $A$.
4. An $F-$algebra $A$ is said to be $F-$central over $F$, if $Z(A) = A$.
5. An $F-$algebra $A$ is said to be simple, if it has no two-sided ideals other than $(0), A$.
6. $F-$algebra $A$ is said to be central simple algebra (CSA) over $F$, if it is both central over $F$ and is simple.

In this chapter, we study central simple algebras over $F$.

Examples.

1. The trivial example is $F$ over $F$.
2. Let $F$ be a vector space of $F$ with $\dim V = n$. Then $End(V) \xrightarrow{\sim} \mathbb{M}_n(R)$ is a CSA over $F$. Proof. Exercise.
3. For two nonzero $a, b \in F$, we have $A = \left(\frac{a,\ b}{F}\right)$ is a CSA over $F$. Proof: Seen before.

Important theorem:

**Theorem 1.1.** *Let $A, B$ be two $F-$algebras.*

1. *Let $A' \subseteq A,\ B' \subseteq B$ be subalgebras. Then,*
$$C_{A\otimes B}(A' \otimes B') = C_A(A') \otimes C_B(B').$$
2. *If $A$ is CSA and $B$ is simple, then $A \otimes B$ is simple.*
3. *In particular, if $A, B$ are CSA over $F$ then so is $A \otimes B$.*

**Proof.** Postponed ■

We proceed to define Brauer Group of a field $F$.

**Definition 1.2.** Let $A, B$ be two CSAs over $F$. We say that $A$ is similar to $B$, if there are vector spaces $U, V$ such that $A \otimes End(U) \approx B \otimes End(V)$. This means, if $A \otimes \mathbb{M}_n(F) \approx B \otimes \mathbb{M}_m(F)$, for some $n, m$. Then, "similarity" is an equivalence relation on the set of all CSAs over $F$.

**Proof.** We write $A \sim A'$ if they are similar. For any CSA $A \sim A$. It is also obvious the $A \sim B \Longrightarrow B \sim A$. Now suppose $A \sim B \sim C$. So, there are vector spaces $U, V, W, Z$ such that
$$A \otimes End(U) \approx B \otimes End(V) \quad \textit{and} \quad B \otimes End(W) \approx C \otimes End(Z).$$

So,

$A \otimes End(U) \otimes End(W) \approx B \otimes End(V) \otimes End(W) \approx B \otimes End(V) \otimes End(Z)$.

Hence

$$A \otimes End(U \otimes W) \otimes B \otimes End(V \otimes Z).$$

The proof is complete. ■

**Definition 1.3.** Let $B(F)$ denote the set of all equivalence classes of CSAs over $F$. Denote the equivalence class of $A$ by $[A]$.

1. Define a "multiplication" as follows:

$$[A] \cdot [B] = [A \otimes B], \quad \text{note} \quad \mathrm{A} \otimes \mathrm{B} \quad \text{is a CSA.}$$

2. It is easy to check that this is a well defined binary structure on $B(F)$.

3. Note $[F] = [\mathbb{M}_n(F)] = [End(V)]$ for all $n$ and vector spaces $V$ with $\dim V = n$.

4. Also, $[A] \cdot [F] = [A]$. So, $[F]$ is the identity.

5. So, $B(F)$ has a monoid structure. In fact it is group, as follows (1.4).

**Proposition 1.4.** *Let $A$ be a CSA over $F$ and $A^{op}$ denote the opposite algebra. Then, $A \otimes A^{op} \approx End_F(A)$, where $End_F(A)$ denotes the $F-$linear homomorphisms $A \longrightarrow A$. So, $[A]^{-1} = [A^{op}]$.*

**Proof.** For convenience denote $A^{op} = \{a^{op} : A \in A\}$. Recall, $a^{op} \cdot b^{op} := ba$. Note, if $I$ is also a two sided ideal in $A^{op}$, then $I$ is a two sided ideal in $A$. So, $A^{op}$ is simple and similarly, it is also central over $F$. Now define

$$\theta_0 : A \times A^{op} \longrightarrow End_F(A) \quad by \quad \theta_0(a, b^{op})(c) := acb.$$

It is easy to see that $\theta$ is a bilinear morphism. So, $\theta_0$ extends to a homomorphism

$$\theta : A \otimes A^{op} \longrightarrow End_F(A).$$

It is easy the check that $\theta$ is an $F-$algebra homomorphism. Since $A \otimes A^{op}$ simple $\ker(\theta) = 0$. By comparing the diemension $\theta$ is an isomorphism. The proof is complete. ■

**Definition 1.5.** This group $B(F)$ is called the Brauer group.

**Theorem 1.6** (Wedderburn Theorem)**.** Suppose $A$ is a CSA over $F$.

1. Then, $A \approx \mathbb{M}_n(D)$ for some central division algebra $D$ over $F$.

2. Also, this division algebra $D$ is uniquely determined (upto isomorphism). This means, for central division algebras $D, \Delta$, we have

$$\mathbb{M}_n(D) \approx \mathbb{M}_m(\Delta) \implies D \approx \Delta.$$

**Proof.** The proof is not difficult. Lam skips the proof and we do the same. ■

**Theorem 1.7.** *The elements of $B(F)$ is in 1-1 correspondance with the isomorphim classes of central division rings over $F$.*

**Proof.** First, suppose $A$ is a CSA. By Wedderburn theorem, for some entral division ring $D$ we have

$$A \approx \mathbb{M}_n(D) \approx D \otimes \mathbb{M}_n(F). \qquad \textit{hence} \quad [A] = [D].$$

Now suppose $[D] = [\Delta]$ for some central division algebras. Then, for doem $m, n$ we have

$$D \otimes \mathbb{M}_n(F) \approx \Delta \otimes \mathbb{M}_m(F). \quad \textit{Hence} \quad \mathbb{M}_n(D) \approx \mathbb{M}_n(\Delta).$$

By Wedderburn theorem $D \approx \Delta$. ■

Follwoing follows:

1. Two non-isomorphic quaternion algebras represent different elementrs of $B(F)$.

2. Quaternion algebras need not form a group.

3. **Examples** Witout proof.

   (a) (Frobenius) $B(\mathbb{R}) = \{\pm 1\}$ where $-1 := \left(\frac{-1,-1}{\mathbb{R}}\right)$.

(b) If $F$ is a finite field then $B(F) = 0$.

(c) Let $\mathbb{C}(X) \hookrightarrow F$ be algebraic then $B(F) = 0$.

(d) Let $F$ be the completion of of a number field at a finite prime. Then $B(F) = \mathbb{Q}/\mathbb{Z}$.

(e) If $K$ is algebraically closed, then $B(K) = 0$.

# 2 Central Simple Graded Algebras

As usual $F$ denotes a filed.

**Definition 2.1.** A $\mathbb{Z}_2$ graded $F-$algebra is an $F-$algebra $A$ such that

1. $\dim_F A < \infty$
2. $A = A_0 \oplus A_1$
3. $F \subseteq A_0$, $A_i A_j \subseteq A_{i+j}$ for $i, j = 0, 1 \in \mathbb{Z}_2$.

We sometimes call them just "graded algebras". Given such a $\mathbb{Z}_2$ graded $F-$algebra $A$ we have:

1. Write $h(A) = A_0 \cup A_1$ to be called homogeneous elements.
2. For $a \in h(A)$ write $\partial(a) = i$ if $a \in A_i$.
3. A subspace $S \subseteq A$ is called graded subspace, if

$$s = s_0 + s_1 \in S \textit{ with } s_i \in A_i \quad \Longrightarrow \quad s_i \in S.$$

   So, $S = S_0 \oplus S_1$ where $S_i = S \cap A_i$.

4. Likewise, we define graded ideals, graded subalgebras etc.
5. Suppose $S \subseteq A$ is a graded subspace. The graded centralizer $\hat{C}_A(S)$ is defined as

$$\hat{C}_A(S) = C_0 \oplus C_1 \quad \textit{where} \quad C_i = \{c \in A_i : cs = (-1)^{i\partial s} sc \quad \forall s \in h(S)\}$$

   Equivalently,

$$c \in h(\hat{C}_A(S)) \Longleftrightarrow cs = (-1)^{\partial(c)\partial(s)} sc \quad \forall s \in h(S)$$

   More explicitly, for $c_0 + c_1 \in \hat{C}_A(S)$ and $s_0 + s_1 \in S$, we have

$$c_0 s_0 = s_0 c_0, \quad c_0 s_1 = s_1 c_0, \quad c_1 s_0 = s_0 c_1, \quad c_1 s_1 = -s_1 c_1.$$

6. $\hat{C}_A(S)$ is a graded subalgebra of $A$.

7. Note for $s_0 + s_1 \in \hat{C}_A(S)$ and $a_0 + a_1 \in A$ we have

$$(s_0 + s_1)(a_0 + a_1) = s_0a_0 + s_0a_1 + s_1a_0 - s_1a_1 \neq (a_0 + a_1)(s_0 + s_1).$$

   So, $\hat{C}_A(S) \nsubseteq C_A(S)$. Similarly, $C_A(S) \nsubseteq \hat{C}_A(S)$.

8. Define the graded centralizer $\hat{Z}(A) := \hat{C}_A(A)$.

9. It is also clear $C_A(S)$ is also graded.

10. Also, $Z(A)_0 = \hat{Z}(A)_0$.

**Definition 2.2.** Let $A$ be a grades algebra over $F$.

1. $A$ is called a graded central algebra (GCA) if $F = \hat{Z}(A)$.

2. $A$ is said to be graded simple algerba (GSA), if it has no nontrivial graded two-sided ideal.

3. $A$ is said to be graded simple central algerba (GSCA), if it is both GSA and GCA.

## 2.1 Graded Tensor Product

**Definition 2.3.** Suppose $A = A_0 \oplus A_1$, $B = B_0 \oplus B_1$ are two graded algebras over $F$. Define

$$A\hat{\otimes}B = (A_0 \otimes B_0 \oplus A_1 \otimes B_1) \bigoplus (A_0 \otimes B_1 \oplus A_1 \otimes B_0)$$

Define the multiplication,that is induced by,

$$(a \otimes b)(x \otimes y) := (-1)^{\partial(b)\partial(x)}(ax \otimes by) \qquad \forall\ a, x \in h(A)\ and\ b, y \in h(B).$$

1. **Remark.** Clearly, as vector spaces $A\hat{\otimes}B$ is same as $A \otimes B$. Only the multiplication structure is different.

2. ALso $A \hookrightarrow A\hat{\otimes}B$ and $B \hookrightarrow A\hat{\otimes}B$ as graded rings.

The following is the graded version of theorem 1.1.

**Theorem 2.4.** *Let $A, B$ be graded $F-$algebras.*

1. *Let $S \subseteq A, T \subseteq B$ be graded subalgebras. Then*

$$\hat{C}_{A\hat{\otimes}B}(S\hat{\otimes}T) = \hat{C}_A(S)\hat{\otimes}\hat{C}_B(T).$$

2. *In particular, if $A, B$ are GSCA then so is $A\hat{\otimes}B$.*

**Proof.** The following is a commutative diagram of injective homomorphism

$$\hat{C}_A(S) \otimes \hat{C}_B(T) \dashrightarrow \hat{C}_{A\hat{\otimes}B}(A\hat{\otimes}B) \quad \text{of graded rings.}$$

The rest of the proof is routine or long, which I skip. ■

# 3 Structure of CSGA

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# 4 Brauer-Wall Group

Give a field $F$, the Brauer-Wall Group $BW(F)$ is constructed in the same way that the Brauer Group $B(F)$ was defined. Only difference:

1. Replace $\otimes$ by $\hat{\otimes}$.

2. Given two GSCAs $A, B$ define $A \sim B$ if

$$A\hat{\otimes}End(V) \approx Bhat\otimes End(W)$$

for some graded vector spaces $V, W$. There is a way to give a graded structure to $End(V)$.