Chapter V
The Clifford Algebras

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1 Construction of Clifford Algebras

In this section, a quadratic space (V, q) need not be regular.

Definition 1.1. Suppose (V, q) is a quadratic space. Let A be an $F-$algebra with $V \subseteq A$. We say that A is compatible with (V, q), if

$$x \in V \quad \Longrightarrow \quad x^2 = q(x).$$

1. In this case,

$$\forall\, x, y \in V \qquad 2B(x, y) = q(x+y) - q(x) - q(y) = (x+y)^2 - x^2 - y^2 = xy + yx.$$

2. In particular,

$$\forall\, x, y \in V \qquad x \perp y \Longleftrightarrow xy = -yx.$$

Lemma 1.2. *For A as above and $0 \neq x \in V$, x is invertible in A if and only if x is anisotropic in V.*

Proof. Suppose x is anisotropic. Then, $q(x) = x^2 \neq 0$. So, $x^{-1} = \frac{x}{q(x)}$. Conversely, suppose x is invertible and $xy = 1$ for some $y \in A$. Then, $q(x)y = x^2 y = x \neq 0$. ■

Lemma 1.3. *let A be as above and $0 \neq u \in V$ be anisotropic. Then, the hyperboloc reflection*

$$\tau_u(x) = -uxu^{-1} \qquad \forall\ x \in V.$$

So, τ_u is negative of the conjugation by u.

Proof. Straight forward computation:

$$\tau_u(x) = x - \frac{2B(x,u)}{q(u)} \cdot u = x - \frac{xu + ux}{u^2} \cdot u = -uxu^{-1}$$

■

Definition 1.4. Given a quadratic form (V, q), the Clifford algebra C of (V, q) is the universal object in the category of all $F-$algebras A containing V. That means, given any $F-$algebra A containing V, there is a unique algebra homomorphism $\varphi : C \longrightarrow A$ such that the diagram

$$\begin{array}{ccc} V & \hookrightarrow & C \\ & \searrow & \downarrow \exists!\,\varphi \\ & & A \end{array}$$

commutes. That means $\varphi(x) = x \ \ \forall\ x \in V$.

1. Since, it is defined by the universal property, "any two" Clifford algebras are naturally isomorphic.

2. **Construction**: Define the tensor algebra

   $$T(V) = \oplus_{n=0}^{\infty} T^n V \quad where\ T^n V = V \otimes V \otimes \cdots \otimes V \quad n-fold\ tensor\ product.$$

   Let $\mathcal{I}(q)$ be the two sider ideal of $T(V)$ generated by $\{x \otimes x - q(x) : x \in V\}$. Now define, $C(q) = \frac{T(V)}{\mathcal{I}(q)}$. Then, $C(q)$ has the universal property of the definition. We also use the notations $C(q) = C(V) = C(V, q)$. ■

3. Note, V generates $C(q)$.

4. The product in $C(q)$ is expressed by juxtaposition.

5. Write $T(V) = T_0(V) \oplus T_1(V)$, where $T_0(V) = \oplus_{n=0}^{\infty} T^{2n}V$ and $T_0(V) = \oplus_{n=0}^{\infty} T^{2n+1}V$. This gives $T(V)$ a $\mathbb{Z}_2$−grading.

6. $\mathcal{I}(q)$ is a $\mathbb{Z}_2$−graded ideal.

7. So, $C(q) = C_0(q) \oplus C_1(q)$ has a $\mathbb{Z}_2$−grading. That means $C_i(q)C_j(q) \subseteq C_{i+j}(q)$.

8. $C_0(q)$ is called the even part of $C(q)$, which is a subalgebra.

Examples. Here is a list of examples.

1. Let $(V, q) = \langle x \rangle$. Write $V = Fx$. Then, $C(q) = \frac{F[x]}{(x^2-a)} = F(\sqrt{a})$, if $a \neq 0$.

2. If $q = 0$ on V, then $\mathcal{I}(q)$ is generated by $x \otimes x$ with $x \in V$. So, $C(q)$ is the exterior algebra.

3. Let $q = \langle a, b \rangle$ with $a, b \in \dot{F}$. Write Let $V = Fx \perp Fy$ with $q(x) = a, q(y) = b$. Then,

$$C(q) = \langle \frac{a, b}{F} \rangle = \text{The graded quarternion}$$

   with usual basis $\{1, i, j, k\}$ and $\partial(1) = \partial(k) = 0$ and $\partial(i) = \partial(j) = 1$. **Proof.** The map $V \longrightarrow C(q)$ with $x \mapsto i,\ y \mapsto j$ has th universal property.

   Alternately: One can check that the ideal $\mathcal{I}$ of the definition of the quaternion algebra and $\mathcal{I}(q)$ are same. ■

4. It follows from above $C(\mathbb{H}) = \langle \frac{1,-1}{F} \rangle$, where $\mathbb{H} = \langle 1, -1 \rangle$ is the hyperbolic plane. Hence also, $C(\mathbb{H}) = \mathbb{M}_2(F)$.

1.1 Dimension and Basis of $C(q)$

Corollary 1.5. Let (V, q) be any quadratic from, with $\dim V = n$. Then $\dim C(q) \leq 2^n$.

Proof. Let $\{x_1, \ldots, x_n\}$ be an orthogonal basis of V.

$$(x_1 + x_2)^2 - q(x_1 + x_2) = 0, \Longrightarrow \quad x_1^2 + x_2^2 + x_1x_2 + x_2x_1 - q(x_1) - q(x_2) = 0$$

$$\Longrightarrow \quad x_1x_2 + x_2x_1 = 0. \quad \textit{hence} \quad x_ix_j = x_jx_i \quad \forall i \neq j.$$

Therefore, $C(q)$ is generated, as a vector space, by

$$\{x_1^{\epsilon_1} x_2^{\epsilon_2} \cdots x_n^{\epsilon_n} : \epsilon_i = 0, 1\}$$

The proof is complete. ■

In fact, we will prove,

$$\{x_1^{\epsilon_1} x_2^{\epsilon_2} \cdots x_n^{\epsilon_n} : \epsilon_i = 0, 1\} \qquad \text{is a basis of} \quad C(q).$$

Following is a more general lemma.

Lemma 1.6. *Let $(V, q), (W, Q)$ be quadratic spaces. Then, there is an isomorphism*

$$C(V \perp W) \xrightarrow{\sim} C(V)\hat{\otimes}C(W) \quad \text{as } \mathbb{Z}_2 - \text{graded algebras.}$$

Proof. Since $V \hookrightarrow C(V), W \hookrightarrow C(W)$ there is a homomorphism of vector spaces $V \oplus W \longrightarrow C(V)\hat{\otimes}C(W)$. By universal propoerty, this map extends to a $F-$algebra homomorphism $C(V \perp W) \longrightarrow C(V)\hat{\otimes}C(W)$ such that the diagram

$$\begin{array}{ccc} V \oplus W & \hookrightarrow & C(V \perp W) \\ & \searrow & \downarrow \varphi \\ & & C(V)\hat{\otimes}C(W) \end{array} \qquad \textit{commutes.}$$

In proof of lemma 1.5 generators of these algebras were given. From this it follows that φ is surjective. It follows from theorem 1.7 that φ is an isomorphism. ■

Theorem 1.7. *Suppose* (V, q) *is a quadratic space and* $\dim V = n$*. Then,* $\dim C(q) = 2^n$*. In particular, if* $\{x_1, \ldots, x_n\}$ *is an orthogonal basis of* V*, then*

$$\{x_1^{\epsilon_1} x_2^{\epsilon_2} \cdots x_n^{\epsilon_n} : \epsilon_i = 0, 1\} \qquad \text{is a basis of} \quad C(q).$$

Proof. The latter statement follows form the first one. We prove the first statement, by induction. In $\dim V = 1$ then, by the example above $\dim C(q) = 2^1$. If $n > 1$, take an orthogonal basis of V and write $V = U \perp W$, where $\dim W = 1$. Since $C(V)$ maps onto $C(U)\hat{\otimes}C(W)$, we have

$$\dim C(V) \geq \dim(C(U)\hat{\otimes}C(W)) = \dim(C(U))\dim(C(W)) = 2^n.$$

Also $\dim C(V) \leq 2^n$. The proof is complete. ∎

Corollary 1.8. $\dim C_0(q) = \dim C_1(q) = 2^{n-1}$.

Proof. Let $\{x_1, \ldots, x_n\}$ be an orthogonal basis of V. Then,

$$\{x_1^{\epsilon_1} x_2^{\epsilon_2} \cdots x_n^{\epsilon_n} : \epsilon_i = 0, 1\} \qquad \text{is a basis of} \quad C(q).$$

$$\{x_1^{\epsilon_1} x_2^{\epsilon_2} \cdots x_n^{\epsilon_n} : \epsilon_i = 0, 1 \quad \sum \epsilon_i \in 2\mathbb{Z}\} \qquad \text{is a basis of} \quad C_0(q).$$

$$\{x_1^{\epsilon_1} x_2^{\epsilon_2} \cdots x_n^{\epsilon_n} : \epsilon_i = 0, 1 \quad \sum \epsilon_i \in 1 + 2\mathbb{Z}\} \qquad \text{is a basis of} \quad C_1(q).$$

If q is totally isotropic, $C(q)$ is the exterior algebra. In this case the corollary holds. Assume q is not totally isotropic. So, we can assume $q(x_1) = a_1 \neq 0$. Consider two multiplication maps

$$x_1 : C_0(q) \longrightarrow C_1(q), \quad x_1 : C_1(q) \longrightarrow C_0(q).$$

Two compositions of these two maps is multiplication by a_1. So, each one is an isomorphism. So,

$$\dim C_0(q) = \dim C_1(q) = \frac{\dim C(q)}{2} = 2^{n-1}$$

The proof is complete. ∎

Corollary 1.9. $C(m\mathbb{H}) \cong \mathbb{M}_2(F)\hat{\otimes}\mathbb{M}_2(F)\hat{\otimes}\cdots\hat{\otimes}\mathbb{M}_2(F)$.

Proof. Follows from (1.6). ∎

1.2 Spinor Norm (Skip)

Proposition 1.10. *There is a unique anti-siomorphism* $\epsilon : C(q) \xrightarrow{\sim} C(q)$ *such that* $\epsilon_{|V} = Id_V$*. Also,* ϵ *stabilizes both* $C_0(q)$ *and* $C_1(q)$*.*

Proof. Since $C(q)^{op}$ is a $\mathbb{Z}_2-$graded algebra, by universal property, there is an homomorphism ϵ as follows:

$$\begin{array}{ccc} V & \hookrightarrow & C(q) \\ & \searrow & \downarrow \epsilon \\ & & C(q)^{op} \end{array} \qquad \text{the diagram commutes.}$$

ϵ is clearly surjective. So, it is an isomorphism, by dimension consideration. Clearly, $\epsilon(u_1 \cdots u_m) = u_m \cdots u_1$. ■

Proposition 1.11. *Let* (V, q) *be a regular quadratic form. Suppose* $u_1, u_2, \ldots, u_r \in V$ *are anisotropic. Then,*

$$\tau_{u_1}\tau_{u_2}\cdots\tau_{u_r} = Id \Longrightarrow q(u_1)q(u_2)\cdots q(u_2) \in \dot{F}^2.$$

Proof. For any anisotropic $u \in V$ let $c(u) : C(q) \xrightarrow{\sim} C(q)$ be conjugation $c(u)(z) = uzu^{-1}$. By (1.3)

$$\tau_u = -c(u)_{|V}. \qquad \text{Also} \qquad c(u_i)c(u_j) = c(u_iu_j).$$

Since $(-1)^r = \det(\tau_{u_1}\tau_{u_2}\cdots\tau_{u_r}) = 1$, *$r$ is even*. So, $x := u_1u_2\cdots u_r \in C_0(q)$. Also,

$$\tau_{u_1}\tau_{u_2}\cdots\tau_{u_r} = Id \Longrightarrow (-1)^r c(u_1u_2\cdots u_r)_{|V} = Id_V.$$

Hence, $xvx^{-1} = v \quad \forall\ v \in V$. That means $xv = vx \quad \forall\ v \in V$. Since V generates $C(q)$, $x \in Z(C(q)) \cap C_0(q)$. We will prove (2.1) $C(q)$ is $F-$central graded algebra. So,

$$x \in Z(C(q)) \cap C_0(q) = F.$$

We use the anti-isomorphism in (1.10). We have $\epsilon(x) = x$. So,

$$q(u_1)q(u_2)\cdots q(u_r) = (u_1u_2\cdots u_r)(u_ru_{r-1}\cdots u_1) = x\epsilon(x) = x^2 \in \dot{F}^2.$$

The first equality holds because $q(u_i) = u_i^{\in} F$. The proof is complete. ■

Definition 1.12. *Let $\sigma \in O(V,q)$. We can write $\sigma = \tau_{u_1}\tau_{u_2}\cdots\tau_{u_m}$ where $u_i \in V$ are anisotropic. Define,*

$$\theta(\sigma) := q(u_1)q(u_2)\cdots q(u_r)\dot{F}^2 \in \frac{\dot{F}}{\dot{F}^2}. \qquad \text{This is well defined by } 1.11.$$

We say $\theta(\sigma)$ is the spenoir norm of σ.

Theorem 1.13. *$\theta : O(V,q) \longrightarrow \frac{\dot{F}}{\dot{F}^2}$ is a group homomorphism.*

2 Structure Theorem

We have the main theorem:

Theorem 2.1. *Suppose* (V, q) *is a regular quadratic form. Then,* $C(q)$ *is a CSGA.*

Proof. If $\dim V = 1$ then $C(q) = F(\sqrt{a})$ where $A = Fx$ and $q(x) = a$. So, $C(q)$ is CSGA. (*Confusing Notations: By definition* $F(\sqrt{a}) = \frac{F[x]}{x^2-a}$, *which need not be a field.* Also, by notation $\sqrt{a} := \overline{x}$.) Here $\partial(\sqrt{a}) = 1$. It follow, $C(q)$ is CGSA. Not the theorem follows from Theorem IV.2.3 that graded tensor product of CGSA is a CGSA. ■

3 The Clifford Invariant

Let $(V, q), (W, Q)$ be two regular quadratic spaces.

1. By lemma 1.6, $C(q \perp Q) \cong C(q)\hat{\otimes}C(Q)$.
2. Then $(V, q) \mapsto \langle C(q) \rangle \in BW(F)$ is a monoid hemomorphism.
3. So, it indueces a homomorphism Γ so that the diagram

$$\begin{array}{ccc} M(F) & \xrightarrow{\iota} & \widehat{W}(F) \\ & \searrow^{\zeta} & \downarrow{\Gamma} \\ & & BW(F) \end{array} \qquad \textit{commutes.}$$

4. In fact $\zeta(\mathbb{H}) = \hat{\mathbb{M}}_2(F) = 0 \in BW(F)$.
5. So, Γ factors through the Witt groups:

$$\begin{array}{ccccc} M(F) & \xrightarrow{\iota} & \widehat{W}(F) & \longrightarrow & W(F) \\ & \searrow^{\zeta} & \downarrow{\Gamma} & \swarrow_{\Gamma} & \\ & & BW(F) & & \end{array} \qquad \textit{commutes.} \qquad \text{(We use same notation } \Gamma)$$

6. Γ is called the Clifford Invariant map.

4 Real Periodicity and Clifford Modules

Let F be any field with $char(F) \neq 2$. Let

$$\varphi_{p,q} = p\langle -1\rangle \perp q\langle 1\rangle = -\sum_{i=1}^{p} X_i^2 + \sum_{j=1}^{q} Y_j^2 \quad and \quad C^{p,q} = C(\varphi_{p,q}).$$

Theorem 4.1. *There is a graded algebra isomorphism*

$$C^{p+n,q+n} \cong \hat{\mathbb{M}}_{2^n}(C^{p,q})$$

Proof. We have

1. Note $\varphi_{n,n} = n\mathbb{H}$. So, $C^{n,n} = \hat{\mathbb{M}}_{2^n}(F)$.
2. $\varphi_{p+n,q+n} \cong \varphi_{p,q} \perp \varphi_{n,n}$
3. So, $C^{p+n,q+n} \cong C^{p,q}\hat{\otimes}C^{n,n} \cong C^{p,q}\hat{\otimes}\hat{\mathbb{M}}_{2^n}(F) \cong \hat{\mathbb{M}}_{2^n}(C^{p,q})$.

The proof is complete. ∎

We prove the following lemma that we use latter.

Lemma 4.2 ([ABS])**.** *We have*

1. $C^{2,0}\hat{\otimes}C^{0,2} \xrightarrow{\sim} C^{0,4}$.
2. *More generally,* $C^{p,0}\hat{\otimes}C^{0,2} \xrightarrow{\sim} C^{0,p+2}$.
3. *And* $C^{0,q}\otimes C^{2,0} \xrightarrow{\sim} C^{q+2,0}$

Proof. Define

$$\psi : F^4 \longrightarrow C^{2,0}\hat{\otimes}C^{0,2} \quad by \quad \begin{cases} \psi(e_1) = 1 \otimes e_1 \\ \psi(e_2) = 1 \otimes e_2 \\ \psi(e_3) = e_1 \otimes e_1e_2 \\ \psi(e_4) = e_2 \otimes e_1e_2 \end{cases}$$

Frm the description of the standard bases of $C^{2,0}, C^{0,2}$, we see that ψ is injective. Also

$$\psi(e_1)^2 - \varphi_{0,4}(e_1) = (1 \otimes e_1)(1 \otimes e_1) - 1 = (-1)^{\partial(e_1)\partial(1)} 1 \otimes e_1^2 - 1 = 1 \otimes \varphi_{0,2}(e_1) - 1 = 0.$$

Also,

$$\psi(e_3)^2 - \varphi_{0,4}(e_3) = (e_1 \otimes e_1 e_2)(e_1 \otimes e_1 e_2) - 1 = (-1)^{\partial(e_1 e_2)\partial(e_1)} e_1^2 \otimes e_1 e_2 e_1 e_2 - 1$$

$$= \varphi_{2,0}(e_1) \otimes (-e_1^2 e_2^2) - 1 = (-1) \otimes (-\varphi_{0,2}(e_1)\varphi_{0,2}(e_2)) - 1 = 0$$

So, there is a homomorphism

$$\eta : C^{0,4} \longrightarrow C^{2,0} \hat{\otimes} C^{0,2} \quad \ni \quad \begin{array}{ccc} F^4 & \longrightarrow & C^{0,4} \\ & \searrow^{\psi} & \downarrow \eta \\ & & C^{2,0} \hat{\otimes} C^{0,2} \end{array} \quad \textit{commutes.}$$

It is easy to check that ψ is surjective. Since they have same dimension, η is an isomorphism. The proof is complete. ■

Proposition 4.3 (Periodicity 8)**.**

$$C^{p+8,q} \cong \hat{\mathbb{M}}_{16}(C^{p,q}) \cong C^{p,q+8}$$

Proof. We prove the first one only. Suppose $p = q = 0$. First note $C^{4,0} \cong C^{4,0}$ (see lemma 4.2). Then

$$C^{8,0} \cong C^{4,0} \hat{\otimes} C^{4,0} \cong C^{4,0} \hat{\otimes} C^{0,4} \cong C^{4,4} \cong \hat{\mathbb{M}}_{16}(F).$$

Therefore,

$$C^{p+8,q} \cong C^{8,0} \hat{\otimes} C^{p,q} \cong \hat{\mathbb{M}}_{16}(C^{p,q})$$

The proof is complete. ■

We will compute

$$C^{p,0}, C^{0,q} \qquad 0 \leq p, q \leq 7.$$

Denote

$$X \cong F\langle\sqrt{-1}\rangle, \ Y := C(\langle -1, -1 \rangle) = \langle \frac{-1, -1}{F} \rangle, \ Z \cong F\langle\sqrt{1}\rangle, \ W := C(\langle 1, 1 \rangle) \cong \langle \frac{1, 1}{F} \rangle$$

Also note, by (III.2.6)

$$W = \langle \frac{1,1}{F} \rangle \cong \langle \frac{1,-1}{F} \rangle \cong \hat{\mathbb{M}}_2(F)$$

Now, we have the following table of $C^{p,q}$.

n	0	1	2	3	4	5	6	7
$C^{n,0}$	F	X	Y	$Y\hat{\otimes}Z$	$Y\hat{\otimes}W$	$\hat{\mathbb{M}}_2(X\hat{\otimes}W)$	$\hat{\mathbb{M}}_4(W)$	$\hat{\mathbb{M}}_8(Z)$
$C^{0,n}$	F	Z	W	$X\hat{\otimes}W$	$Y\hat{\otimes}W$	$\hat{\mathbb{M}}_2(Y\hat{\otimes}Z)$	$\hat{\mathbb{M}}_4(Y)$	$\hat{\mathbb{M}}_8(X)$

We rewrite it:

n	0	1	2	3	4	5	6	7
$C^{n,0}$	F	X	Y	$Y\times Y$	$\hat{\mathbb{M}}_2(Y)$	$\hat{\mathbb{M}}_4(X)$	$\hat{\mathbb{M}}_8(F)$	$\hat{\mathbb{M}}_8(F)\times\hat{\mathbb{M}}_8(F)$
$C^{0,n}$	F	$F\times F$	$\hat{\mathbb{M}}_2(F)$	$\hat{\mathbb{M}}_2(X)$	$\hat{\mathbb{M}}_2(Y)$	$\hat{\mathbb{M}}_2(Y)\times\hat{\mathbb{M}}_2(Y)$	$\hat{\mathbb{M}}_4(Y)$	$\hat{\mathbb{M}}_8(X)$

Proof. We will establish one by one.

1. $C^{0,0} = F$ is more like a convention. It makes more sense to compute $C^{0,8} \cong \hat{\mathbb{M}}_{16}(F)$.

2. $C^{0,1} = C(Y_1^2) = \frac{F[[x]]}{x^2-1} = \frac{F[x}{x^2-1} = Z$. Similalry, $C^{1,0} \cong X$.

3. $C^{0,2} = C(Y_1^2 + Y_2^2) = \frac{F[[x_1,x_2]]}{(x_1^2-1,x_2^2-1)} \cong W$. Similalry, $C^{2,0} \cong Y$. by one of the examples above.

4. Now $C^{0,3} = C(Y_1^2 + Y_2^2 + Y_3^3) = C^{1,0}\hat{\otimes}C^{0,2} \cong X \otimes W$. Similalry, $C^{3,0} \cong Y \otimes Z$.

5. Similarly, $C^{0,4} = C^{2,0}\hat{\otimes}C^{0,2} \cong Y \otimes W$, Similalry, $C^{4,0}Y \otimes W$. (*In fact, we proved* $C^{0,4} \cong C^{4,0}$.)

6. Now, for $q \leq 3$ we have

$$C^{0,q+4} \cong C^{0,q}\hat{\otimes}C^{0,4} \cong C^{0,q}\hat{\otimes}C^{4,0} \cong C^{4,q} \cong \hat{\mathbb{M}}_{2^q}(C^{4-q,0}) \quad by \quad (4.1).$$

   This completely establishes the last line. We establish the first line similalry. The proof is complete. ■

Complex Case: More generally, when $-1 \in \dot{F}^2$. In this case

1. $X = F\sqrt{-1} \cong \frac{F[x]}{X^2+1} \cong F \times F$.
2. Also $\varphi_{n,0} \cong \varphi_{0,n}$ and so $C^{n,0} \cong C^{0,n}$.
3. Also, $\varphi_{2,0} = \langle -1,-1\rangle \cong \langle 1,-1\rangle \cong \mathbb{H}$. Similarly, $\varphi_{0,2} = \langle 1,1\rangle \cong \langle 1,-(\sqrt{-1})^2\rangle \cong \mathbb{H}$. Therefore
$$Y = W = \hat{\mathbb{M}}_2(F)$$
4. So,
$$C^{p+2,0} \cong C^{0,p}\hat{\otimes}C^{2,0} \cong C^{p,0}\hat{\otimes}C(\mathbb{H}) \cong C^{p,0}\hat{\otimes}\hat{\mathbb{M}}_2(F) \cong \hat{\mathbb{M}}_2(C^{p,0}).$$

Therefore,

n	0	1	2	3	4	5	6	7
$C^{0,n}$	F	$F\times F$	$\hat{\mathbb{M}}_2(F)$	$\hat{\mathbb{M}}_2(F)\times\hat{\mathbb{M}}_2(F)$	$\hat{\mathbb{M}}_4(F)$	$\hat{\mathbb{M}}_4(F)\times\hat{\mathbb{M}}_4(F)$	$\hat{\mathbb{M}}_8(F)$	$\hat{\mathbb{M}}_8(F)\times\hat{\mathbb{M}}_8(F)$

Lam is ignoring thae grading. We should discuss if my grading is alright.

Case: $-1 \notin \dot{F}^2$ **and** $-1 = \alpha^2+\beta^2$ In this case,

1. $X = F\langle\sqrt{-1}\rangle$ is a field.
2. By (I.5.1), since $\langle 1,1\rangle \cong \langle -1,-1\rangle$.
3. So, $Y \cong W \cong \hat{\mathbb{M}}_2(F)$.
4. Also $C^{4,0} \cong \langle -1,-1\rangle \perp \langle -1,-1\rangle \cong \langle 1,1\rangle \perp \langle 1,1\rangle = C^{0,4}$.
5. Hence $C^{p+4,0} \cong C^{p,0}\hat{\otimes}C^{4,0} \cong C^{p,0}\hat{\otimes}C^{0,4} \cong \hat{\mathbb{M}}_2(C^{p,0})$

The table reduces to:

n	0	1	2	3	4	5	6	7
$C^{n,0}$	F	X	$\hat{\mathbb{M}}_2(F)$	$\hat{\mathbb{M}}_2(F)\times\hat{\mathbb{M}}_2(F)$	$\hat{\mathbb{M}}_4(F)$	$\hat{\mathbb{M}}_4(X)$	$\hat{\mathbb{M}}_8(F)$	$\hat{\mathbb{M}}_8(F)\times\hat{\mathbb{M}}_8$
$C^{0,n}$	F	$F\times F$	$\hat{\mathbb{M}}_2(F)$	$\hat{\mathbb{M}}_2(X)$	$\hat{\mathbb{M}}_4(F)$	$\hat{\mathbb{M}}_4(F)\times\hat{\mathbb{M}}_4(F)$	$\hat{\mathbb{M}}_8(F)$	$\hat{\mathbb{M}}_8(X)$

The Real Case: More generally -1 **is not sum of two squares**: In this case, we have no furhter simplication of the original table.

References

[ABS] M. F. Atiyah, R. Bott, and A. Shapiro, *Clifford modules,* Topology 3 (1964), 3-38.