Chapter X
Milnor $K-$theory, Milnor Conjecture
Gersten Conjecture

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1 Pfister Forms

Definition 1.1. For n elements $a_1, a_2, \ldots, a_n \in \dot{F}$ define

$$\langle\langle a_1, \ldots, a_n \rangle\rangle := \otimes_{i=1}^{n} \langle 1, a_i \rangle.$$

This form has dimension 2^n. It is called an $n-$fold Pfister Form. By convention, $0-$fold Pfister Form is defined to be $\langle 1 \rangle$.

1. An $1-$fold Pfister Form $\langle\langle a \rangle\rangle = \langle 1, a \rangle$.
2. A $2-$fold Pfister Form $\langle\langle a_1, a_2 \rangle\rangle = \langle 1, a_1, a_2, a_1 a_2 \rangle = \left(\frac{-a_1, -a_2}{F}\right)$.
3. If $a_i = -1$ for some i, then $\langle\langle a_1, a_2, \ldots, a_n \rangle\rangle = 2^{n-1}\mathbb{H}$.
4. Also, $\langle\langle 1, a_2, \ldots, a_n \rangle\rangle = 2\langle\langle a_2, \ldots, a_n \rangle\rangle$.

Recall, the fundamental ideal of $W(F)$, was defined to be the ideal $I = I(F)$ of all even dimensional forms in $W(F)$.

Proposition 1.2. *The ideal $I(F)^n$ of $W(F)$ is additively generated, as an abelian group, by all the $n-$fold Pfister forms.*

Proof. By II.1.2 $I(F)$ is generated, additively, by $\langle\langle a\rangle\rangle$. Therefore, $I(F)^n$ is additively generated by $n-$fold Pfister forms. ■

Proposition 1.3. We have the following:

1. First,
$$\forall\ x \in D\langle\langle a\rangle\rangle, \quad \langle\langle a, b\rangle\rangle \cong \langle\langle a, bx\rangle\rangle$$

2. Also,
$$\forall\ y \in D\langle ab\rangle, \quad \langle\langle a, b\rangle\rangle \cong \langle\langle y, ab\rangle\rangle$$

Proof. We have

$$\langle\langle a, b\rangle\rangle \cong \langle 1, a\rangle \otimes \langle 1, b\rangle \cong \langle 1, a, b, ab\rangle \cong \langle 1, a\rangle \perp \langle b, ab\rangle \cong \langle 1, a\rangle \perp \langle b\rangle\langle 1, a\rangle$$

$$\cong \langle 1, a\rangle \perp \langle b\rangle\langle x, ax\rangle \cong \langle 1, a\rangle \perp \langle xb, abx\rangle \cong \langle 1, axb, abx\rangle \cong \langle\langle a, xb\rangle\rangle$$

Similarly,

$$\langle\langle a, b\rangle\rangle \cong \langle 1, ab\rangle \perp \langle a, b\rangle \cong \langle 1, ab\rangle \perp \langle y, yab\rangle \cong \langle 1, ab, y, yab\rangle \cong \langle\langle y, ab\rangle\rangle$$

The proof is complete. ■

1.1 one and two fold to $n-$fold

Definition 1.4. Let $\langle\langle a_1, \ldots, a_n\rangle\rangle$ and $\langle\langle b_1, \ldots, b_n\rangle\rangle$ be two $n-$fold Pfister forms. We say that they are simply P-equivalent, if there exists $i \leq j$ such that

1. $\langle\langle a_i, a_j\rangle\rangle \cong \langle\langle b_i, b_j\rangle\rangle$ and
2. $a_k = b_k \ \ \forall\ k \neq i, j$.

More generally, two forms φ, γ are said to be chain P-equivalent, if there is a sequence of forms:

$$\varphi = \varphi_0, \varphi_1, \cdots, \varphi_{m-1}, \varphi_m = \gamma$$

such that $\forall\, i \;\; \varphi_i$ is simply P-equivalent to φ_{i+1}. In this case, we write $\varphi \simeq \gamma$.

1. $\simeq$ is an equivalence relation.
2. $\simeq \Longrightarrow \cong$.
3. ALso, recall, we worked with chain equivalence in simple equivalence in section I.5.

Definition 1.5. Suppose φ is an $n-$fold Pfister form and it represents 1. Then $\varphi \cong \langle 1 \rangle \perp \varphi'$. By cancellation. φ' is uniquely determined, upto an isometry. This form φ' is called the pure subform of φ. We will use this notation φ'. Note, by direct expansion, we can see a diagonal form of φ'.

Theorem 1.6 (Pure Subform)**.** *Suppose $\varphi = \langle\langle a_1, \ldots, a_n \rangle\rangle$ is an $n-$fold Pfister form and $b \in D_F(\varphi')$. Then,*

$$\varphi \approx \langle\langle b, b_2, \ldots, b_n \rangle\rangle \quad \text{for some} \quad b_i \in \dot{F}.$$

Proof. Use induction on n. If $n = 1$, $\varphi = \langle 1, a_1 \rangle$. Then, $\varphi' = \langle a_1 \rangle$. Then $b \in D(\varphi') \Longrightarrow b = ax^2$. So, $\varphi = \langle 1, a_1 \rangle = \langle 1, b \rangle$. Now, assume that the theorem holds for $(n-1)-$fold forms. Write

$$\tau = \langle\langle a_1, \ldots, a_{n-1} \rangle\rangle. \;\; So, \quad \varphi \cong \tau \otimes \langle 1, a_n \rangle \cong \tau \perp \langle a_n \rangle \tau.$$

Therefore,

$$\varphi' = \tau' \perp \langle a_n \rangle \tau \;\; So, \;\; b \in D(\varphi') \implies b = x + a_n y \quad where \quad x \in D(\tau') \cup \{0\}, \;\; y \in D(\tau) \cup \{0\}.$$

Case 1. Suppose $y = 0$. Then, $b = x \in D(\tau')$. By induction,

$$\tau \approx \langle\langle b, b_2 \ldots, b_{n-1} \rangle\rangle \quad and \quad hence \quad \varphi = \tau \otimes \langle\langle a_n \rangle\rangle \approx \langle\langle b, b_2 \ldots, b_{n-1}, a_n \rangle\rangle$$

Case 2. Suppose $y \neq 0$. We will prove

$$\varphi \approx \langle\langle a_1, \ldots, a_{n-1}, a_n y\rangle\rangle.$$

Since $y \in D(\tau)$, we can write $y = t^2 + y_0$ with $y_0 \in D(\tau')) \cup \{0\}$. If $y_0 = 0$ then $y = t^2$ and there is nothing to prove. So, assume $y_0 \neq 0$ and hence $y_0 \in D(\tau')$. By induction

$$\tau \approx \langle\langle y_0, c_2, \ldots, c_{n-1}\rangle\rangle \quad where \quad c_i \in \dot{F}.$$

. Therefore,

$$\varphi \approx \langle\langle y_0, c_2, \ldots, c_{n-1}, a_n\rangle\rangle$$

Since, $y = t^2 + y_0 \in \langle\langle y_0\rangle\rangle$, by (1.3(1)), $\langle\langle y_0, a_n\rangle\rangle \approx \langle\langle y_0, a_n y\rangle\rangle$. Hence,

$$\varphi \approx \langle\langle y_0, c_2, \ldots, c_{n-1}, a_n\rangle\rangle \approx \langle\langle y_0, c_2, \ldots, c_{n-1}, a_n y\rangle\rangle \approx \langle\langle a_1, 2_2, \ldots, 2_{n-1}, a_n y\rangle\rangle$$

This establishes our claim above.

To complete the proof, if $x = 0$ then $a_n y = b$ and we are done. So, assume $x \neq 0$ and so $x \in D(\tau')$. By induction,

$$\tau \approx \langle\langle x, d_2, \ldots, d_{n-1}\rangle\rangle \quad for\ some \quad d_i \in \dot{F}.$$

Since $x + a_n y \in \langle x, a_n\rangle$, by (1.3(2)), $\langle\langle x, a_n y\rangle\rangle \cong \langle\langle x + a_n y, a_n x y\rangle\rangle$. Therefore,

$$\varphi = \tau \otimes \langle\langle a_n y\rangle\rangle \approx \langle\langle x, d_2, \ldots, d_{n-1}, a_n y\rangle\rangle \approx \langle\langle x + a_n y, d_2, \ldots, d_{n-1}, a_n x y\rangle\rangle$$
$$\approx \langle\langle b, d_2, \ldots, d_{n-1}, a_n x y\rangle\rangle$$

The proof is complete. ∎

The following follows from the proof of (1.6.

Proposition 1.7. *Suppose* $\tau = \langle\langle a_1, a_2, \ldots, a_{n-1}\rangle\rangle$ *and* $y \in D(\tau)$. *Then, for any* $a_n \in \dot{F}$, *we have*

$$\langle\langle a_1, a_2, \ldots, a_{n-1}, a_n\rangle\rangle \approx \langle\langle a_1, a_2, \ldots, a_{n-1}, a_n y\rangle\rangle$$

In particular,

$$2\tau = \langle\langle a_1, a_2, \ldots, a_{n-1}, 1\rangle\rangle \approx \langle\langle a_1, a_2, \ldots, a_{n-1}, y\rangle\rangle$$

and

$$\langle\langle a_1, a_2, \ldots, a_{n-1}, -y\rangle\rangle \approx \langle\langle a_1, a_2, \ldots, a_{n-1}, -1\rangle\rangle \qquad \text{is hyperbolic.}$$

Theorem 1.8. *If a Pfister form φ is isotopic, then it is hyperbolic.*

Proof. In this case, φ contains a hyperboloc plane $\mathbb{H}$. So, $\varphi = \langle 1 \rangle \varphi'$ and $-1 \in \varphi'$. So, by (1.6), $\varphi \approx \langle\langle -1, b_2, \ldots \rangle\rangle$, which is hyperbolic. ■

Definition 1.9. Let q be a quadratic form. Define $G(q) = G_F(q) = \{c \in \dot{F} : \langle c \rangle q \cong q\}$. Note $G(q)$ is a subgroup of $\dot{F}$. $G(q)$ is called the group of similarity factors of q. Also note, $\dot{F}^2 \subseteq G(q)$.

Definition 1.10. *For any Pfister form φ over F, $D(\varphi) = G(\varphi)$. In particular, φ is a group form.*

Proof. Since, φ represents 1, $G(\varphi) \subseteq D(\varphi)$. Suppose $c \in D(\varphi)$. Then $\langle\langle c \rangle\rangle \varphi \cong \varphi \perp \langle c \rangle \varphi$ contains the hyperbolic $\mathbb{H} \cong \langle c, -c \rangle$. So, by (1.7) $\varphi \perp \langle c \rangle \varphi$ is hyperbolic space. By I.1.4(3), $\varphi \cong \langle c \rangle \varphi$. The proof is complete. ■

Corollary 1.11. *For integers $n \geq 0$, the nonzero sums of 2^n squares in F form a subgroup of $\dot{F}$.*

Proof. Follows form (1.10), by application of $\langle\langle 1, 1, \ldots, 1 \rangle\rangle$.The proof is complete. ■

Theorem 1.12.

$$\text{Let} \quad \tau = \langle\langle b_1, b_2, \ldots, b_r \rangle\rangle \ (r \geq 0), \quad \gamma = \langle\langle d_1, d_2, \ldots, d_s \rangle\rangle \ (s \geq 0).$$

And $e_1 \in D(\tau\gamma')$. Then, $\exists\ e_2, \ldots, e_s \in \dot{F}$ suchthat

$$\tau\gamma = \langle\langle b_1, b_2, \ldots, b_r, d_1, d_2, \ldots, d_s \rangle\rangle \approx \langle\langle b_1, b_2, \ldots, b_r, e_1, e_2, \ldots, e_s \rangle\rangle.$$

Proof. Prove induction on s. If $s = 1$ and $\gamma' = \langle d_1 \rangle$. So, $e_1 = d_1 x$, with $x \in D(\tau)$. By (1.7),

$$\tau\gamma = \langle\langle b_1, b_2, \ldots, b_r, d_1 \rangle\rangle \approx \langle\langle b_1, b_2, \ldots, b_r, d_1 x \rangle\rangle = \langle\langle b_1, b_2, \ldots, b_r, e_1 \rangle\rangle$$

Now, assume the result holds for $\langle\langle b_1, b_2, \ldots, b_r, d_1, d_2, \ldots, d_{s-1} \rangle\rangle$. Write $\sigma := \langle\langle d_1, d_2, \ldots, d_{s-1} \rangle\rangle$. So,

$$\gamma = \sigma\langle d_s, 1 \rangle \cong \langle d_s \rangle \sigma \perp \sigma \quad \textit{and} \quad \gamma' = \langle d_s \rangle \sigma \perp \sigma'. \quad \textit{So,} \quad \tau\gamma' = \langle d_s \rangle \tau\sigma \perp \tau\sigma'.$$

So,

$$e_1 = d_s x + y \qquad for\ some \quad x \in D(\tau\sigma) \cup \{0\}, \quad y \in D(\tau\sigma') \cup \{0\}$$

Case $x \neq 0, y \neq 0$. We have two steps

1. By (1.7),

$$\langle\langle b_1, b_2, \ldots, b_r, d_1, d_2, \ldots, d_s \rangle\rangle \approx \langle\langle b_1, b_2, \ldots, b_r, d_1, d_2, \ldots, d_s x \rangle\rangle$$

2. By induction,

$$\langle\langle b_1, b_2, \ldots, b_r, d_1, d_2, \ldots, d_{s-1} \rangle\rangle \approx \langle\langle b_1, b_2, \ldots, b_r, y, e_2, \ldots, e_{s-1} \rangle\rangle \qquad (*)$$

Combining these two equations

$$\langle\langle b_1, b_2, \ldots, b_r, d_1, d_2, \ldots, d_s \rangle\rangle \approx \langle\langle b_1, b_2, \ldots, b_r, d_1, d_2, \ldots, d_{s-1}, d_s x \rangle\rangle$$

$$\approx \langle\langle b_1, b_2, \ldots, b_r, y, e_2, \ldots, e_{s-1}, d_s x \rangle\rangle \approx \langle\langle b_1, b_2, \ldots, b_r, e_1, e_2, \ldots, e_{s-1}, d_s xy \rangle\rangle \quad \text{by } (1.3(2)).$$

Corollary 1.13. *Let q be a Pfister form. Write $q = \langle 1, b, e \rangle \perp \langle b_1 \ldots, b_* \rangle$. Then, $q = \langle\langle b, e, e_2 \ldots, e_2 \rangle\rangle$.*

Proof. By Pure Subform Theorem 1.6, $q \cong \langle\langle b \rangle\rangle \gamma$ for sone Pfister form $\gamma = \langle\langle b_1, \ldots, b_s \rangle\rangle$. So, we have

$$q = \langle 1, b \rangle \perp \langle e, * \ldots, * \rangle \cong \langle\langle b \rangle\rangle \perp \langle\langle b \rangle\rangle \gamma'. \quad \text{By Cancellation} \quad e \in \langle\langle b \rangle\rangle \gamma'$$

By theorem 1.12, $\langle\langle b \rangle\rangle \gamma \approx \langle\langle b, e, e_2, \ldots, e_s \rangle\rangle$. The proof is complete. ∎

Theorem 1.14 (P-Equivalence)**.** *Let φ, ψ be two $n-$fold Pfister forms. Then, $\varphi \cong \psi \iff \varphi \approx \psi$.*

Proof. Clearly, $\varphi \approx \psi \Longrightarrow \varphi \cong \psi$. Now, assume $\varphi \cong \psi$. Write

$$\varphi = \langle\langle a_1, \ldots, a_n \rangle\rangle \quad and \quad \psi = \langle\langle b_1, \ldots, b_n \rangle\rangle$$

For integers $0 \leq r \leq n$ we prove

$$(\mathrm{A_r}) \quad \exists \quad c_{r+1}, \ldots, c_n \in \dot{F} \quad \ni \quad \varphi \approx \langle\langle b_1, \ldots, b_r, c_{r+1}, \ldots, c_n \rangle\rangle.$$

Theorem would be established with $r = n$. There is nothing to prove with $r = 0$, with $c_i = a_i \ \forall \ i$. Assume A_r is true. Write

$$\tau = \langle\langle b_1, \ldots, b_r \rangle\rangle, \ \beta = \langle\langle b_{r+1}, \ldots, b_n \rangle\rangle, \ \ \gamma = \langle\langle d_{r+1}, \ldots, d_n \rangle\rangle.$$

Write $s = n - r$. Then, γ is an $s-$fold Pfister form. By induction $\varphi \approx \tau\gamma$. So,

$$\tau\beta = \psi \cong \varphi \cong \tau\gamma. \quad \text{Hence} \quad \tau \perp \tau\beta' \cong \tau \perp \tau\gamma'. \quad \text{Hence} \quad \tau\beta' \cong \tau\gamma'.$$

$$\text{Hence} \quad b_{r+1} \in D(\beta') \subseteq D(\tau\beta') = D(\tau\gamma').$$

By theorem 1.12,

$$\langle\langle b_1, \ldots, b_r, d_{r+1}, \ldots, d_n \rangle\rangle \approx \langle\langle b_1, \ldots, b_r, b_{r+1}, c_{r+2}, \ldots, c_n \rangle\rangle \quad \text{for some } c_{r+2}, \ldots, c_n \in \dot{F}.$$

This establishes (A_{r+1}). The proof is complete. ∎

2 Milnor Conjecture

Definition 2.1. *Suppose F is a field. The Milnor $K-$theory is defined as:*

$$K_{\bullet}^{M}(F) = \oplus_{n=0}^{\infty} K_n(F) := \frac{T_{\mathbb{Z}}(\dot{F})}{(\langle a \otimes (1-a) : a \in \dot{F} \rangle)}$$

where $T_{\mathbb{Z}}(\dot{F})$ denotes the tensor algebra of $\dot{F}$ over $\mathbb{Z}$. Note $K_0^M(F) = \mathbb{Z}, K_1^M(F) = \dot{F}$.

Proposition 2.2. *Let F be a field and $I := I(F) \subseteq W(F)$ be the fundamental ideal. Consider the graded algebra $R(I) := \bigoplus_{n=0}^{\infty} \frac{I^n}{I^{n+1}}$ Then, there is a ring homomorphism of graded rings*

$$\varphi : T_{\mathbb{Z}}(F) \longrightarrow R(I).$$

Proof. First, note $R_0(I) = \mathbb{Z},\ R_1(I) = I$. Define an map

$$\varphi_0 : K_1^M(F) = \dot{F} \longrightarrow R(I) \qquad \textit{by} \qquad \varphi_0(a) = [\langle\langle -a \rangle\rangle] := [\langle 1, -a \rangle]$$

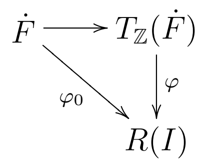
We wish to prove that this is a homomorphism of $\mathbb{Z}-$modules. We have

$$0 = [\langle 1, -a \rangle \langle 1, -b \rangle] = [\langle 1, -a, -b, ab \rangle].$$

So,

$$\varphi_0(ab) = [\langle 1, -ab \rangle] = [\langle 1, -a, -b, ab \rangle] + [\langle 1, -ab \rangle]$$
$$= [\langle 1, 1, -a, -b \rangle] + [\langle ab, -ab \rangle] = [\langle 1, 1, -a, -b \rangle] = \varphi_0(a) + \varphi_0(b)$$

This established that φ_0 is $\mathbb{Z}-$linear homomorphism. So, by unversal property of tensor algebra, φ_0 extends to φ as follows:



The proof is complete. ∎

Proposition 2.3. *With Notations as in (2.2), we have*

$$\forall \ \ a \in \dot{F} \quad \varphi(a \otimes (1-a)) = 0$$

Proof. Since φ is a homomorphism of rings,

$$\varphi(a \otimes (1-a)) = \varphi(a)\varphi(1-a) = [\langle 1, -a \rangle][\langle 1, -(1-a) \rangle] = [\langle 1, -a, -(1-a), a(1-a) \rangle]$$

Since $1 \in D(\langle a, 1-a \rangle)$, we have $\langle a, 1-a \rangle \cong \langle 1, a(1-a) \rangle$. Adding $\langle -a, -(1-a) \rangle$ to both sides,

$$\textit{in} \ \ W(F) \quad 0 = \overline{\langle a, 1-a \rangle} + \overline{\langle -a, -(1-a) \rangle} = \overline{\langle 1, a(1-a) \rangle} + \overline{\langle -a, -(1-a) \rangle}$$
$$= \overline{\langle 1, -a, -(1-a), a(1-a) \rangle}.$$

The proof is complete. ∎

Theorem 2.4. *There is a homomorphism*

$$\psi : K_\bullet^M(F) \longrightarrow R(I) \qquad \textit{sending} \quad \langle a \rangle \mapsto [\langle\langle -a \rangle\rangle]$$

of graded rings.

Proof. Follows from propostions 2.2, 2.3. The proof is complete. ∎

Theorem 2.5. *In fact ψ factors through*

$$\Psi : \frac{K_\bullet^M(F)}{2K_\bullet^M(F)} \longrightarrow R(I) \quad \ni \quad \begin{array}{ccc} K_\bullet^M(F) & \longrightarrow & \frac{K_\bullet^M(F)}{2K_\bullet^M(F)} \\ & \searrow^{\psi} & \downarrow{\scriptstyle \Psi} \\ & & R(I) \end{array} \qquad \textit{commutes.}$$

Proof. For $2 \in \mathbb{Z} = K_0(F)$ we only need to prove $\psi(2) = 0$. But $R_0(I) = \frac{W(F)}{I} = \mathbb{Z}_2$. So, the proof is complete. ∎

Milnor Conjecture: This homomorphim Ψ in theorem 2.5 is an isomorphism. The conjecture was proved by Voevodsky. So, for each n we have

$$\Psi_n : \frac{K_n^M(F)}{2K_n^M(F)} \xrightarrow{\sim} \frac{I(F)^n}{I(F)^{n+1}} \quad \text{is an isomorphism.}$$

3 Gersten Complex for $K-$theory

This is partly or mostly from paper of Milnor ([M]).

It is customary to use $\ell : \dot{F} \leftrightarrow K_1F$ by $a \mapsto \ell(a)$, and treart K_1F as an additive group. With this new notations

$$K(F) = \frac{T_{\mathbb{Z}}K_1F}{(\langle l(a) \otimes l(1-a) : a \in \dot{F}\rangle)}$$

We have

1. Clearly, $K_0(F) = \mathbb{Z}$
2. $K_n(F) = \frac{K_1F \otimes K_1F \cdots \otimes K_1F}{(\sum \ell(a_1)\ell(a_2)\cdots\ell(a_n) : \exists\ i<n\ \ni\ a_i + a_{i+1} = 1)}$

Lemma 3.1. *For $a, b \in \dot{F}$, the follwoing holds in $K_2(F)$:*

1. $a + b = 0 \Longrightarrow \ell(a)\ell(b) = 0$

2. $\ell(a)\ell(b) = -\ell(b)\ell(a)$

3. $\ell(a)\ell(a) = \ell(a)\ell(-1) = \ell(-1)\ell(a)$

4. $a + b \neq 0 \Longrightarrow \ell(a + b)\ell(-b/a) = \ell(a)\ell(b)$

Proof.

1. To prove (1), we can assume $a \neq 1$. Then $\ell(a^{-1})\ell(1 - a^{-1}) = 0$. So,

$$\ell(a)\ell(-a) = \ell(a)\ell(-a)+\ell(a)\ell(1-a^{-1}) = \ell(a)(\ell(-a)+\ell(1-a^{-1})) = \ell(a)\ell(1-a) = 0.$$

2. We use (1)

$$\ell(a)\ell(b) + \ell(b)\ell(a) = \ell(a)\ell(-a) + \ell(a)\ell(b) + \ell(b)\ell(a) + \ell(b)\ell(-b)$$

$$= \ell(a)\ell(-ab) + \ell(b)\ell(-ab) = \ell(ab)\ell(-ab) = 0.$$

3. For (3)

$$\ell(a)\ell(a) - \ell(a)\ell(-1) = \ell(a)\ell(-a) = 0$$

4. Write $c = a + b$. Then $ac^{-1} + bc^{-1} = 1$. So, $0 = \ell(ac^{-1})\ell(bc^{-1})$. We have

$$\ell(a)\ell(b) - \ell(c)\ell(b) + \ell(a)\ell(c) - \ell(c)\ell(c) = \ell(ac^{-1})\ell(b) - \ell(ac^{-1})\ell(c) = \ell(ac^{-1})\ell(bc^{-1}) = 0$$

So, solve for $\ell(a)\ell(b)$ and use (2), (3):

$$\ell(a)\ell(b) = \ell(c)\ell(b) - \ell(c)\ell(a) + \ell(c)\ell(-1) = \ell(c)\ell(-ba^{-1}))$$

The proof is complete. ∎

4 Milnor's Paper ([M])

4.1 Residue Homomorphism

Suppose A is DVR and $F = Q(A)$. Let π denote a prime, not fixed. Note

$$K_1F = \{\ell(u) + n\ell(\pi) : u \in U(A), n \in \mathbb{Z}\}$$

So,

$$K_n(F) = \left\{ \sum \ell(\pi)^r \ell(u_{r+1}) \cdots \ell(u_n) : r \geq 0,\ u_i \in U(A) \right\}$$

Theorem 4.1. *There is a unique homomorphism,* $\partial : K_n(F) \longrightarrow K_{n-1}F_0 \ \ni$

$$\Big\{\ \partial(\ell(\pi)\ell(u_2)\cdots\ell(u_n)) \ = \ell(\overline{u_2})\cdots\ell(\overline{u_n}) \qquad \forall\ u_i \in U(A),\ \pi \text{ any prime}$$

Further,

1. *In this case* $\partial(\ell(v_1)\ell(v_2)\cdots\ell(v_n)) = 0$ *whenever* $u_i \in U(A)$.
2. *This homomorphims is independent of choice of* π.

Proof. Uniquesness: Let π be any prime. K_nF is generated by $x := \ell(\pi)^r\ell(u_{r+1})\cdots\ell(u_n)$, with $r \geq 0$. If $r = 0$, then

$$\partial(l(u_1)l(u_2)\cdots l(u_n)) = \partial[l(u_1\pi)l(u_2)\cdots l(u_n)) - l(\pi)l(u_2)\cdots l(u_n))]$$
$$= l(\bar{u_2})\cdots l(\bar{u_n}) - l(\bar{u_2})\cdots l(\bar{u_n}) = 0$$

If $r \geq 2$ then by (3.1),

$$x = \ell(\pi)^r\ell(u_{r+1})\cdots\ell(u_n) = \ell(\pi)\ell(-1)^{r-1}\ell(u_{r+1})\cdots\ell(u_n)$$

So,

$$\partial(x) = \ell(-1)^{r-1}\ell(\overline{u_{r+1}})\cdots\ell(\overline{u_n})$$

is independent of π. Similalry, if $r = 1$, $\partial(x)$ is independent of π, by hypothesis.

Now we prove existence of ∂. Now, we fix a prime π. Let X be an indeterminate and consider $K_*\overline{F}[X]$ with $Xy = -yX\ \forall\ y \in K_i\overline{F}$. Given

$$\ell(\pi^{i_1}u_1), \ldots, \ell(\pi^{i_1}u_1) \in K_1F$$

define $\varphi_j \in K_j\bar{F}$ such that

$$(Xi_1 + \ell(\bar{u_1}))\cdots(Xi_n + \ell(\bar{u_n})) = X^n\varphi_0 + X^{n-1}\varphi_1 + \cdots + \varphi_n.$$

It is easy to see that $\varphi_j : K_1F \times \cdots \times K_1F \longrightarrow K_j\overline{F}$ are $n-$(multi)linear on K_1F.

Define

$$\varphi = \ell(\overline{-1})^{n-1}\varphi_0 + \ell(\overline{-1})^{n-2}\varphi_1 + \cdots + \varphi_{n-1} \in K_{n-1}\bar{F}$$

So, $\varphi\left(\ell(\pi^{i_1}u_1), \ldots, \ell(\pi^{i_1}u_1)\right)$ is multilinear, so it factors through the tensor product:

$$\begin{array}{ccc} K_1F \times \cdots \times K_1F & \longrightarrow & T^nF \\ & \searrow^{\varphi} & \downarrow \\ & & K_{n-1}\overline{F} \end{array}$$

Now suppose $\pi^{i_j}u_j + \pi^{i_{j+1}}u_{j+1} = 1$. We can assume $i = 1$. So $\pi^{i_1}u_1 + \pi^{i_2}u_2 = 1$. Assume $i_1 \le i_2$. Then, By routine calculation, it follows that $i_1 = 0 \le i_2$. So,

$$u_1 + \pi^{i_2}u_2 = 1$$

1. If $i_2 = 0$ then $u_1 + u_2 = 1$ and $\ell(\overline{u_1})\ell(\overline{u_2}) = 0$. So, $(xi_1 + \ell(\overline{u_1}))((xi_2 + \ell(\overline{u_2})) \equiv 0$. So, $\varphi_j(*, \ldots, *) = 0$ for all j.
2. It $i_2 > 0$, then $\ell(\overline{u_1}) = \ell(\bar{1}) = 0$. So, $(xi_1 + \ell(\overline{u_1}))((xi_2 + \ell(\overline{u_2})) \equiv 0$. So, $\varphi_j(*, \ldots, *) = 0$ for all j.

This proves that all φ_j factor through K_jF. Also φ is defined on K_nF. We define $\partial = \varphi$. For $x = \ell(\pi u_1)\ell(u_2)\cdots\ell(u_n)$, $i_1 = 1, i_2 = 0, \ldots, i_{=}0$. So, the defining equation gives

$$(X + \ell(\overline{u_1}))\ell(\bar{u_2})\cdots\ell(\bar{u_n}) = \sum X^r\varphi_{n-r} \Longrightarrow$$

$$\varphi_{n-1}(x) = \ell(\bar{u_2})\cdots\ell(\bar{u_n}), \quad \varphi_j(x) = 0 \quad \forall j \neq n-1.$$

Since, it does not nvolve u_1, this is indpendent of π. So, $\partial(x) = \varphi(x)$ is as desired. ∎

Remark. Note, if $v : F \longrightarrow \mathbb{Z}$ denotes the valuation, then

$$\delta(\ell(a)\ell(u_2)\cdots\ell(u_n)) = v(a)\ell(\overline{u_2})\cdots\ell(\overline{u_n}))$$

Lemma 4.2. *Let* (A,π) *be a DVR. There is a uniques ring homomorphism*

$$\psi : K_*F \longrightarrow K_*\overline{F} \qquad \textit{where} \quad \psi(l(\pi^{i_1}u_i)\cdots(\pi^{i_n}u_n) = l(\overline{u_1})\cdots l(\overline{u_n}).$$

This depends on the prime π.

Proof. Similar to that of theorem 4.1. ∎

4.2 Milnon'r Theorem

Now let F be a field and $F(t)$ be a field of rational functions. Each monic irreducible polynomial $\pi \in F[t]$ gives rise to a $(\pi)-$adic valaution on $F(t)$, with residue field $F_\pi = F[t]/(\pi)$. This provides a surjection

$$\partial_\pi : K_n(F(t)) \longrightarrow K_{n-1}F_\pi$$

Theorem 4.3. *There is a split exact sequence:*

$$0 \longrightarrow K_nF \longrightarrow K_nF(t) \xrightarrow{\partial} \bigoplus K_{n-1}F_\pi \longrightarrow 0 \qquad \textit{where} \quad \partial = \oplus\partial_\pi$$

and the direct sum extends over all non-zero prime ideals.

proof. For $n = 1$ the $\partial = \oplus Ord_\pi$ It is easy so see that

$$Ord_\pi(f) = 0 \quad \forall\pi \Rightarrow f \in F^\bullet.$$

Keep n fixed. Let $L_d = L_d^n \subseteq K_nF(t)$ be the subsgroup generated by products $l(f_1)l(f_2)\cdots l(f_n)$ such that $degree(f_i) \leq d$. Clearly,

$$L_0 \subseteq L_1 \subseteq L_2 \subseteq \cdots, \qquad K_nF(t) = \bigcup L_d.$$

By lemma 4.2, we have

$$fix\ any\ linear\ monic \quad \pi.\ Then, \quad \begin{array}{ccccc} K_nF & \twoheadrightarrow & L_0 & \hookrightarrow & K_nF(t) \\ \| & & & \swarrow_{\psi_\pi} & \\ K_nF & & & & \end{array} \quad commutes.$$

So, $K_nF \approx L_0$ is a direct summand. Since this is a split, we only need to ptove that the sequence is exact.

Lemma 4.4. *Let π be a monic prime with degree$(\pi) = d$.*

1. Given an element $y \in F[t]/(\pi)$, by division algorithm, the is an unique $g \in F[t]$ with $\bar{g} = y$ and $degree(g) < d$.

2. *There is a unique homomorphism*

$$h_\pi : K_{n-1}\left(\frac{F[t]}{(\pi)}\right) \longrightarrow \frac{L_d}{L_{d-1}}, \quad where \quad h_\pi(l(\bar{g_2})\cdots l(\bar{g_n})) = \overline{l(\pi)l(g_2)\cdots l(g_n)}$$

with $degree(g_i) < d$.

Proof. Consider the same map on

$$(K_1 F[t]/(\pi))^n \qquad (l(\bar{g_2}), \cdots, l(\bar{g_n})) \mapsto \overline{l(\pi)l(g_2)\cdots l(g_n)}.$$

First, we prove it is mulitlinear. We will only prove for the first coordiante. Suppose

$$g_2 = g_2' g_2" \quad mod(\pi) \qquad degree(g_2), degree(g_2'), degree(g_2") < d.$$

So,

$$g_2 = \pi f + g_2' g_2" \qquad where \qquad degree(f) < d.$$

If $f = 0$, then

$$\overline{l(\pi)l(g_2)\cdots l(g_n)} = \overline{l(\pi)(l(g_2') + l(g_2"))\cdots l(g_n)} = \overline{l(\pi)l(g_2')\cdots l(g_n)} + \overline{l(\pi)l(g_2")\cdots l(g_n)}$$

If $f \neq 0$, then

$$1 = \frac{\pi f}{g_2} + \frac{g_2' g_2"}{g_2}$$

So,

$$(l(\pi) + l(f) - l(g_2))(l(g_2') + l(g_2") - l(g_2)) = 0.$$

Or

$$l(\pi)l(g_2') + l(\pi)l(g_2") - l(\pi)l(g_2)$$
$$+l(f)l(g_2') + l(f)l(g_2") - l(f)l(g_2) + l(g_2)l(g_2') + l(g_2)l(g_2") - l(g_2)l(g_2) = 0$$

Multiply by $l(g_3)\cdots l(g_n)$ and mod by L_{d-1}(only first 3 terms survive):

$$\overline{l(\pi)l(g_2')l(g_3)\cdots l(g_n)} + \overline{l(\pi)l(g_2")l(g_3)\cdots l(g_n)} - \overline{l(\pi)l(g_2)l(g_3)\cdots l(g_n)} = 0.$$

Hence

$$\overline{l(\pi)l(g_2)l(g_3)\cdots l(g_n)} = \overline{l(\pi)l(g_2')l(g_3)\cdots l(g_n)} + \overline{l(\pi)l(g_2")l(g_3)\cdots l(g_n)}$$

This establishes the desired map, at the $n-$fold tesor product level,

$$T^n\left(K_1\left(\frac{F[t]}{(\pi)}\right)\right) \longrightarrow \frac{L_d}{L_{d-1}}.$$

Also,

$$\overline{g_j} + \overline{g_{j+1}} = 1, degree(g_j) < d, degree(g_{j+1}) < d \Longrightarrow g_j + g_{j+1} = 1.$$

This completes the proof. ■

Lemma 4.5. *Let $\mathcal{P}_d$ be the set of all monic primes π of degree d Then,*

$$\partial^d = \bigoplus_{\pi\in\mathcal{P}_d} \partial_\pi : K_nF(t) \longrightarrow \bigoplus_{\pi\in\mathcal{P}_d} K_{n-1}\left(\frac{F[t]}{(\pi)}\right)$$

induces an isomorphism on $\frac{L_d}{L_{d-1}}$. Diagramatically

$$\begin{array}{ccccc} L_d & \hookrightarrow & K_nF(t) & & \\ \downarrow & & & \overset{\partial}{\searrow} & \\ \frac{L_d}{L_{d-1}} & \overset{\sim}{\longrightarrow} & \bigoplus_{\pi\in\mathcal{P}_d} K_{n-1}\left(\frac{F[t]}{(\pi)}\right) & \hookrightarrow & \bigoplus K_{n-1}\left(\frac{F[t]}{(\pi)}\right) \end{array}$$

proof. Let π be a prime and $degree(\pi) = d$. For $g \in K[t]$ with $degree(g) < d$, we have g is an unit in the DVR $K[t]_{(\pi)}$. So, $\partial_\pi(L_{d-1}) = 0$ and ∂_π factors through $\frac{L_d}{L_{d-1}}$. We also have

$$\begin{array}{ccc} K_{n-1}\left(\frac{F[t]}{(\pi)}\right) & \overset{h_\pi}{\longrightarrow} & \frac{L_d}{L_{d-1}} \\ \| & \overset{\partial_\pi}{\swarrow} & \\ K_{n-1}\left(\frac{F[t]}{(\pi)}\right) & & \end{array} \qquad \textit{and if} \quad \pi \neq \pi' \qquad \begin{array}{ccc} K_{n-1}\left(\frac{F[t]}{(\pi)}\right) & \overset{h_\pi}{\longrightarrow} & \frac{L_d}{L_{d-1}} \\ 0\downarrow & \overset{\partial_{\pi'}}{\swarrow} & \\ K_{n-1}\left(\frac{F[t]}{(\pi)}\right) & & \end{array}$$

Write $h = \bigoplus_{\pi\in\mathcal{P}_d} h_\pi$. The above shows $\partial h = Id$. If we show that h is surjective, the proof will be complete.

Generator of $\frac{L_d}{L_{d-1}}$ are given by the image of $l(f_1)\cdots l(f_s)l(g_{s+1})\cdots l(g_n)$ where $degree(f_i) = d$ and $degree(g_i) < d$. We want to prove that $\frac{L_d}{L_{d-1}}$ is generated by such expressions with $s = 1$ and f_1 is a prime. We can write

$$f_2 = -af_1 + g \qquad a \in \dot{F}, \quad degree(g) < d.$$

If $g \neq 0$, we have

$$1 = \frac{af_1}{g} + \frac{f_2}{g} \qquad so \qquad (l(a) + l(f_1) - l(g))(l(f_2) - l(g)) = 0$$

So,

$$l(f_1)l(f_2) = -l(a)l(f_2) + l(a)l(g) + l(f_1)l(g) + l(g)l(f_2) - l(g)^2$$

So,

$$l(f_1)l(f_2)\cdots l(f_s)l(g_{s+1})\cdots l(g_n) \equiv l(f_1)l(g)\cdots l(f_s)l(g_{s+1})\cdots l(g_n) + two{-}terms$$

If $g = 0$ we have $f_2 + af_1 = 0$. (It is possible that $f_1 = f_2$ and $a = -1$.) Then $(l(a) + l(f_1))l(f_2)) = 0$. So,

$$l(f_1)l(f_2) = -l(a)l(f_2)$$

So

$$l(f_1)l(f_2)\cdots l(f_s)l(g_{s+1})\cdots l(g_n) \equiv -l(f_1)l(a)\cdots l(f_s)l(g_{s+1})\cdots l(g_n)$$

By induction on s it follows that $\frac{L_d}{L_{d-1}}$ is generated by images of

$$y = l(f)l(g_2)\cdots\cdots l(g_n) \qquad degree(f_1) = d, degree(g_i) < d.$$

If f is prime, then $f = a\pi$ for some monic π. In this case,

$$y = l(f)l(g_2)\cdots\cdots l(g_n) = l(\pi)l(g_2)\cdots l(g_n) + l(a)l(g_2)\cdots l(g_n)$$

Hence

$$h_\pi(l(\overline{g_2})\cdots l(\overline{g_n})) = \overline{y}.$$

So, y is in the image of h_π.

If f is not a prime, then f factors further into polynomilas of degree $< d$. So, $y \equiv 0 \pmod{L_{d-1}}$, which is in the image of h. This complete the proof. ■

Proof of theorem 4.3: The sequence is clearly a complex. Now, let $\partial(x) = 0$. Then, $\partial_\pi(x) = 0$ for all prime π. Suppose $x \notin L_0$. If $x \neq 0$ then $x \in L_d \setminus L_{d-1}$. So, $\overline{\partial^d(x)} \neq 0$ (as in 4.5). This is a contradiction.

For the surjectivity on the right side, let $x \in K_{n-1}\left(\frac{L_d}{L_{d-1}}\right)$. Note $\overline{\partial_\pi} h_\pi(x) = x$ and $h_{\pi'}(h_\pi(x)) = 0$. So, $\partial(y) = x$ for any $y \in L_d$ that lifts x. The proof is complete. ■

5 Norm and Residue Homomorphisms

My main reference for this section is ([EKM]).

5.1 Norm Homomorphism

Recall the following.

Definition 5.1. Suppose $F \hookrightarrow L$ is a finite field extension.

1. Suppose $F \hookrightarrow L$ is a Galois extension. Then, norm is defined as

$$N_{L/F} : L \longrightarrow F \quad defined\ by \quad N_{L/F}(\alpha) = \prod_{\sigma \in Emb_F(L)} \sigma(\alpha).$$

In fact, we want to define the norm homomorphism $C_{L/F} : K_n L \longrightarrow K_n F$.

1. Suppose $L = F(y)$ be simple. Then, $L = \frac{F[T]}{(\pi)}$, where π is the irreducible polynomial of y.
2. Suppose $\alpha \in K_n(L) = K_n\left(\frac{F[T]}{(\pi)}\right)$. By Milnor's theorem 4.3, there is a $\beta \in K_{n+1}F(T)$ such that

$$\partial_p(\beta) = \begin{cases} \alpha & if\ p = \pi \\ 0 & otherwise \end{cases}$$

3. Let v_∞ be a discrete valuation $v_\infty : K(T) \longrightarrow \mathbb{Z}$, where $v_\infty(T^{-1}) = 1$. We set

$$c_{L/F}(\alpha) = \partial_{v_\infty}(\beta)$$

4. More formally, Recall Milnor's sequence is a split exact sequence. Let γ be a split of ∂.

5.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_{n+1}F & \longrightarrow & K_{n+1}F(T) & \xrightarrow{\partial} & \bigoplus K_n F_p & \longrightarrow & 0 \\
 & & & & \| & \overset{\gamma}{\swarrow} & \downarrow{\scriptstyle \oplus c_{F_\pi/F}} & & \\
 & & & & K_{n+1}F(T) & \xrightarrow[\partial_{v_\infty}]{} & \bigoplus K_n F & &
\end{array}$$

In fact, $c_{L/F}$ is independent of choice of γ.

Proof. Let η be another split. Then $\partial(\gamma - \eta) = 0$. So, $Image(\gamma - \eta) \subseteq K_{n+1}F$. So, $\partial_{v_\infty}(\gamma - \eta) = 0$. ∎

Now suppose $F \longrightarrow L$ be any finite extension. Then we can choose a chain of simple extensions:

$$F = F_0 \hookrightarrow F_1 \hookrightarrow \cdots \hookrightarrow F_n = L.$$

We define

$$c_{L/F} := c_{F_1/F_0} c_{F_2/F_1} \cdots c_{F_n/F_{n-1}}$$

It is stated without proof, that $c_{L/F}$ independent of this choice of the sequence of simple extensions, is well defined.

Proof. See ([GS, §7.3]).

∎

5.2 Residue Homomorphism for local rings

Suppose (A, m, k) is a local ring (excellent domain) of dimension one. Write $K = Q(A)$ the field of fractions of A. We define a groups homomorphism $W(K) \longrightarrow W(k)$ as follows.

1. Let B be the integral closure of A, in K. Then,
   (a) $Q(B) = K$, $\dim B = 1$
   (b) B is semilocal. Let $Max(B) = \{m_1, \ldots, m_r\}$.
   (c) B normal. So, B is a Dedekind domain.
   (d) So, B_{m_i} are DVR.
2. Write $k_i = B/m_i$.
3. By theorem 4.1, there are residue class maps $\Delta_i : K_n(K) \longrightarrow K_n(k_i)$
4. Note $k \hookrightarrow k_i$ is a finite extension. Then, there are norm homomorphisms $C_i := C_{K/k_i} : K_{n-1}(k_i) \longrightarrow K_{n-1}(k)$.
5. Let
$$\Delta_A = \oplus_{i=1}^{m} \Delta_i : K_n(K) \longrightarrow \oplus_{i=1}^{n} K_{n-1}(k_i)$$
$$\text{and} \quad \Psi_A = \oplus_{i=1}^{m} c_i : \oplus_{i=1}^{m} K_{n-1}(k_i) \longrightarrow K_{n-1}(k).$$

Definition 5.2. *Now define a residue class homomorphism*

$$\partial_A : K_n(K) \longrightarrow K_{n-1}(k) \quad \ni \quad \begin{array}{ccc} K_n(K) & \xrightarrow{\partial_A} & K_{n-1}(k) \\ \downarrow{\scriptstyle \Delta_A} & \nearrow{\scriptstyle \Psi_A} & \\ \oplus_{i=1}^{m} K_{n-1}(k_i) & & \end{array} \quad \text{commutes.}$$

5.3 Gersten Complex for $K-$theory

Let A be any commutative noetherian ring and $X = spec(A)$. Denote $X^{(r)} = \{\wp \in Spec(X) : height(\wp) = r\}$. Assume $d := \dim A = \dim A_m \ \forall \ m \in MAx(A)$.

1. For $\wp \in Spec(A)$ denote $\kappa(\wp :) = \frac{A_\wp}{\wp A_\wp}$.

2. Let $x \subseteq y$ be two primes, with $height(y) = height(x) + 1$. By (5.2), there is a residue class homomorphism
$$\partial_y^x : K_n(\kappa(x)) \longrightarrow K_{n-1}(y)$$
For any other pairs of prime ideals (x, y), define $\partial_y^x : K_n(\kappa(x)) \longrightarrow K_{n-1}(y) = 0$.

3. Accordingly, ∂_y^x induce homomorphisms
$$\partial_n = \oplus \partial_y^x : \coprod_{x \in X^{(n)}} K_n(\kappa(x)) \longrightarrow \coprod_{x \in X^{(n+1)}} K_n(\kappa(x)) \quad \forall \ n.$$
We deonte
$$C^n(X) := C_{d-n}(X) := \coprod_{x \in X^{(n)}} K_n(\kappa(x))$$
Also, denote
$$C(X) = \oplus C^n(X)$$
Further, denote
$$C_n^p(X) := C_{d-p,n}(X) := \coprod_{x \in X^{(p)}} K_{d-p+n}(\kappa(x)) \quad i.e \ C_{r,n}(X) := \coprod_{x \in X_{(r)}} K_{r+n}(\kappa(x))$$
So, we have
$$\partial_n : C^n(X) \longrightarrow C^{n+1}(X)$$

4. For all integers $n \geq 0$, this gives rise to a sequences $\mathcal{K}^n$:
$$0 \longrightarrow \coprod_{x \in X^{(0)}} K_n(\kappa(x)) \longrightarrow \coprod_{x \in X^{(1)}} K_{n-1}(\kappa(x)) \longrightarrow \cdots \longrightarrow$$
$$\coprod_{x \in X^{(r)}} K_{n-r}(\kappa(x)) \xrightarrow{\partial_r^n} \cdots \longrightarrow \coprod_{x \in X^{(n)}} K_0(\kappa(x)) \longrightarrow 0$$

In C^r_k-notation this sequence is written as:

$$\begin{array}{ccccccccccc} 0 \longrightarrow & C_{d,n-d}(X) & \longrightarrow & C_{d-1,n-d}(X) & \longrightarrow \cdots \longrightarrow & C_{d-r,n-d}(X) & \xrightarrow{\partial^n_r} & C^{d-n,n-d}(X) & \longrightarrow 0 \\ & \| & & \| & & \| & & \| & \\ 0 \longrightarrow & C^0_{n-d}(X) & \longrightarrow & C^1_{n-d}(X) & \longrightarrow \cdots \longrightarrow & C^r_{n-d}(X) & \longrightarrow \cdots & C^n_{n-d}(X) & \longrightarrow 0 \end{array}$$

This is known as $n^{th}-$ Gersten sequence for Milnor $K-$theory. This is also known as $n^{th}-$Rost sequence.

Theorem 5.3. Suppose A is an excellent ring. Then

1. The Gersten sequence is a complex.

2. (**Conjecture**) If A is regular local, then Gersten sequence is exact at degree $n \geq 1$. Also, $\ker(\partial^n_0) = K_n(A)$, which we did not define.

   When A contains an infinite field, this conjecture is known to be true. It was proved by Moritz Kerz ([K]).

A E

xcellent Rings

Definition A.1. *A ring A is called excellent, if the follwing conditions are satified:*

1. *A is noetherian,*

2. *A is universally catenary,*

3. *(G$-$rings): $\forall\ \wp \in Spec(A)$ the homomorphism $A_\wp \longrightarrow \hat{A}$ is regular.*

4. *$(J-2)$: Given any finitley generate $A-$algebra B, the locus $reg(B)$ is open.*

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