Chow Groups and Chow Witt

Satya Mandal

University of Kansas, Lawrence KS 66045 USA

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1 Chow Groups

Suppose X is a noetherian scheme. We will assume $X = Spec(A)$, where A is a noetherian commutative ring with $\dim A = d$. We start with Gersten Complexes: $\mathcal{K}^n$:

$$0 \longrightarrow \coprod_{x \in X^{(0)}} K_n(\kappa(x)) \longrightarrow \coprod_{x \in X^{(1)}} K_{n-1}(\kappa(x)) \longrightarrow \cdots \longrightarrow$$

$$\coprod_{x \in X^{(r)}} K_{n-r}(\kappa(x)) \xrightarrow{\partial_r^n} \cdots \longrightarrow \coprod_{x \in X^{(n)}} K_0(\kappa(x)) \longrightarrow 0$$

For now, we are interested in the right tail of these sequences:

$$\begin{array}{ccccc} \bigoplus_{x \in X^{(r)}} K_1(\kappa(x)) & \xrightarrow{\partial_1^n} & \bigoplus_{x \in X^{(n)}} K_0(\kappa(x)) & \longrightarrow & 0 \\ \| & & \| & & \\ \bigoplus_{x \in X^{(r)}} \kappa(x)^\times & \xrightarrow{\partial_1^n} & \bigoplus_{x \in X^{(n)}} \mathbb{Z}x & \longrightarrow & 0 \end{array}$$

The last term being the free $\mathbb{Z}-$module.

Preliminary Definitions and Comments

1. The last term, as in many books ([EKM, F]), is denoted by

$$Z^n(X) := Z_{d-n}(X) := \bigoplus_{x \in X^{(n)}} \mathbb{Z}x$$

It is called the group of codimension $n-$cycles or the group of dimension $(d-n)-$cycles of X.

2. Also,

$$\mathcal{R}at^n(X) := Image(\partial_0^n) \subseteq Z^n(X)$$

is called the subgroup of rational equivalence of codimension $n-$cycles.

3. The cokernel

$$CH^n(X) := CH_{d-n}(X) := \frac{Z^n(X)}{\mathcal{R}at^n(X)}$$

is called the Chow group of codimension $n-$cycles. Notations $A^n(X) := A_{d-n}(X) := CH^n(X)$ are also used.

4. **Superscript and subscript:** In a sense, it is nicer to follow the subscript notation $CH_k(X)$. Because dimenation of a cycle, or of a prime ideal x does not depend of the embedding space. That means: let $z = \sum r_i x_i Z_m(X)$ be a cycle of dimension $m = d = n$. Suppose $y \subseteq x_i \subseteq A$ be another prime. and $Y = V(y)$. Then, $z \in Z_m(Y) \subseteq Z_m(X)$. However $z \in Z^n(X)$ and $z \in Z^{\dim y - m}(Y)$.

   On the other hand, $CH(X) = \oplus_{n=0}^{d} CH^n(X)$, in some cases, would have a graded ring structure.

5. Now, suppose (A, m) is regular local. Take any set of generators $m = (f_1, \ldots, f_d)$. Then $x = (f_1, \ldots, f_{d-1}) \in X^{(d-1)}$. Then, $m = \partial_1(\overline{f_d})$. Therfore, $CH^d(X) = 0$.

Lemma 1.1. *Suppose $F \hookrightarrow L$ is a finite field extension. Then, at degree zero, the norm homomorphism is givnen by*

$$c_{L/F} : \mathbb{Z} = K_0(L) \longrightarrow K_0(F) = \mathbb{Z} \qquad 1 \mapsto [L : F].$$

Proof. It is enough to assume $L = F(y) = \frac{F[T]}{p(T)}$. Consider the diagram

$$\begin{array}{ccc} K_1(F(T)) & \xrightarrow{\partial_p} & K_0\left(\frac{F[T]}{p(T)}\right) \\ & \searrow^{\partial_\infty} & \downarrow c_{L/F} \\ & & K_0\left(\frac{F[T^{-1}]}{T^{-1}}\right) = \mathbb{Z} \end{array}$$

Note $c_{L/F}(1) = c_{L/F}\partial_p(p) = \partial_\infty(p) = degree(p)$.

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Lemma 1.2. *Let $A \longrightarrow B$ be a homomorphism of commutative noetherian rings. Let M be an $B-$module with $\ell_B(M) < \infty$. Then,*

$$\ell_A(M) = \sum_{m \in Max(B)} \ell_{B_m}(M_m)\ell_A\left(\frac{B}{m}\right)$$

(Both sides can be infinite.)

Proof. Take a decomposition

$$0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_r = M, \qquad with \quad \frac{M_i}{M_{i-1}} \cong \frac{B}{m_i}, \quad m_i \in Max(B).$$

Since both sides are additive in M, we can assume $M = \frac{B}{m}$ with $m \in Max(B)$. In this case,

$$LHS = \ell_A\left(\frac{B}{m}\right), \quad and \quad RHS = \ell_{B_m}\left(\frac{B}{m}\right)\ell_A\left(\frac{B}{m}\right) = \ell_A\left(\frac{B}{m}\right)$$

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Lemma 1.3. *Let $X = Spec(A)$ be as above. Let $x = \wp_0 \in X^{(r)}$ and $f \in A \in \frac{A}{\wp_0}$. Then,*

$$\partial_1(f) = \sum_{\wp \in X^{(r+1)} \cap V(\wp_0)} \ell\left(\frac{A_\wp/\wp_0}{(f)}\right)\wp = \sum_{\wp_i \in MinV(f)} \ell\left(\frac{A_{\wp_i/\wp_0}}{(f)}\right)\wp_i$$

Proof. Replacing A by $\frac{A}{\wp_0}$ we can assume $A = \frac{A}{\wp_0}$ is an integral domain and $\wp_0 = 0$. Write $K = Q(A)$ the fraction field. For all $\wp \in X^{(1)}$, we need to consider that maps

$$\partial_\wp : K_1(K) = \kappa(0)^\times \longrightarrow K_0(\kappa(\wp)) = \mathbb{Z} \quad \textit{and} \quad \textit{prove} \quad \partial_\wp(f) = \ell\left(\frac{A_\wp}{(f)}\right).$$

If $f \notin \wp$ then f represents a unit in $A_\wp$ and that $\Longrightarrow \partial_\wp(f) = 0$. Now let $f \in A_\wp$. By replacing A by $A_\wp$, we assume (A, m) is local.

Let B be the integral closure of A in K. Let $m_1, \ldots, m_r$ be the maximal ideals of B. $\partial_\wp$ was defined by the cummuting diagram

$$\begin{array}{ccc} K_n(K) & \xrightarrow{\partial_\wp} & K_0(k) \\ \downarrow{\scriptstyle \Delta_B} & \nearrow{\scriptstyle \Psi_A} & \\ \oplus_{i=1}^m K_0(k_i) & & \end{array} \quad \text{with} \quad k_i = A/m_i.$$

We have

$$\Delta_A(F) = \oplus ord_{B_{m_i}}(f) = \oplus \ell\left(\frac{B_{m_i}}{(f)}\right)$$

So,

$$\partial_\wp(f) = \Psi_A \Delta_B(f) = \sum \left(\frac{B_{m_i}}{(f)}\right) c_{k_i/k} = \sum \left(\frac{B_{m_i}}{(f)}\right)[k_i : k] = \sum \left(\frac{B_{m_i}}{(f)}\right)\ell_A(B/m_i)$$

By lemma 1.2, $\partial_\wp(f) = \ell_A\left(\frac{B}{fB}\right)$. Since $\frac{B}{A}$ is an $A-$module of finite length $m^n\frac{B}{A} = 0$. Hence, $f^n\frac{B}{A} = 0$. We have an exact sequence

$$0 \longrightarrow \frac{A \cap fB}{fA} \longrightarrow \frac{A}{fA} \longrightarrow \frac{B}{fB} \longrightarrow \frac{B}{A} \longrightarrow 0$$

It follows

$$\partial_\wp(f) = \ell_A\left(\frac{B}{(f)}\right) = \ell_A\left(\frac{A}{(f)}\right)$$

2 Chow Witt Groups

1. Recall the n^{th} Milnor/Gersten K-theory complex (also called the n-th Rost complex defined as

$$\mathcal{K}^n \qquad 0 \longrightarrow \oplus_{x\in X^{(0)}} K_n\left(k(x)\right) \longrightarrow \oplus_{x\in X^{(1)}} K_{n-1}\left(k(x)\right) \longrightarrow \oplus_{x\in X^{(2)}} K_{n-2}\left(k(x)\right)$$

$$\cdots \longrightarrow \oplus_{x\in X^{(n-1)}} K_1\left(k(x)\right) \longrightarrow \oplus_{x\in X^{(n)}} K_0\left(k(x)\right) \longrightarrow 0$$

2. Other than the $\mathcal{K}^n$ complexes, we consider the following complexes, for ieach $n \geq 0$, from the Witt-side:

$$\mathcal{W}^n \qquad 0 \longrightarrow \oplus_{x\in X^{(0)}} I^n\left(k(x)\right) \longrightarrow \oplus_{x\in X^{(1)}} I^{n-1}\left(k(x)\right) \longrightarrow \oplus_{x\in X^{(2)}} I^{n-2}\left(k(x)\right)$$

$$\cdots \longrightarrow \oplus_{x\in X^{(n-1)}} I^1\left(k(x)\right) \longrightarrow \oplus_{x\in X^{(n)}} I^0\left(k(x)\right) \longrightarrow \oplus_{x\in X^{(n+1)}} I^{-1}\left(k(x)\right)$$

   Here for $r > 0$ we denote $I^{-r}(k) := W(k)$.

3. The $r^{th}-$term will be denoted by

$$W_r^n(X) = \oplus_{x\in X^{(r)}} I^{i-r}\left(k(x)\right)$$

4.

5. By Voevodsky's theorem, for $r \geq 0$ we have

$$\frac{K_r^M(k(x))}{2K_r^M(k(x))} \xrightarrow{\sim} \frac{I^r(k(x))}{I^{r+1}(k(x))}.$$

6. The $r^{th}-$term is given by

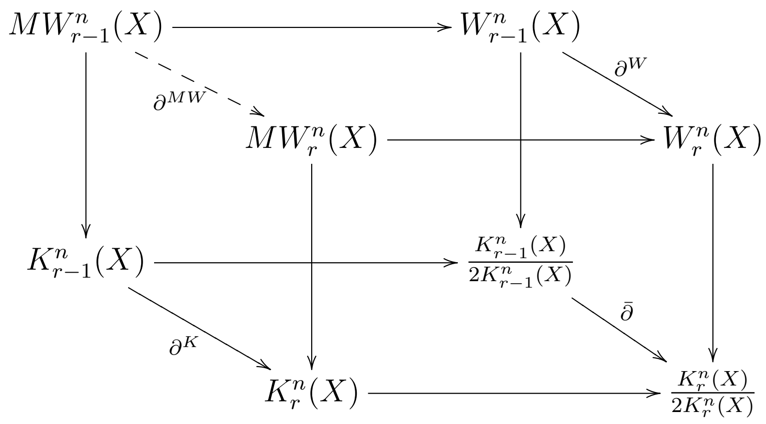
$$\begin{array}{ccccc} MW_r^n(X) & \longrightarrow & & & \oplus_{x\in X^{(r)}} I^{n-r}(x) \\ \downarrow & & & & \downarrow \\ \oplus_{x\in X^{(r)}} K_{n-r}(x) & \longrightarrow & \oplus_{x\in X^{(r)}} \frac{K_{n-r}(x)}{2K_{n-r}(x)} & \xrightarrow{\sim} & \oplus_{x\in X^{(r)}} \frac{I^{n-r}(x)}{I^{n-r+1}(x)} \end{array}$$

7. By taking the fiber product of these two sets of complexes, for $n \geq 0$, we get conplexes $MW^n_\bullet(X)$. We have the fiber product diagram:

$$\begin{array}{ccccc}
MW(X)^n_r(X) & \longrightarrow & & & \oplus_{x\in X^{(r)}} I^{n-r}(x) \\
\downarrow & & & & \downarrow \\
\oplus_{x\in X^{(r)}} K_{n-r}(x) & \longrightarrow & \oplus_{x\in X^{(r)}} \frac{K_{n-r}(x)}{2K_{n-r}(x)} & \xrightarrow{\sim} & \oplus_{x\in X^{(r)}} \frac{I^{n-r}(x)}{I^{n-r+1}(x)}
\end{array}$$

Note the middle isomorphism is not necessary to defin $MW^n(X)$.

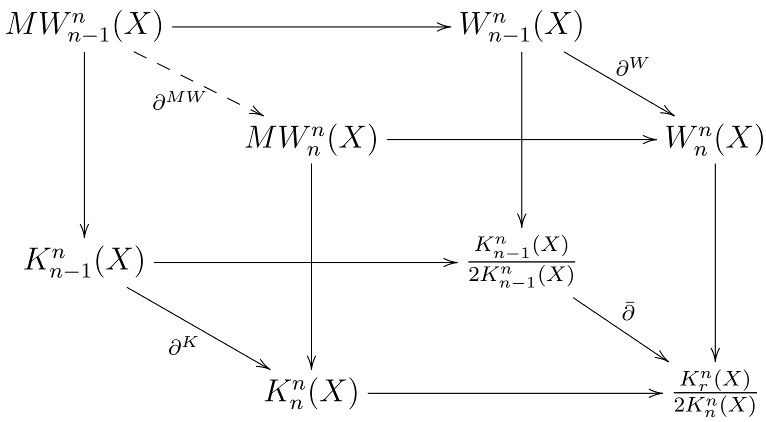
8. The properties of fiber product diagrams diggeremtials of the complex $MW^n_\bullet$:



Definition 2.1. *The Chow-Witt group*

$$\widetilde{CH}^n(X) := H_n(MW^n_\bullet)$$

n^{th}−homology of the $MW^n_\bullet$ The respective diagram is:



Some clarifications: The upper right, upper left and lower left complexes, respectively looks like:

$$\begin{array}{ccccc} \oplus_{x\in X^{(n-1)}} I^1\left(k(x)\right) & \longrightarrow & \oplus_{x\in X^{(n)}} W\left(k(x)\right) & \longrightarrow & \oplus_{x\in X^{(n+1)}} W\left(k(x)\right) \\ \uparrow & & \uparrow & & \wr\uparrow \\ MW^n_{n-1}(X) & \longrightarrow & MW^n_n(X) & \longrightarrow & MW^n_{n+1}(X) \\ \downarrow & & \downarrow & & \downarrow \\ \oplus_{x\in X^{(n-1)}} K_1\left(k(x)\right) & \longrightarrow & \oplus_{x\in X^{(n)}} K_0\left(k(x)\right) & \longrightarrow & 0 \end{array}$$

Lemma 2.2. *Using Voevodsky's theorem, it follows that all the vertical maps are surjective.*

Proof. Use Voevodsky. ∎

Computing homologies, we have surjective homomorphisms:

$$H_n(\mathcal{W}^n_\bullet) \twoheadleftarrow \widetilde{CH}^n(X) \twoheadrightarrow CH^n(X)$$

Usually, $n = d : \dim X$ (zero cycle case, i.e. that of points, height= n) is easier:

$$\oplus_{x\in X^{(d-1)}} I^1\left(k(x)\right) \longrightarrow \oplus_{x\in X^{(d)}} W\left(k(x)\right) \longrightarrow 0$$

$$\oplus_{x\in X^{(d-1)}} K_1\left(k(x)\right) \longrightarrow \oplus_{x\in X^{(d)}} K_0\left(k(x)\right) \longrightarrow 0$$

Again, in this case of $n = d = \dim X$, resulats are easier to obtian.

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