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Derived Witt group formalism

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ABSTRACT

In this article we establish some formalism of derived Witt theory for resolving subcategories of abelian categories. Results directly apply to noetherian schemes.

1. Introduction

Suppose X is a noetherian scheme with $\dim X = d$ and $D^b(\mathscr{V}(X))$ denotes the bounded derived category of locally free sheaves (*see Notations 2.1 for other details*). The main objective of this article and the preceding article [12] is to study Witt groups of the subcategory $D^b_{\mathbb{M}(X)}(\mathscr{V}(X))$ of complexes $\mathcal{E}_\bullet$ in $D^b(\mathscr{V}(X))$ whose homologies $\mathcal{H}_i(\mathcal{E}_\bullet)$ have finite locally free dimension, particularly when X is non-regular. Such a study differs in spirit from the existing literature because $D^b_{\mathbb{M}(X)}(\mathscr{V}(X))$ does not have a triangulated structure and the Witt group has to be suitably defined. Even the fact that $D^b_{\mathbb{M}(X)}(\mathscr{V}(X))$ is closed under duality is non-trivial and is a result (Section 4) in this article.

The focus of this article is the formalism of dévissage type of theorems for Witt groups in the context of resolving subcategories of abelian categories, which encompasses similar Witt theory for noetherian schemes X. In [12], analogous results for affine Cohen–Macaulay schemes $X = Spec(A)$ were proved and some of the methods are similar to that in [12].

The foundational motivation for this line of research comes from the fact that K-theoretically, the category $\mathscr{V}(X)$ of locally free sheaves is equivalent to the category $\mathbb{M}(X)$ of coherent $\mathcal{O}_X$-modules with finite $\mathscr{V}(X)$-dimension (see [15, 2.3.12]). In the context of obstruction theory of projective modules, the motivation comes from the fact that, for a locally free sheaf $\mathcal{E}$, its obstruction class $e(\mathcal{E})$ (i.e. Euler class), in the Chow–Witt group or Grothendieck Witt group [9,8,14], is given by its Koszul complex $\mathcal{K}_\bullet$, which lies in $D^b_{\mathbb{M}(X)}(\mathscr{V}(X))$. Further, when $X = Spec(A)$ is affine, the image of the homomorphism, $E(A) \longrightarrow \widetilde{CH}^d(X)$, from the Euler class group $E(A)$ [7] to the Chow–Witt group $\widetilde{CH}^d(X)$, is given

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by the forms (Koszul complexes) in $D^b_{\mathbb{M}(X)}(\mathscr{V}(X))$. If X is regular $D^b_{\mathbb{M}(X)}(\mathscr{V}(X)) = D^b(\mathscr{V}(X))$ and when X is non-regular, both $D^b_{\mathbb{M}(X)}(\mathscr{V}(X))$ and $D^b(\mathscr{V}(X))$ could be a candidate to take the place of $D^b(\mathscr{V}(X))$, depending on the objective. Extending Balmer's result [2], it would not be difficult to prove that $W(X) \cong W^0(D^b_{\mathbb{M}(X)}(\mathscr{V}(X))) \cong W^0(D^b(\mathscr{V}(X)))$. While the category $D^b_{\mathbb{M}(X)}(\mathscr{V}(X))$ may be worthwhile to look at, lack of literature on this may be due to the crucial fact that $D^b_{\mathbb{M}(X)}(\mathscr{V}(X))$ does not seem to fit into any well studied formalism (e.g. triangulated, *dg*-category). The category $D^b_{\mathbb{M}(X)}(\mathscr{V}(X))$ occupies an important place in the study of the Witt theory, Chow–Witt theory, Euler class theory and their relationship.

The concept of resolving subcategories dates back to the paper of Auslander and Bridger [1]. More recently, there has been considerable amount of activities (e.g. [16,17]) on resolving subcategories of the category $Mod(A)$ of finitely generated modules over noetherian commutative rings A. Many of these are directed toward the classification of such resolving subcategories under various conditions, which is encompassed by similar classification of variety of types of subcategories of the module categories. The work of Benson, Iyengar and Krause (e.g. [6]) would be one of the stimulus, where they consider subcategories of group algebras. Not much is available in the literature regarding resolving subcategories of abelian categories or that of the category $Coh(X)$ of coherent sheaves over noetherian non-affine schemes X. However, informed readers would know that the concept of resolvability is fairly prevalent in K-theory and other areas (e.g. see [11], [15, 2.3.12]). This provides further justification for formalism. While Witt theory of noetherian schemes is of fundamental interest to many, formalism on resolving subcategories unifies the theory and has its significance by its own rights. In this introduction, we will state the formal versions of our results (*as opposed to noetherian scheme-versions*), for resolving subcategories. Before we do that, we set up some notations.

Suppose $\mathscr{V}$ is a resolving subcategory (see Definition 3.1) of an abelian category $\mathcal{C}$. Let ω be an object in $\mathcal{C}$ with an injective resolution. For objects M in $\mathcal{C}$, denote $M^* := Mor_{\mathcal{C}}(M, \omega)$. Assume $\mathscr{V}$ inherits a duality structure from * and $\mathscr{V}$ is totally ω-reflexive (see Definition 7.1). The following is a list of a few notations and facts.

1. Let $\mathbb{M}(\mathscr{V}, \omega)$ denote the full subcategory of objects in $\mathcal{C}$ with finite $\mathscr{V}$-dimension. Assume $d := \max\{\dim_{\mathscr{V}} M : M \in \mathbb{M}(\mathscr{V}, \omega)\} < \infty$.
2. Let $\mathcal{A}(\omega) := \{M \in \mathbb{M}(\mathscr{V}, \omega) : Ext^i(M, \omega) = 0 \ \forall i < d\}$. It follows that $M \mapsto M^\vee := Ext^d(M, \omega)$ is a duality on $\mathcal{A}(\omega)$, to be denoted by $\tilde{\varpi}$.
3. $D^b(\mathscr{V})$ would denote the bounded derived category of complexes of objects in $\mathscr{V}$ and $D^b_{\mathcal{A}(\omega)}(\mathscr{V})$ would be the subcategory of complexes with homologies in $\mathcal{A}(\omega)$.
4. There is a canonical functor $\zeta : \mathbb{M}(\mathscr{V}, \omega) \longrightarrow D^b(\mathscr{V})$ given by (choices of) $\mathscr{V}$-resolutions (see Proposition 3.3). This is, indeed, a standard fact for familiar categories.
5. The restriction of $\zeta : \mathcal{A}(\omega) \longrightarrow D^b_{\mathcal{A}(\omega)}(\mathscr{V})$ is a duality preserving functor.

First, we prove isomorphisms of Witt groups, as follows:

$$W\big(\mathcal{A}(\omega), {}^\vee, \tilde{\varpi}\big) \xrightarrow{\sim} W\big(D^b_{\mathcal{A}(\omega)}(\mathcal{A}(\omega), {}^\vee, \tilde{\varpi})\big) \xrightarrow{\sim} W\big(D^b(\mathcal{A}(\omega), {}^\vee, \tilde{\varpi})\big).$$

The same holds for skew duality also. It is a consequence of a theorem of Balmer [4] that the composition of these two homomorphisms is an isomorphism. We further prove that the functor ζ induces isomorphism of Witt groups

$$W_{St}\big(\mathcal{A}(\omega)\big) \xrightarrow{\sim} W^d\big(D^b_{\mathcal{A}(\omega)}(\mathscr{V}), *, 1, \varpi\big), \qquad W^-_{St}\big(\mathcal{A}(\omega)\big) \xrightarrow{\sim} W^{d-2}\big(D^b_{\mathcal{A}(\omega)}(\mathscr{V}), *, 1, \varpi\big)$$

where subscript "St" corresponds to "standard" sign convention of the duality. Also for $n = d-1, d-3$, we have $W^n(D^b_{\mathcal{A}(\omega)}(\mathscr{V}, *, 1, \varpi)) = 0$. By 4-periodicity all the shifted Witt groups are determined.

These results can be applied to noetherian schemes X, with $\mathcal{C} = Coh(X)$ and $\mathscr{V}$ as the subcategory of locally free sheaves on X, provided $\mathscr{V}$ is a resolving subcategory. This will be the case for a wide variety of schemes X (see [10]), including those that have an ample invertible sheaf. In these applications, we assume $d := \dim X = \max\{depth(\mathcal{O}_{X,x}) : x \text{ is a closed point}\}$ (see Remark 6.5). As a consequence, the following decomposition theorem follows.

Theorem 1.1. *Suppose X is a noetherian scheme, with $\dim X = d$, as in Notations 2.1 and $X^{(d)}$ will denote the set of all closed points of codimension d in X. We assume $d = \max\{depth(\mathcal{O}_{X,x}) : x \text{ is a closed point}\} = \dim X$. Then, the homomorphisms*

$$W^d\big(D^b_{\mathcal{A}(X)}\big(\mathscr{V}(X)\big), *, 1, \varpi\big) \xrightarrow{\sim} \bigoplus_{x \in X^{(d)}} W^d\big(D^b_{\mathcal{A}(\mathcal{O}_{X,x})}\big(\mathscr{V}(\mathcal{O}_{X,x})\big), *, 1, \varpi\big) \quad \text{and}$$

$$W^{d-2}\big(D^b_{\mathcal{A}(X)}\big(\mathscr{V}(X)\big), *, 1, \varpi\big) \xrightarrow{\sim} \bigoplus_{x \in X^{(d)}} W^{d-2}\big(D^b_{\mathcal{A}(\mathcal{O}_{X,x})}\big(\mathscr{V}(\mathcal{O}_{X,x})\big), *, 1, \varpi\big)$$

are isomorphism.

When X is regular, this is a theorem of Balmer and Walter [5]. In this case, first isomorphism follows from such a decomposition of the corresponding categories and the second isomorphism would have zeros on both sides.

We close this introduction with a few comments on the layout of this article. In the interest of broader readership, we made a choice to give complete proofs of results on Witt groups, in the special case, of noetherian schemes (Sections 5, 6) and the proofs of the results on resolving subcategories (Section 7) would follow similarly. While the main isomorphism theorem was dealt with in Section 5, we summarize our results on noetherian schemes in Section 6. In Section 7, we state our results on resolving subcategories.

2. Some notations

First, we borrow some notations from [12].

Notations 2.1. What follows would be our standard setup, throughout this article.

Throughout this paper, $X := (X, \mathcal{O}_X)$ will denote a noetherian scheme, with $\dim X = d$. We also assume 2 is invertible in $\mathcal{O}_X$. Unless stated otherwise, we assume that every coherent sheaf on X is quotient of a locally free sheaf on X. This hypothesis is satisfied in the following two cases: (1) when X is integral, locally factorial and separated (see [10, II Ex. 6.8]) and (2) when X has an ample invertible sheaf. To avoid technicalities (see Remark 6.5), we assume

$$d := \dim X = \max\big\{depth(\mathcal{O}_{X,x}) : x \text{ is a closed point}\big\}.$$

This ensures that $\mathcal{A}(X)$ has nonzero symmetric forms. We set up further notations:

1. $Coh(X)$ will denote the category of coherent $\mathcal{O}_X$-modules and $\mathscr{V}(X)$ will denote the full subcategory of locally free sheaves on X.
2. The full subcategory of objects $\mathcal{F}$ in $Coh(X)$, so that $\mathcal{F}$ has finite resolution by locally free sheaves, will be denoted by $\mathbb{M} := \mathbb{M}(X)$.
3. Also, let $\mathcal{A} := \mathcal{A}(X) \subseteq \mathbb{M}(X)$ be the full subcategory of objects $\mathcal{F} \in \mathbb{M}(X)$ such that $\mathcal{E}xt^i(\mathcal{F}, \mathcal{O}_X) = 0$ for all $i < d$. This is a local property. A coherent sheaf $\mathcal{F} \in \mathcal{A}(X)$ if and only if, for all closed points

$x \in X$, $\mathcal{F}_x$ has finite projective dimension and $Ann(\mathcal{F}_x)$ has an $\mathcal{O}_{X,x}$-regular sequence of length d (see [13]). Also note, given a short exact sequence in $Coh(X)$, if two of them are in $\mathcal{A}(X)$ (*resp.* $\mathbb{M}(X)$) then so is the third one. In particular, they are exact subcategories in $Coh(X)$. (*For our subsequent discussions, these two notations* $\mathbb{M}, \mathcal{A}$ *will be of some importance.*)

4. For any exact category $\mathcal{C}$, $Ch^b(\mathcal{C})$, $D^b(\mathcal{C})$ will denote the category of bounded chain complexes, and respectively, derived category. If $\mathcal{C}$ is a subcategory in an ambient abelian category $\mathcal{C}'$ and $\mathcal{H}$ is a subcategory of $\mathcal{C}'$, then $Ch^b_{\mathcal{H}}(\mathcal{C})$ denotes the full subcategory of $Ch^b(\mathcal{C})$ consisting of complexes with homologies in $\mathcal{H}$. The derived category of $Ch^b_{\mathcal{H}}(\mathcal{C})$ will be denoted by $D^b_{\mathcal{H}}(\mathcal{C})$, which is obtained by inverting quasi-isomorphisms in $Ch^b_{\mathcal{H}}(\mathcal{C})$. However, $D^b_{\mathcal{H}}(\mathcal{C})$ can also be viewed as the full subcategory of $D^b(\mathcal{C})$ consisting of objects from $Ch^b_{\mathcal{H}}(\mathcal{C})$. Also, $K^b(\mathcal{C})$, $K^b_{\mathcal{H}}(\mathcal{C})$ would denote the corresponding homotopy categories. For other similar notations, readers are referred to [18].
5. Denote objects $\mathcal{E}_\bullet := (\mathcal{E}_\bullet, \partial_\bullet)$ in $Ch^b(Coh(X))$ as:

$$\cdots 0 \longrightarrow \mathcal{E}_m \xrightarrow{\partial_m} \mathcal{E}_{m-1} \longrightarrow \cdots \longrightarrow \mathcal{E}_n \longrightarrow 0 \cdots \quad \text{with } m > n.$$

6. A complex $\mathcal{E}_\bullet$ is said to be supported on $[m, n]$ if $\mathcal{E}_i = 0$ unless $m \geq i \geq n$.
7. "#" will be a generic notation for the duality functor in any triangulated category. We mostly work with the shifted categories $T^n D^b(\mathscr{V}(X))$, where $n \in \mathbb{Z}$ and $\#$ is induced by $\mathcal{H}om(-, \mathcal{O}_X)$. Also, $\varpi : \mathcal{E}_\bullet \xrightarrow{\sim} \mathcal{E}_\bullet^{\#\#}$ will denote the evaluation map.
8. Let $B_r = B_r(\mathcal{E}_\bullet) := \partial_{r+1}(\mathcal{E}_{r+1}) \subseteq \mathcal{E}_r$ denote the module of r-boundaries and $\mathcal{Z}_r = \mathcal{Z}_r(\mathcal{E}_\bullet) := \ker(\partial_r) \subseteq \mathcal{E}_r$ denote the module of r-cycles.
9. The rth-homology of $\mathcal{E}_\bullet$ will be denoted by $\mathcal{H}_r := \mathcal{H}_r(\mathcal{E}_\bullet)$ and is defined by the exact sequence $0 \longrightarrow B_r(\mathcal{E}_\bullet) \longrightarrow \mathcal{Z}_r(\mathcal{E}_\bullet) \longrightarrow \mathcal{H}_r \longrightarrow 0$.

3. A formal resolution functor

In this section we give a definition of a resolving subcategory and record the existence of the resolution functor (Proposition 3.3), in such a context. While we are particularly interested in Witt theory of noetherian schemes, much of the arguments can be formulated in the realm of resolving subcategories of abelian categories, which we define now.

Definition 3.1. Suppose $\mathcal{C}$ is an abelian category. An exact subcategory $\mathscr{V}$ of $\mathcal{C}$ is called a *resolving subcategory* if, (1) $\mathscr{V}$ is closed under direct summand and direct sum, (2) every epimorphism $\mathscr{V}$ is admissible (*i.e. the kernel of any epimorphism between objects in* $\mathscr{V}$ *is in* $\mathscr{V}$), and (3) given any object M in $\mathcal{C}$ there is an epimorphism $\mathcal{E} \twoheadrightarrow \mathcal{G}$, for some $\mathcal{E} \in \mathscr{V}$.

The most natural example, and the one of our particular interest, of a resolving subcategory is the subcategory $\mathscr{V}(X)$ of locally free sheaves, of the category $Coh(X)$ of the coherent sheaves over a noetherian scheme X. Apparently, the resolving subcategories of the module categories have been studied more extensively than others. While they were not called as such, the concept of resolvability has had an important and widespread use in the literature. Indeed, condition (3) of Definition 3.1 is precisely the condition C1 in the article of Keller [11, §12], and the condition C2 of [11, §12] also holds for resolving categories, in a natural way. Subsequently, the results in [11] will be used in the proof of Proposition 3.3 below.

Remark 3.2. Suppose $\mathscr{V}$ is a resolving subcategory of an abelian category $\mathcal{C}$. Following are some immediate and obvious comments:

(1) Any object M in $\mathcal{C}$ has a resolution by objects in $\mathscr{V}$. (2) The categories of chain complexes $Ch^b(\mathscr{V})$, $Ch^+(\mathscr{V})$, $Ch(\mathscr{V})$ of objects in $\mathscr{V}$ and the homotopy categories $K^b(\mathscr{V})$, $K^+(\mathscr{V})$, $K(\mathscr{V})$ are defined, as usual.

(3) The derived categories $D^b(\mathscr{V})$, $D^+(\mathscr{V})$, and $D(\mathscr{V})$ are defined by inverting the quasi-isomorphisms in the corresponding homotopy category (*without any regard to any other structure*). Consult the proof of [18, 10.4.1] on localization.

The existence of the following resolution functor ζ would be a standard fact for familiar resolving categories.

Proposition 3.3. *Suppose $\mathcal{C}$ is an abelian category and $\mathscr{V}$ is a resolving subcategory. Let $\mathbb{M} := \mathbb{M}(\mathscr{V}) := \{\mathcal{F} \in \mathcal{C} : \mathcal{F}$ has a finite $\mathscr{V}$-resolution$\}$ be the full subcategory of $\mathcal{C}$. Assume, given an object $\mathcal{F} \in \mathbb{M}$ and a $\mathscr{V}$-resolution $\mathcal{E}_\bullet \twoheadrightarrow \mathcal{F}$, the cycle objects $Z_n := \ker(\mathcal{E}_n \longrightarrow \mathcal{E}_{n-1}) \in \mathscr{V}\ \forall n \gg 0$. Then,*

1. *$\mathbb{M}$ is an exact subcategory and every epimorphism in $\mathbb{M}$ is admissible.*
2. *There is a natural functor $\zeta : \mathbb{M}(\mathscr{V}) \longrightarrow D^b(\mathscr{V})$, where for an object $M \in \mathbb{M}(\mathscr{V})$, $\zeta(M)$ is defined to be a (choice of) finite resolution of M.* (For simplicity, we make a convention that if the length of a minimal $\mathscr{V}$-resolution of M is r, then by choice $\zeta(M)_i = 0\ \forall i > r$.)
3. *Further, suppose $P_\bullet \in D^b(\mathscr{V})$ is a resolution of $H_0(P_\bullet) = M$, $Q_\bullet \in D^b(\mathscr{V})$ is a resolution of $H_0(Q_\bullet) = N$, and $g : M \longrightarrow N$ is a morphism in $\mathbb{M}(\mathscr{V})$. Then, there is a unique map in $\varphi : P_\bullet \longrightarrow Q_\bullet$ such that $H_0(\varphi) = g$.*

Proof. To prove (1), let $0 \longrightarrow K \xrightarrow[f]{} M \xrightarrow[g]{} C \longrightarrow 0$ be an exact sequence in $\mathcal{C}$. Inductively, define a resolution (possibly infinite) of this sequence in $Ch^{\geq 0}(\mathscr{V})$, as follows. Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & L_0 \arrow[r] \arrow[d, "\partial_0"'] & P_0 \arrow[r] \arrow[d, "\varphi_0"'] & Q_0 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K \arrow[r] \arrow[d, equal] & \Gamma_0 \arrow[r, "\gamma_0"'] \arrow[d, "d'_0"'] & Q_0 \arrow[r] \arrow[d, "d_0"] & 0 \\
0 \arrow[r] & K \arrow[r, "f"'] & M \arrow[r, "g"'] & C \arrow[r] & 0
\end{tikzcd}$$

In this diagram, $d_0 : Q_0 \twoheadrightarrow C$ is a surjective morphism, Γ_0 is the pullback of (d_0, g) and $\varphi : P_0 \twoheadrightarrow \Gamma_0$ is a surjective morphism, with $Q_0, P_0 \in \mathscr{V}$. Also, $L_0 := \ker(\gamma_0\varphi_0) \in \mathscr{V}$. By Snake lemma ∂_0 is surjective. Write $\delta_0 = d'_0\varphi_0$. Again, by Snake lemma $0 \longrightarrow \ker(\partial_0) \longrightarrow \ker(\delta_0) \longrightarrow \ker(d_0) \longrightarrow 0$ is exact and the process continues.

Now, if $K, C \in \mathbb{M}(\mathscr{V})$ then, by hypothesis, the resolutions $L_\bullet \twoheadrightarrow K, Q_\bullet \twoheadrightarrow C$ terminate, hence process stops and $P_\bullet \in Ch^b(\mathscr{V})$. So, $\mathbb{M}(\mathscr{V})$ is an exact subcategory. Similarly, every epimorphism in $\mathbb{M}(\mathscr{V})$ is admissible. So, the proof of (1) is complete.

To prove (2), we apply the theorem of Keller [11, Theorem 12.1] to the inclusion $\mathscr{V} \subseteq \mathbb{M}(\mathscr{V})$. Note that the condition C1 in [11] follows from the definition of the resolving categories and C2 follows as in the proof of (1). It follows [11] that $\iota : D^b(\mathscr{V}) \xrightarrow{\sim} D^b(\mathbb{M})$ is an equivalence of categories. Now ζ is obtained by the composition of the functor $\mathbb{M} \longrightarrow D^b(\mathbb{M})$, followed by the inverse of ι. This completes the proof of (2).

The existence of φ in (3) follows from (2). To prove the uniqueness (3) of the lift of g, note that $D^b_{\mathbb{M}(\mathscr{V})}(\mathscr{V}) \longrightarrow D^b(\mathcal{C})$ is a faithful functor. So, we will prove that any two lifts of g coincide in $D^b(\mathcal{C})$. Consider N as a complex, concentrated at degree zero and let $P_\bullet \xleftarrow{t} \Gamma_\bullet \xrightarrow{G} Q_\bullet$ be the pullback of the maps $P_\bullet \longrightarrow N \longleftarrow Q_\bullet$. Then t is a quasi-isomorphism and Gt^{-1} is a lift of g. Further, given any lift $\gamma\tau^{-1} : P_\bullet \xleftarrow{\tau} L_\bullet \xrightarrow{\gamma} Q_\bullet$ of g, it factors through a quasi-isomorphism $\epsilon : L_\bullet \longrightarrow \Gamma_\bullet$. So, $\gamma\tau^{-1} = (G\epsilon)(t\epsilon)^{-1} = Gt^{-1}$. This completes the proof of (3). $\square$

4. Duality properties

In this section X will denote a noetherian scheme, as in Notations 2.1, with $\dim X = d$. Also, $\mathcal{A}(X)$ will denote the exact subcategory of $Coh(X)$, as defined in Notations 2.1. With a view of Section 7 on formalism, we take the formal approach to the proofs, as opposed to local. First, we define a duality on $\mathcal{A}(X)$.

Definition 4.1. Suppose X is a noetherian scheme with $\dim X = d$, as in Notations 2.1. We fix a choice of an injective resolution $\mathscr{I}_\bullet$ of $\mathcal{O}_X$. For a coherent sheaf $\mathcal{F}$, and integers $n \geq 0$, we define $\mathcal{E}xt^n(\mathcal{F}, \mathcal{O}_X) := \mathcal{H}^n(\mathcal{H}om(\mathcal{F}, \mathscr{I}_\bullet))$.

Lemma 4.2. *Suppose X is a noetherian scheme with $\dim X = d$ and $\mathcal{A}(X)$ is as in Notations 2.1. Define a functor $^\vee : \mathcal{A} \longrightarrow \mathcal{A}$ by $\mathcal{F}^\vee := \mathcal{E}xt^d(\mathcal{F}, \mathcal{O}_X)$. Then, $^\vee$ defines a duality on $\mathcal{A}$.*

Proof. We need to establish that, for objects $\mathcal{F} \in \mathcal{A}(X)$, there is a natural isomorphism $\tilde{\varpi} : \mathcal{F} \xrightarrow{\sim} \mathcal{F}^{\vee\vee}$. Let $\mathcal{E}_\bullet := \zeta(\mathcal{F})$ as defined in Proposition 3.3. We denote this complex as

$$0 \longrightarrow \mathcal{E}_d \xrightarrow{\partial_d} \cdots \longrightarrow \mathcal{E}_1 \xrightarrow{\partial_1} \mathcal{E}_0 \xrightarrow{\partial_0} \mathcal{F} \longrightarrow 0$$

So, there is a natural isomorphism $co\ker(\partial_d^*) \xrightarrow{\sim} \mathcal{F}^\vee$ (see [10]). Since $\mathcal{F} \in \mathcal{A}(X)$ we have $\mathcal{E}xt^i(\mathcal{F}, \mathcal{O}_X) = 0$ for all $i \neq d$. Therefore, the dual $\mathcal{E}^\#$ yields a resolution:

$$0 \longrightarrow \mathcal{E}_0^* \longrightarrow \cdots \longrightarrow \mathcal{E}_{d-1}^* \longrightarrow \mathcal{E}_d^* \longrightarrow \mathcal{F}^\vee \longrightarrow 0$$

Dualizing again, we have the following diagram of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{E}_d \arrow[r] \arrow[d, "ev"'] & \cdots \arrow[r] & \mathcal{E}_1 \arrow[r] \arrow[d, "ev"'] & \mathcal{E}_0 \arrow[r] \arrow[d, "ev"'] & \mathcal{F} \arrow[r] \arrow[d, dashed, "\wr\ \varpi_0"] & 0 \\
0 \arrow[r] & \mathcal{E}_d^{**} \arrow[r] & \cdots \arrow[r] & \mathcal{E}_1^{**} \arrow[r] & \mathcal{E}_0^{**} \arrow[r] & co\ker(\partial_0^{**}) \arrow[r] & 0
\end{tikzcd}$$

The isomorphism ϖ_0 is induced by the evaluation maps. There is also a natural isomorphism $\omega_1 : co\ker(\partial_0^{**}) \xrightarrow{\sim} \mathcal{F}^{\vee\vee}$. Hence, $\tilde{\varpi} := \varpi_1\varpi_0 : \mathcal{F} \xrightarrow{\sim} \mathcal{F}^{\vee\vee}$ is a natural isomorphism. The proof is complete. □

The derived category $D^b(\mathscr{V}(X))$ has a triangulated structure, with the duality induced by the duality $-^* := \mathcal{H}om(-, \mathcal{O}_X)$, which we will denote by $\#$. We have particular interest in the subcategory $D^b_{\mathcal{A}}(\mathscr{V}(X))$. It was pointed out in [12] that $D^b_{\mathcal{A}}(\mathscr{V}(X))$ may not be closed under cone construction, but it was stable under duality. This works for any noetherian scheme, as follows.

Lemma 4.3. *Suppose X is a noetherian scheme, with $\dim X = d$* (the condition in Notations 2.1 that every coherent sheaf on X is quotient of a sheaf in $\mathscr{V}(X)$, is not needed). *Also, let $\mathcal{A} := \mathcal{A}(X)$ be as in Notations 2.1. Let $\mathcal{E}_\bullet$ be a complex in $Ch^b_{\mathcal{A}}(\mathscr{V}(X))$. Then, for all integers $r \in \mathbb{Z}$, there is natural isomorphism*

$$\mathcal{E}xt^i\left(\frac{\mathcal{E}_r}{B_r}, \mathcal{O}_X\right) \xrightarrow{\sim} \begin{cases} \mathcal{E}xt^d(\mathcal{H}_{r+i-d}(\mathcal{E}_\bullet), \mathcal{O}_X) & 1 \leq i \leq d \\ 0 & i \geq d+1 \end{cases} \tag{1}$$

where $B_r \subseteq \mathcal{E}_r$ denote the boundary, as in Notations 2.1.

Proof. We assume $\mathcal{E}_r = 0\ \forall r < 0$. Since (1) holds for $r = -1$, assume that Eq. (1) holds for degree r. We prove it for degree $r+1$. We have two exact sequences

$$0 \longrightarrow B_r \longrightarrow \mathcal{E}_r \longrightarrow \tfrac{\mathcal{E}_r}{B_r} \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow \mathcal{H}_{r+1} \longrightarrow \tfrac{\mathcal{E}_{r+1}}{B_{r+1}} \longrightarrow B_r \longrightarrow 0$$

The long exact sequence of the first exact sequence yields the following isomorphisms:

$$\forall i \geq 1 \quad \mathcal{E}xt^i(B_r, \mathcal{O}_X) \xrightarrow{\sim} \mathcal{E}xt^{i+1}\left(\frac{\mathcal{E}_r}{B_r}, \mathcal{O}_X\right) \xrightarrow{\sim} \begin{cases} \mathcal{E}xt^d(\mathcal{H}_{r+(i+1)-d}, \mathcal{O}_X) & 1 \leq i \leq d-1 \\ 0 & i \geq d \end{cases}$$

Whether $\mathcal{H}_{r+1} = 0$ or $\mathcal{H}_{r+1} \neq d$, $\mathcal{E}xt^i(\mathcal{H}_{r+1}, \mathcal{O}_X) = 0\ \forall i \neq d$. It follows from the second exact sequence

$$\mathcal{E}xt^i\left(\frac{\mathcal{E}_{r+1}}{B_{r+1}}, \mathcal{O}_X\right) \xrightarrow{\sim} \begin{cases} \mathcal{E}xt^i(B_r, \mathcal{O}_X) & \text{if } i \leq d-1 \\ \mathcal{E}xt^d(\mathcal{H}_{r+1}, \mathcal{O}_X) & \text{if } i = d \\ 0 & \text{if } i \geq d+1. \end{cases}$$

This establishes the lemma. $\square$

Theorem 4.4. *Let X be a noetherian scheme as in Lemma 4.3 and $\mathcal{E}_\bullet$ be a complex in $Ch^b_{\mathcal{A}}(\mathscr{V}(X))$. Then, for $r \in \mathbb{Z}$, there is a canonical isomorphism $\eta_{\mathcal{E}_\bullet} : \mathcal{H}_{-r}(\mathcal{E}_\bullet^\#) \xrightarrow{\sim} \mathcal{H}_{r-d}(\mathcal{E}_\bullet)^\vee$. In particular, $Ch^b_{\mathcal{A}}(\mathscr{V}(X))$ is stable under duality. Further, $\eta_{\mathcal{E}_\bullet}$ is natural with respect to morphisms $f : \mathcal{E}_\bullet \longrightarrow \mathcal{E}'_\bullet$ in $Ch^b_{\mathcal{A}}(\mathscr{V}(X))$.*

Proof. First, we have the following commutative diagram of exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & (\frac{\mathcal{E}_{r-1}}{B_{r-1}})^* \arrow[r] \arrow[d, equal] & (\mathcal{E}_{r-1})^* \arrow[r] \arrow[d, equal] & (B_{r-1})^* \arrow[r] \arrow[d, "\wr"] & \mathcal{E}xt^1(\frac{\mathcal{E}_{r-1}}{B_{r-1}}, \mathcal{O}_X) \arrow[r] \arrow[d, dashed, "\wr", "\psi"'] & 0 \\
0 \arrow[r] & (\frac{\mathcal{E}_{r-1}}{B_{r-1}})^* \arrow[r] & (\mathcal{E}_{r-1})^* \arrow[r] & (\frac{\mathcal{E}_r}{B_r})^* \arrow[r] & H_{-r}(\mathcal{E}_\bullet^*) \arrow[r] & 0
\end{tikzcd}$$

The isomorphism ψ is induced. Now, the first part of the theorem follows by composing ψ with the isomorphism, as follows: $\mathcal{E}xt^1(\frac{\mathcal{E}_{r-1}}{B_{r-1}}, \mathcal{O}_X) \xrightarrow{\sim} \mathcal{E}xt^d(\mathcal{H}_{r-d}, \mathcal{O}_X)$

$$\begin{tikzcd}
\mathcal{E}xt^1(\frac{\mathcal{E}_{r-1}}{B_{r-1}}, \mathcal{O}_X) \arrow[r, "\sim"] \arrow[d, "\psi"] & \mathcal{E}xt^d(\mathcal{H}_{r-d}, \mathcal{O}_X) \arrow[dl, "\eta_{\mathcal{E}_\bullet}"] \\
H_{-r}(\mathcal{E}_\bullet^\#) &
\end{tikzcd}$$

The latter part follows because all the isomorphisms are natural. The proof is complete. $\square$

5. The main isomorphism theorem

Having established that $D^b_{\mathcal{A}}(\mathscr{V}(X))$ is stable under duality, we discuss the Witt group of $D^b_{\mathcal{A}}(\mathscr{V}(X))$, which is a subcategory of the derived category $D^b(\mathscr{V}(X))$. Before we proceed, we recall the definition of the Witt group of subcategories of triangulated categories with duality, from [12].

Definition 5.1. Suppose $K := (K, \#, \delta, \varpi)$ is a triangulated category with translation T and δ-duality $\#$. Let $MW(K)$ denote the monoid of isometry classes symmetric forms in K and $NW(K)$ denote the submonoid of neutral spaces in $MW(K)$. As usual, the Witt group of K is defined as $W(K) = \frac{MW(K)}{NW(K)}$. Now, suppose K_0 is a full subcategory of K that is closed under isomorphism, translation and orthogonal sum.

Let $MW(K_0) = \{[(P,\varphi)] \in MW(K) : P \in Ob(K_0)\}$, which is submonoid of $MW(K)$. A symmetric space (P,φ), with $P \in Ob(K_0)$ is said to be K_0-*neutral space*, if it has a lagrangian (L, α, w) in K such that $L, L^{\#} \in Ob(K_0)$. Denote $NW(K_0) = \{[(P,\varphi)] \in MW(K) : (P,\varphi)$ is K_0-neutral$\}$, which is also a submonoid. Define the Witt group of K_0 as:

$$W(K_0) := \frac{MW(K_0)}{NW(K_0)}.$$

Note that this definition is analogous to that of Witt groups of exact categories in [4], requiring the lagrangians to be admissible monomorphisms. We set up some more basic framework, analogous to [12].

1. The standard translation, in $D^b(\mathscr{V}(X))$, which changes the sign of the differential will be denoted by T or T_s and the unsigned translation will be denoted by T_u. For a complex $\mathcal{E}_\bullet$ in $Ch^b(\mathscr{V}(X))$, $T_u(\mathcal{E}_\bullet)$ will denote the unsigned translation, and $T_s(\mathcal{E}_\bullet)$ or $T(\mathcal{E}_\bullet)$.
2. Denote the shifted derived categories: $T^n D^b_{\mathcal{A}}(\mathscr{V}(X))_u := (D^b_{\mathcal{A}}(\mathscr{V}(X)), T^n_u o\#, 1, \varpi)$ and $T^n D^b_{\mathcal{A}}(\mathscr{V}(X)) := T^n D^b_{\mathcal{A}}(\mathscr{V}(X))_s := (D^b_{\mathcal{A}}(\mathscr{V}(X)), T^n_s o\#, 1, \varpi)$ where ϖ is the evaluation map.
3. As was pointed out, $D^b_{\mathcal{A}}(\mathscr{V}(X))$ may not have a triangulated structure. As usual, we denote $W^n(D^b_{\mathcal{A}}(\mathscr{V}(X))_u) := W(T^n D^b_{\mathcal{A}}(\mathscr{V}(X))_u)$ and $W^n(D^b_{\mathcal{A}}(\mathscr{V}(X))_s) := W(T^n D^b_{\mathcal{A}}(\mathscr{V}(X))_s)$.
4. In this section, we will exclusively work with unsigned translation and sometimes will drop the subscript or superscript "u" and write $T = T_u$ (*which would be an abuse of notation due to the notations in* (1)).
5. In this section, we work in the category $T^d D^b(\mathscr{V}(X))$ and $\#$ will denote the respective duality.

Now, we prove that the functor $\zeta : \mathcal{A}(X) \longrightarrow D^b_{\mathcal{A}}(\mathscr{V}(X))$ induces isomorphisms of Witt groups. First, we prove ζ induces a homomorphism of Witt groups.

Theorem 5.2. *Suppose X is a noetherian scheme, as in Notations 2.1, with $d = \dim X$. Let $\mathcal{A} := \mathcal{A}(X)$ be as in Notations 2.1. Then, the functor $\zeta : \mathcal{A} \longrightarrow D^b_{\mathcal{A}}(\mathscr{V}(X))$ induces a well defined homomorphism $W(\zeta) : W(\mathcal{A}, ^\vee, \tilde{\varpi}) \longrightarrow W^d(D^b_{\mathcal{A}}(\mathscr{V}(X))_u)$. The same hold for skew duality also.*

Proof. We will only give the proof for the plus duality. We first define the homomorphism $W(\zeta)$: $W(\mathcal{A}, ^\vee, \tilde{\varpi}) \longrightarrow W^d(D^b_{\mathcal{A}}(\mathscr{V}(X))_u)$ and then prove that it is well defined. Suppose $(\mathcal{F}, \varphi_0)$ is a symmetric form in $(\mathcal{A}, ^\vee, \tilde{\varpi})$. Write $\mathcal{X}_\bullet := \zeta(\mathcal{F})$. By choice, $\mathcal{X}_i = 0\ \forall i > d$. Then, $\mathcal{X}_\bullet^{\#}$ gives a resolution of $\mathcal{F}^\vee$. It is also easy to see that ζ preserves the duality on $\mathcal{A}$. By the uniqueness Proposition 3.3 of the lifts $\zeta(f)$ of the morphisms f in $\mathcal{A}(X)$, the identity $\varphi_0 = \varphi_0^\vee \tilde{\varpi}$ produces a symmetric from $\zeta(\varphi_0) : \mathcal{X}_\bullet \xrightarrow{\sim} \mathcal{X}_\bullet^{\#}$. Therefore, the association $(\mathcal{F}, \varphi_0) \mapsto (\mathcal{X}_\bullet, \zeta(\varphi_0))$ induces a homomorphism $MW(\zeta) : MW(\mathcal{A}, ^\vee, \tilde{\varpi}) \longrightarrow MW(T^d(D^b_{\mathcal{A}}(\mathscr{V}(X))_u))$ of monoids of symmetric spaces.

Now we want to prove that $MW(\zeta)$ maps neutral spaces to neutral spaces. Suppose $(\mathcal{F}, \varphi_0)$ is a neutral space in $(\mathcal{A}, ^\vee, \tilde{\varpi})$. So, there is an exact sequence (i.e. a lagrangian) $0 \longrightarrow \mathcal{G} \xrightarrow{\alpha_0} \mathcal{F} \xrightarrow{\alpha_0^\vee \varphi_0} \mathcal{G}^\vee \longrightarrow 0$. As above, write $\mathcal{X}_\bullet := \zeta(\mathcal{F})$. We would like to show $(\mathcal{X}_\bullet, \zeta(\varphi_0))$ is neutral. It will be convenient to work with forms without denominators. By Proposition 3.3 $\zeta(\varphi_0) = \gamma\tau^{-1} : \mathcal{X}_\bullet \xrightarrow{\sim} \mathcal{X}_\bullet^{\#}$, where τ, γ are quasi-isomorphisms in $Ch^b(\mathscr{V}(X))$, $\mathcal{E}_\bullet \in Ch^+(\mathscr{V}(X))$, such that the diagram

$$\begin{tikzcd} \mathcal{X}_\bullet \arrow[d, two heads] & \mathcal{E}_\bullet \arrow[l, "\tau"'] \arrow[r, "\gamma"] \arrow[d, two heads] & \mathcal{X}_\bullet^{\#} \arrow[d, two heads] \\ \mathcal{F} \arrow[r, equal] & \mathcal{F} \arrow[r, "\varphi_0"'] & \mathcal{F}^\vee \end{tikzcd} \quad \text{commutes. With } \varphi = \tau^{\#}\gamma \text{ the diagram} \quad \begin{tikzcd} \mathcal{E}_\bullet \arrow[r, two heads] \arrow[d, "\varphi"'] & \mathcal{F} \arrow[d, "\varphi_0"] \\ \mathcal{E}_\bullet^{\#} \arrow[r, two heads] & \mathcal{F}^\vee \end{tikzcd} \qquad (2)$$

commutes and φ is a quasi-isomorphism in $Ch^b(\mathscr{V}(X))$, without denominator. Clearly, $(\mathcal{E}_\bullet, \varphi)$ and $(\mathcal{X}_\bullet, \zeta(\varphi_0))$ are isometric.

Now that $MW(\zeta)(\mathcal{F}, \varphi_0)$ is given by a denominator free quasi-isomorphism $\varphi : \mathcal{E}_\bullet \longrightarrow \mathcal{E}_\bullet^\#$, the rest of the argument is borrowed from [2], *for extra details see* [12]. We outline the proof to point out the subtleties. Let $\mathcal{L}_\bullet := \zeta(\mathcal{G})$. By choice, $\mathcal{L}_i = 0$ unless $d \geq i \geq 0$. Let $\alpha : \mathcal{L}_\bullet \longrightarrow \mathcal{E}_\bullet$ be a lift of α_0. Since the homotopy category $K^b(\mathscr{V}(X))_u$ has a triangulated structure, we can embed α in an exact triangle in $T^d K^b(\mathscr{V}(X))_u$ and obtain the following diagram and a morphism s, as follows:

$$\begin{tikzcd} \mathcal{L}_\bullet \arrow[r, "\alpha"] \arrow[d] & \mathcal{E}_\bullet \arrow[r, "j"] \arrow[d, "\alpha^\# \varphi"] & \mathcal{V}_\bullet \arrow[r, "k"] \arrow[d, dashed, "s"] & T(\mathcal{L}_\bullet) \arrow[d, "0"] \\ 0 \arrow[r] & \mathcal{L}_\bullet^\# \arrow[r, equal] & \mathcal{L}_\bullet^\# \arrow[r] & 0. \end{tikzcd} \quad \text{in } T^d K^b\big(\mathscr{V}(X)\big)_u.$$

From the long exact sequence of homologies, it follows $\mathcal{H}_0(\mathcal{V}_\bullet) \cong \mathcal{G}^\vee$ and $\mathcal{H}_i(\mathcal{V}_\bullet) = 0\ \forall i \neq 0$. So, s is an isomorphism in $D^b_{\mathcal{A}}(\mathscr{V}(X))_u$. Therefore, we get the exact triangle

$$\mathcal{L}_\bullet \xrightarrow{\alpha} \mathcal{E}_\bullet \xrightarrow{\alpha^\# \varphi} \mathcal{L}_\bullet^\# \xrightarrow{ks^{-1}} T(\mathcal{L}_\bullet) \quad \text{in } D^b_{\mathcal{A}}\big(\mathscr{V}(X)\big)_u.$$

Translating this triangle, and with $w = -T^{-1}(ks^{-1})$, we get the exact triangle

$$T^{-1}\mathcal{L}_\bullet^\# \xrightarrow{w} \mathcal{L}_\bullet \xrightarrow{\alpha} \mathcal{E}_\bullet \xrightarrow{\alpha^\# \varphi} \mathcal{L}_\bullet^\# \quad \text{in } T^d D^b_{\mathcal{A}}(\mathscr{V}(X))_u.$$

It remains to show that $T^{-1}w^\# = w$. Routine analysis shows that this is equivalent to showing $T(k^\#)s = T(s^\#)k$. Indeed, $s : \mathscr{V}_n = \mathcal{E}_n \oplus \mathcal{L}_{n-1} \longrightarrow \mathcal{L}^*_{d-n}$ is a morphism of chain complexes. Therefore all the maps in this equation $T(k^\#)s = T(s^\#)k$ are morphisms of chain complexes. It would suffice to show that this identity holds in $T^d K^b(\mathscr{V}(X))$, which can be done exactly as in [2,12]. This establishes that ζ induces a homomorphism of the Witt groups. $\square$

The following lemma will be helpful for subsequent discussion.

Lemma 5.3. *Suppose X is a noetherian scheme with* $\dim X = d$. *Suppose $(\mathcal{E}_\bullet, \varphi)$ is a symmetric from in $T^d D^b_{\mathcal{A}}(\mathscr{V}(X))$. Assume, for some $n \geq 0$, $\mathcal{H}_{-n}(\mathcal{E}_\bullet) \neq 0$ and $\mathcal{H}_i(\mathcal{E}_\bullet) = 0\ \forall i < -n$. Then, $(\mathcal{E}_\bullet, \varphi)$ is isometric, to a symmetric from $(\mathcal{E}'_\bullet, \varphi')$ such that $\mathcal{E}'_i = 0$, unless $n + d \geq i \geq -n$.*

Proof. (*We give a proof that applies to* Section 7.) We use the notations from Notations 2.1. Note that the cycle sheaf $\mathcal{Z}_{-n}(\mathcal{E}_\bullet) \in Ch^b(\mathscr{V}(X))$. Replacing $\mathcal{E}_i$ by zero, when $i < -n$ and $\mathcal{E}_{-n}$ by $\mathcal{Z}_{-n}(\mathcal{E}_\bullet)$, we obtain a complex $\mathcal{E}'_\bullet$, with $\mathcal{E}'_i = 0\ \forall i < -n$ and a quasi-isomorphism $\eta : \mathcal{E}'_\bullet \longrightarrow \mathcal{E}_\bullet$. Therefore, replacing $(\mathcal{E}_\bullet, \varphi)$ by $(\mathcal{E}'_\bullet, \eta^\# \varphi \eta)$, we assume that $\mathcal{E}_i = 0\ \forall i < -n$.

Now, $(\mathcal{E}_\bullet, \varphi)$ and $(\mathcal{E}_\bullet^\#, \varphi^{-1})$ are isometric. So, replacing $(\mathcal{E}_\bullet, \varphi)$ by $(\mathcal{E}_\bullet^\#, \varphi^{-1})$, we can assume $\mathcal{E}_i = 0$ for all $i \geq n + d + 1$. Applying the same argument, as above, on this new $(\mathcal{E}_\bullet, \varphi)$ the lemma is established. $\square$

As in [2,4], the proof of surjectivity of $W(\zeta)$ is done by reduction of support of the symmetric forms by application of the sublagrangian theorem [3, 4.20]. In the non-affine case, both the construction and the proof that it is a sublagrangian require further finesse. The following lemma will be useful.

Lemma 5.4. *Suppose X is as in Notations 2.1. Let $\mathcal{L}_\bullet, \mathcal{G}_\bullet \in Ch^b(\mathscr{V}(X))$ be complexes and $\eta_\bullet : \mathcal{L}_\bullet \longrightarrow \mathcal{G}_\bullet$ be a morphism, as in the diagram*

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] & \mathcal{L}_n \arrow[r] \arrow[d, "\eta"'] & \mathcal{L}_{n-1} \arrow[r] \arrow[d, "\eta"] & \cdots \arrow[r] & \mathcal{L}_0 \arrow[r] \arrow[d, "\eta_0"] & 0 \arrow[r] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & \mathcal{G}_n \arrow[r] & \mathcal{G}_{n-1} \arrow[r] & \cdots \arrow[r] & \mathcal{G}_0 \arrow[r] & \mathcal{G}_{-1} \arrow[r] & \cdots
\end{tikzcd}$$

such that

1. $\mathcal{H}_r(\mathcal{G}_\bullet) = 0 \ \forall r \geq 0$.
2. $\mathcal{H}_r(\mathcal{L}_\bullet) = 0 \ \forall r \neq 0$ *and* $\mathcal{L}_r = 0 \ \forall r < 0$.

Then, $\eta = 0$ in $D^b(\mathscr{V}(X))$.

Proof. We use Notations 2.1. Let $\Gamma_\bullet$ be the complex:

$$\cdots \longrightarrow \mathcal{L}_2 \oplus \mathcal{G}_2 \longrightarrow \mathcal{L}_1 \oplus \mathcal{G}_1 \longrightarrow \mathcal{L}_0 \oplus \mathcal{Z}_0(\mathcal{G}_\bullet) \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

Let $t : \Gamma_\bullet \longrightarrow \mathcal{L}_\bullet$ and $g : \Gamma_\bullet \longrightarrow \mathcal{G}_\bullet$ be the projection maps. Then, t is a quasi-isomorphism. We have the following commutative diagram of morphisms:

$$\begin{tikzcd}
& \mathcal{L}_\bullet \arrow[dl, "1"'] \arrow[d, "(1,\eta)"] \arrow[dr, "\eta"] & \\
\mathcal{L}_\bullet & \Gamma_\bullet \arrow[l, "t"'] \arrow[r, "g"] & \mathcal{G}_\bullet \\
& \mathcal{L}_\bullet \arrow[ul, "1"] \arrow[u, "(1,0)"] \arrow[ur, "0"'] &
\end{tikzcd}$$

Since, t is quasi-isomorphism, so is $(1, \eta)$. Now, $\eta = g(1, \eta) = 0(1, 0)^{-1}(1, \eta) = 0$. The proof is complete. □

Now we prove that $W(\zeta)$ is surjective, as follows.

Proposition 5.5. *Let X be a noetherian scheme, with $\dim X = d$, as in Notations 2.1. Then, the homomorphism $W(\zeta) : W(\mathcal{A}, ^\vee, \tilde{\varpi}) \longrightarrow W^d(D^b_{\mathcal{A}}(\mathscr{V}(X))_u)$ is surjective. The same holds for skew duality.*

Proof. We point out the subtleties involved here in the non-affine case, beyond the arguments in [12]. We will only consider the case of +duality and the case of skew duality follows similarly. As usual, the proof is done by reducing the length of the width of the forms. Suppose $x = [(\mathcal{E}_\bullet, \varphi)] \in W^d(D^b_{\mathcal{A}}(\mathscr{V}(X))_u)$, represented by the symmetric form $(\mathcal{E}_\bullet, \varphi)$. First, $\varphi = fs^{-1}$ for some quasi-isomorphism $s : \mathcal{E}_\bullet, \longrightarrow \mathcal{E}_\bullet$ for some $\mathcal{E}'_\bullet \in Ch^b(\mathscr{V}(X))$. Replacing $(\mathcal{E}_\bullet, \varphi)$ by $(\mathcal{E}'_\bullet, s^\# \varphi s)$, we assume that $\varphi : \mathcal{E}_\bullet \longrightarrow \mathcal{E}^\#_\bullet$ is a quasi-isomorphism (without denominator) in $Ch^b(\mathscr{V}(X))$.

Assume, for some $n > 0$, $\mathcal{H}_{-n}(\mathcal{E}_\bullet) \neq 0$ and $\mathcal{H}_i(\mathcal{E}_\bullet) = 0 \ \forall i < -n$. We will prove that there is a symmetric form $(\mathcal{R}_\bullet, \psi)$ such that $x = [(\mathcal{E}_\bullet, \varphi)] = [(\mathcal{R}_\bullet, \psi)]$ and $\mathcal{H}_i(\mathcal{R}_\bullet) = 0$ unless $n - 1 \geq i \geq -(n - 1)$. By Lemma 5.3, we assume that $\mathcal{E}_i = 0$ unless $n + d \geq i \geq -n$. Further, by duality Theorem 4.4, $\mathcal{H}_i(\mathcal{E}_\bullet) \cong \mathcal{H}_i(\mathcal{E}^\#_\bullet) \cong \mathcal{H}_{-i}(\mathcal{E}_\bullet)^\vee = 0$ for all $i > n$ (*note that the notation "#" in Theorem 4.4 and here differ by d-shift*). So, the left tail $\mathcal{E}_\bullet$ is a resolution of $\frac{\mathcal{E}_n}{B_n(\mathcal{E}_\bullet)}$. Let $\mathcal{L}_\bullet$ in $D^b(\mathscr{V}(X))$ (*labeled as in the diagram below*) be a resolution of $\mathcal{H}_n(\mathcal{E}_\bullet)$. By Proposition 3.3, the morphism $\mathcal{H}_n(\mathcal{E}_\bullet) \hookrightarrow \frac{\mathcal{E}_n}{B_n(\mathcal{E}_\bullet)}$ lifts to a morphism of complexes, as follows:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{L}_{n+d} \arrow[r] \arrow[d, "\nu_{n+d}"'] & \mathcal{L}_{n+d-1} \arrow[r] \arrow[d, "\nu_{n+d-1}"] & \cdots \arrow[r] & \mathcal{L}_n \arrow[r] \arrow[d, "\nu_n"] & \mathcal{H}_n(\mathcal{E}_\bullet) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \mathcal{E}_{n+d} \arrow[r] & \mathcal{E}_{n+d-1} \arrow[r] & \cdots \arrow[r] & \mathcal{E}_n \arrow[r] & \frac{\mathcal{E}_n}{B_n(\mathcal{E}_\bullet)} \arrow[r] & 0
\end{tikzcd}$$

where $\mathcal{L}_i \in \mathscr{V}(X)$ and both the lines are exact. Since ν_n maps to $\mathcal{Z}_n(\mathcal{E}_\bullet)$, with $\mathcal{L}_i = 0\ \forall i \le n-1$, it extends to a morphism $\nu : \mathcal{L}_\bullet \longrightarrow \mathcal{E}_\bullet$ in $Ch^b(\mathscr{V}(X))$. We claim ν is a sublagrangian of $(\mathcal{E}_\bullet, \varphi)$. To see this, write $\eta = \nu^\# \varphi \nu : \mathcal{L}_\bullet \longrightarrow \mathcal{L}_\bullet^\#$. We can write η as follows

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{L}_{n+d} \arrow[r] \arrow[d, "\eta_{n+d}"'] & \mathcal{L}_{n+d-1} \arrow[r] \arrow[d, "\eta_{n+d-1}"] & \cdots \arrow[r] & \mathcal{L}_n \arrow[r] \arrow[d, "\eta_n"] & 0 \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & \mathcal{L}^\#_{-n} \arrow[r] & \mathcal{L}^\#_{-(n-1)} \arrow[r] & \cdots \arrow[r] & \mathcal{L}^\#_{d-n} \arrow[r] & \mathcal{L}^\#_{d-n+1} \arrow[r] & \cdots
\end{tikzcd}$$

Since $\mathcal{H}_i(\mathcal{L}_\bullet^\#) = 0$ for all $i \neq -n$, the second line is exact at degrees $i \ge n$. By Lemma 5.4, $\eta = 0$ in $D^b(\mathscr{V}(X))$. The rest of the arguments in [12] works, which we outline briefly for completeness. As in [3, 4.20], embed ν in an exact triangle as in the first line in the diagram (3) and complete the commutative diagram

$$\begin{tikzcd}
T^{-1}\mathcal{N}_\bullet \arrow[r, "\nu_0"] \arrow[d, "T^{-1}\mu_0^\#"'] & \mathcal{L}_\bullet \arrow[r, "\nu"] \arrow[d, "\mu_0"] & \mathcal{E}_\bullet \arrow[r, "\nu_1"] \arrow[d, "\varphi"] & \mathcal{N}_\bullet \arrow[d, "\mu_0^\#"] \\
T^{-1}\mathcal{L}_\bullet^\# \arrow[r] & \mathcal{N}_\bullet^\# \arrow[r] \arrow[d, "\mu_1"] & \mathcal{E}_\bullet^\# \arrow[r] & \mathcal{L}_\bullet^\# \\
 & \mathcal{R}_\bullet \arrow[d, "\mu_2"] & & \\
 & T\mathcal{L}_\bullet & &
\end{tikzcd} \tag{3}$$

where the second line is the dual of the first line, by choice μ_0 is "very good" (see [3] for definition and existence) and $\mathcal{R}_\bullet$ is the cone of μ_0. We consider this as a diagram in $D^b(\mathscr{V}(X))$, which is a triangulated category. By [3, 4.20] there is a symmetric form $\psi : \mathcal{R}_\bullet \xrightarrow{\sim} \mathcal{R}_\bullet^\#$ such that $(\mathcal{R}_\bullet, -\psi)$ is Witt equivalent to $(\mathcal{E}_\bullet, \varphi)$ in $D^b(\mathscr{V}(X))$. This is shown by exhibiting a lagrangian $\lambda : \mathcal{N}_\bullet^\# \longrightarrow (\mathcal{E}_\bullet, -\varphi) \perp (\mathcal{R}_\bullet, \psi)$. We would have to show that $\mathcal{R}_\bullet$ is in $D^b_{\mathcal{A}}(\mathscr{V}(X))$ and λ is a lagrangian in $D^b_{\mathcal{A}}(\mathscr{V}(X))$. It suffices to show,

1. $\mathcal{N}_\bullet$, $\mathcal{N}_\bullet^\#$ and $\mathcal{R}_\bullet$ are in $D^b_{\mathcal{A}}(\mathscr{V}(X))$, and
2. $\mathcal{H}_i(\mathcal{R}_\bullet) = 0$ unless $n-1 \ge i \ge -(n-1)$.

These are established by writing down the long exact sequences of homologies of the three exact triangles in the diagram (3) and using the fact that $\mathcal{L}_\bullet, \mathcal{L}_\bullet^\#$ have only one nonzero homology. (*With a view to Section 7 on formalism, we avoid local argument.*) By Lemma 5.3, we can further assume that $\mathcal{R}_i = 0$ unless $(n-1)+d \ge i \ge -(n-1)$. Using induction there is a symmetric from $(\mathcal{Q}_\bullet, \omega)$ such that $x = [(\mathcal{E}_\bullet, \varphi)] = [(\mathcal{Q}_\bullet, \omega)]$, with $\mathcal{Q}_i = 0$ unless $d \ge i \ge 0$. By Theorem 4.4, $\mathcal{Q}_\bullet$ is a resolution of $\mathcal{H}_0(\mathcal{Q}_\bullet)$. Further, ω induces a symmetric from $\omega_0 : \mathcal{H}_0(\mathcal{Q}_\bullet) \xrightarrow{\sim} \mathcal{H}_0(\mathcal{Q}_\bullet)^\vee$. By definition, $W(\zeta)([(\mathcal{H}_0(\mathcal{Q}_\bullet, \omega_0))]) = [(\mathcal{Q}_\bullet, \omega)] = x$. So, $W(\zeta)$ is surjective. The proof is complete. □

The following is the main theorem for noetherian schemes.

Theorem 5.6. *Let X be a noetherian scheme, with* $\dim X = d$*, as in Notations 2.1. Then, the homomorphism* $W(\zeta) : W(\mathcal{A}(X), ^\vee, \tilde{\varpi}) \longrightarrow W^d(D^b_{\mathcal{A}}(\mathscr{V}(X))_u)$ *defined in Theorem 5.2, is an isomorphism. The same holds for skew duality also.*

Proof. It follows from Proposition 5.5 that $W(\zeta)$ is surjective. We outline the proof of injectivity, in the case of plus duality. Suppose $(\mathcal{F}, \varphi_0)$ is a symmetric form in $(\mathcal{A}, ^\vee, \tilde{\varpi})$ and $W(\zeta)([(\mathcal{F}, \varphi_0)]) = 0$. By definition, (see diagram (2)) there is a finite resolution $\mathcal{E}_\bullet \twoheadrightarrow \mathcal{F}$, with $\mathcal{E}_\bullet \in Ch^b(\mathscr{V}(X))$ and a quasi-isomorphism (without denominator) $\varphi : \mathcal{E}_\bullet \longrightarrow \mathcal{E}_\bullet^\#$, which lifts φ_0 and $W(\zeta)([(\mathcal{F}, \varphi_0)]) = [(\mathcal{E}_\bullet, \varphi)] = 0$. So, $(\mathcal{E}_\bullet, \varphi)$ is neutral in $D^b_{\mathcal{A}}(\mathscr{V}(X))$. Going through the same argument in [12, 5.12] there is a hyperbolic form $(\mathcal{Q}, \varphi_1) \perp (\mathcal{Q}, -\varphi_1)$ such that $(\mathcal{E}, \varphi) \perp (\mathcal{Q}, \varphi_1) \perp (\mathcal{Q}, -\varphi_1)$ is neutral in $D^b_{\mathcal{A}}(\mathscr{V}(X))$. Therefore, it follows that

$$(\mathcal{U}_\bullet, \beta) := \left(\mathcal{E}_\bullet \oplus \mathcal{Q}_\bullet \oplus \mathcal{Q}_\bullet^\#, \begin{pmatrix} \varphi_0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\right) \quad \text{is neutral in } \left(D^b_{\mathcal{A}}(\mathscr{V}(X))^{\pm}_u\right).$$

Therefore, $(U_\bullet, \varphi)$ has a lagrangian $(\mathcal{L}_\bullet, \alpha)$ given by the following exact triangle

$$T^{-1}\mathcal{L}_\bullet^\# \xrightarrow{w} \mathcal{L}_\bullet \xrightarrow{\alpha} \mathcal{U}_\bullet \xrightarrow{\alpha^\# \varphi} \mathcal{L}_\bullet^\# \quad \text{with } T^{-1}w^\# = w.$$

By the duality Theorem 4.4, $\mathcal{H}_{-r}(\mathcal{U}_\bullet) \xrightarrow{\sim} \mathcal{H}_{d-r}(\mathcal{U}_\bullet)^\vee$ and $\mathcal{H}_{-r}(\mathcal{L}_\bullet) \xrightarrow{\sim} \mathcal{H}_{d-r}(\mathcal{L}_\bullet)^\vee$, for all $r \in \mathbb{Z}$. With this identification, the exact sequence of the homologies of the exact triangle reduces to

$$\longrightarrow \mathcal{H}_{-2}(\mathcal{L}_\bullet)^\vee \xrightarrow{\mathcal{H}_1(w)} \mathcal{H}_1(\mathcal{L}_\bullet) \xrightarrow{\mathcal{H}_1(\alpha)} \mathcal{H}_1(\mathcal{U}_\bullet) \xrightarrow{\mathcal{H}_{-1}(\alpha)^\vee \circ \mathcal{H}_1(\beta)} \mathcal{H}_{-1}(\mathcal{L}_\bullet)^\vee$$

$$\xrightarrow{\mathcal{H}_0(w)} \mathcal{H}_0(\mathcal{L}_\bullet) \xrightarrow{\mathcal{H}_0(\alpha)} \mathcal{H}_0(\mathcal{U}_\bullet) \xrightarrow[\mathcal{H}_0(\alpha)^\vee \circ \mathcal{H}_0(\beta)]{} \mathcal{H}_0(\mathcal{L}_\bullet)^\vee \xrightarrow[\mathcal{H}_0(w)^\vee]{} \mathcal{H}_{-1}(\mathcal{L}_\bullet)^{\vee\vee}$$

$$\xrightarrow[\mathcal{H}_1(\beta)^\vee \circ \mathcal{H}_{-1}(\alpha)^{\vee\vee}]{} \mathcal{H}_1(\mathcal{U}_\bullet)^\vee \xrightarrow[\mathcal{H}_1(\alpha)^\vee]{} \mathcal{H}_1(\mathcal{L}_\bullet)^\vee \xrightarrow[\mathcal{H}_1(w)^\vee]{} \mathcal{H}_{-2}(\mathcal{L}_\bullet)^{\vee\vee} \xrightarrow[\mathcal{H}_2(\beta)^\vee \circ \mathcal{H}_{-2}(\alpha)^{\vee\vee}]{}$$

This exact sequence is "symmetric" and hence [4, 4.1] applies. Since the sequence is exact, it follows $[(\mathcal{H}_0(\mathcal{U}_\bullet), \mathcal{H}_0(\varphi))] = [(0, 0)] = 0$ in $W(\mathcal{A}(X), ^\vee, \tilde{\varpi})$. However,

$$\begin{aligned}
\left(\mathcal{H}_0(\mathcal{U}_\bullet), \mathcal{H}_0(\beta)\right) &= \left(\mathcal{H}_0(\mathcal{E}_\bullet) \oplus \mathcal{H}_0(\mathcal{Q}_\bullet) \oplus \mathcal{H}_0\left(\mathcal{Q}_\bullet^\#\right), \begin{pmatrix} \mathcal{H}_0(\varphi_0) & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\right) \\
&= (\mathcal{F}, \varphi_0) \perp \left(\mathcal{H}_0(\mathcal{Q}_\bullet) \oplus \mathcal{H}_0(\mathcal{Q}_\bullet)^\vee, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right).
\end{aligned}$$

So, we have

$$\left[(\mathcal{F}, \varphi_0)\right] = \left[(\mathcal{F}, \varphi_0) \perp \left(\mathcal{H}_0(\mathcal{Q}_\bullet) \oplus \mathcal{H}_0(\mathcal{Q}_\bullet)^\vee, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)\right] = \left[\left(\mathcal{H}_0(\mathcal{U}_\bullet), \mathcal{H}_0(\varphi)\right)\right] = 0.$$

The proof is complete. □

6. The final results

So far we have been working with unsigned translation, in particular in the statement of Theorem 5.6. To conform to the literature, in this section we would present our results with respect to the standard signed translation. The readers are referred to [3], [12, §6] for unexplained notations. We recall the following notations.

1. Denote $W_{St}(\mathcal{A}(X)) := W^{+}_{St}(\mathcal{A}(X)) := W(\mathcal{A}(X), ^{\vee}, (-1)^{\frac{d(d-1)}{2}}\tilde{\varpi})$, and $W^{-}_{St}(\mathcal{A}(X)) := W(\mathcal{A}(X), ^{\vee}, -(-1)^{\frac{d(d-1)}{2}}\tilde{\varpi})$.
2. From now on, $T := T_s : D^b(\mathscr{V}(X)) \longrightarrow D^b(\mathscr{V}(X))$ will denote the signed ("standard") translation. For $n \in \mathbb{Z}$, denote $\zeta_n = T^{-n} o \zeta : \mathcal{A}(X) \longrightarrow D^b(\mathscr{V}(X))$, where ζ is as in Proposition 3.3. Note that $\zeta_0 = \zeta$.

Now, we state the results.

Theorem 6.1. *Suppose X is a noetherian scheme, with $\dim X = d$, as in Notations 2.1. Then,*

1. *The functor $\zeta_0 : \mathcal{A}(X) \longrightarrow D^b_{\mathcal{A}(X)}(\mathscr{V}(X))$ induces an isomorphism*
$$W(\zeta_0) : W_{St}\big(\mathcal{A}(X)\big) \xrightarrow{\sim} W^d\big(D^b_{\mathcal{A}(X)}\big(\mathscr{V}(X)\big), *, 1, \varpi\big).$$
2. *The functor $\zeta_1 : \mathcal{A}(X) \longrightarrow D^b_{\mathcal{A}(X)}(\mathscr{V}(X))$ induces an isomorphism*
$$W^{-}_{St}\big(\mathcal{A}(X)\big) \xrightarrow{\sim} W^{d-2}\big(D^b_{\mathcal{A}(X)}\big(\mathscr{V}(X)\big), *, 1, \varpi\big).$$
3. *For $n = d-1, d-3$, we have $W^n(D^b_{\mathcal{A}(X)}(\mathscr{V}(X)), *, 1, \varpi) = 0$.*

Further, 4-periodicity determines all the shifted Witt groups $W^n(D^b_{\mathcal{A}(X)}(\mathscr{V}(X)), *, 1, \varpi)$.

Proof. Follows from Theorem 5.6, as in [12, §6]. □

We would also consider the Witt groups of the derived category $D^b(\mathcal{A}(X))$ and of its full subcategory $D^b_{\mathcal{A}}(\mathcal{A}(X))$. As was the case with $D^b_{\mathcal{A}(X)}(\mathscr{V}(X))$, $D^b_{\mathcal{A}(X)}(\mathcal{A}(X))$ may not have a triangulated structure. The following follows from the formalism given in [12].

Theorem 6.2. *Suppose X a noetherian scheme, as in Notations 2.1, with $\dim X = d$. The duality $^{\vee}$: $\mathcal{A} \longrightarrow \mathcal{A}$ induces a duality on the derived category $D^b(\mathcal{A})$, which we continue to denote by $^{\vee}$. Then, $D^b_{\mathcal{A}}(\mathcal{A})$ is stable under this duality. Further, the functors $\mathcal{A} \longrightarrow D^b_{\mathcal{A}}(\mathcal{A}) \hookrightarrow D^b(\mathcal{A})$ induce the following triangle of isomorphisms*

$$\begin{tikzcd} W(\mathcal{A}, ^{\vee}, \tilde{\varpi}) \arrow[r, "\sim"] \arrow[dr, "\sim"'] & W(D^b_{\mathcal{A}}(\mathcal{A}, ^{\vee}, \tilde{\varpi})) \arrow[d, "\wr"] \\ & W(D^b(\mathcal{A}, ^{\vee}, \tilde{\varpi})). \end{tikzcd}$$

The same holds for skew duality also.

Proof. Since $\mathcal{A}$ has 2-out-of-3 property, the theorem follows from [12, A.1, A.2]. □

The following decomposition of the derived Witt groups is in line with the regular case.

Theorem 6.3. *Suppose X is a noetherian scheme, with* $\dim X = d$, *as in Notations 2.1 and $X^{(d)}$ will denote the set of all closed points of codimension n in X. Then, the homomorphisms*

$$W^d\big(D^b_{\mathcal{A}}(\mathscr{V}(X)), *, 1, \varpi\big) \xrightarrow{\sim} \bigoplus_{x\in X^{(d)}} W^d\big(D^b_{\mathcal{A}(\mathcal{O}_{X,x})}(\mathscr{V}(\mathcal{O}_{X,x})), *, 1, \varpi\big) \quad \textit{and}$$

$$W^{d-2}\big(D^b_{\mathcal{A}}(\mathscr{V}(X)), *, 1, \varpi\big) \xrightarrow{\sim} \bigoplus_{x\in X^{(d)}} W^{d-2}\big(D^b_{\mathcal{A}(\mathcal{O}_{X,x})}(\mathscr{V}(\mathcal{O}_{X,x})), *, 1, \varpi\big)$$

are isomorphisms.

Proof. We prove only the first isomorphism. Consider the diagram:

$$\begin{tikzcd}
W_{St}(\mathcal{A}(X)) \arrow[r, "W(\zeta)", "\sim"'] \arrow[d, "\eta"'] & W^d(D^b_{\mathcal{A}(X)}(\mathscr{V}(X)), *, 1, \varpi) \arrow[d, "\gamma"] \\
\bigoplus_{x\in X^{(d)}} W_{St}(\mathcal{A}(\mathcal{O}_{X,x})) \arrow[r, "\oplus W(\zeta)"'] & \bigoplus_{x\in X^{(d)}} W^d(D^b_{\mathcal{A}(\mathcal{O}_{X,x})}(\mathscr{V}(\mathcal{O}_{X,x})), *, 1, \varpi)
\end{tikzcd}$$

By Theorem 6.1, it follows that the two horizontal homomorphisms are isomorphisms. Also note that $\mathcal{A}(X) \longrightarrow \coprod_{x\in X^{(d)}} \mathcal{A}(\mathcal{O}_{X,x})$ is a duality preserving equivalence of categories. Therefore, η is an isomorphism. Now, since three other maps in this rectangle are isomorphisms, so is γ. The proof is complete. □

Remark 6.4. When X is regular, Theorem 6.3 is a result of Balmer and Walter [5]. In this case, Theorem 6.3 would follow from the equivalence of the corresponding categories $D^b_{\mathcal{A}}(X) \longrightarrow \coprod_{x\in X^{(d)}} \mathcal{A}(\mathcal{O}_{X,x})$ [5, 7.1].

Remark 6.5. It was implicitly assumed in the statement of Theorem 6.1 and others, that $\mathcal{A}(X)$ has nonzero objects. This would be false, if $depth(\mathcal{O}_{X,x}) < d = \dim X$ for all closed points x. In such cases, a version of these results can be given by replacing d by $\delta := \max\{depth(\mathcal{O}_{X,x}) : x \text{ is a closed point of } X\}$ (see Section 7).

7. Witt formalism for resolving subcategories

In this section, we give formal versions of results in Section 6, for resolving subcategories (see Definition 3.1) of abelian categories. There is a list of examples of resolving subcategories of $Mod(A)$ in [16] which can be adapted to subcategories of $Coh(X)$. The one that is of our particular interest is the one corresponding to the notion of semi-dualizing A-modules ω. Such semi-dualizing A-modules ω naturally give rise to resolving subcategories of $Mod(A)$. With that in mind, we define the following and establish our setup for this section.

Definition 7.1. Suppose $\mathscr{V}$ is a resolving subcategory of an abelian category $\mathcal{C}$. Let ω be a fixed object on $\mathcal{C}$ and write $M^* := Mor(M, \omega)$. Assume that ω has an injective resolution, and fix one such resolution $\mathscr{I}_\bullet$, as follows:

$$0 \longrightarrow \omega \longrightarrow \mathscr{I}_0 \longrightarrow \mathscr{I}_1 \longrightarrow \mathscr{I}_2 \longrightarrow \cdots \tag{4}$$

1. Given any object M in $\mathcal{C}$, define $Ext^i(M, \omega) := H^i(Mor(M, \mathscr{I}_\bullet))$. It immediately follows that $\{M \mapsto Ext^i(-, \omega) : i \geq 0\}$ is a contravariant δ-functor (see [10, § III.1]).
2. We say that $\mathscr{V}$ inherits a ω-*duality structure* if, for all objects P in $\mathscr{V}$, (1) $P^* \in \mathscr{V}$, (2) and there is a natural equivalence $ev : P \xrightarrow{\sim} P^{**}$, from the identity functor $1_{\mathscr{V}}$ to the double dual.

3. Also, we say that $\mathscr{V}$ is *totally ω-reflexive* if in addition, for all objects P in $\mathscr{V}$ and integers $i \geq 1$, $Ext^i(P, \omega) = 0$. It is easy to see that if $P_\bullet \twoheadrightarrow M$ is a $\mathscr{V}$-resolution of an object in $\mathcal{C}$ that there are natural isomorphisms from the homologies $H_i(P_\bullet^\#) \xrightarrow{\sim} Ext^i(M, \omega)$, where $P_\bullet^\#$ denotes the dual of $P_\bullet$ induced by $*$.
4. We also recall some standard definitions. For objects M in $\mathcal{C}$, $\dim_\mathscr{V} M$ will denote the length of the shortest $\mathscr{V}$-resolution of M (which can be infinite). As in Section 3, $\mathbb{M} := \mathbb{M}(\mathscr{V}) := \{M \in \mathcal{C} : \dim_\mathscr{V} M < \infty\}$ denotes the full subcategory of such objects. Also denote, $d := \dim_\mathscr{V}(\mathbb{M}) := \max\{\dim_\mathscr{V} M : M \in \mathbb{M}\}$. If $d = \dim_\mathscr{V}(\mathbb{M}) < \infty$, let $\mathcal{A}(\omega) \subseteq \mathbb{M}(\mathscr{V})$ be the full subcategory of objects such that $Ext^i(M, \omega) = 0 \ \forall i < d$. It follows $\mathbb{M}(\mathscr{V})$ and $\mathcal{A}(\omega)$ are exact subcategories of $\mathcal{C}$.
5. *Setup*: In what follows, we will have the following setup:

   $\mathcal{C}$ will denote an abelian category and $\mathscr{V}$ will be a resolving subcategory of $\mathcal{C}$. We fix an object ω with a (fixed) injective resolution $\omega \hookrightarrow \mathscr{I}_\bullet$. The δ-functor $\{Ext^i(-, \omega)\}$ is defined as above. We assume: (1) $\mathscr{V}$ inherits a ω-duality structure and is totally ω-reflexive. (2) Given an object $\mathcal{F} \in \mathbb{M}$ and a $\mathscr{V}$-resolution $\mathcal{E}_\bullet \twoheadrightarrow \mathcal{F}$, the cycle objects $Z_n := \ker(\mathcal{E}_n \longrightarrow \mathcal{E}_{n-1}) \in \mathscr{V} \ \forall n \gg 0$. (3) Further, $d := \dim_\mathscr{V}(\mathbb{M}) < \infty$. We observed that Proposition 3.3 applies and the functor ζ is defined.

Most of what is in Section 6, work for resolving subcategories $\mathscr{V}$, of abelian categories $\mathcal{C}$, as in the setup (5) of Definition 7.1. We state them below.

Lemma 7.2. *Suppose $(\mathcal{C}, \mathscr{V}, \omega)$ is as in (5) of Definition 7.1. Then, the association $M \mapsto M^\vee := Ext^d(M, \omega)$ defines a duality $^\vee : \mathcal{A}(\omega) \longrightarrow \mathcal{A}(\omega)$.*

Proof. Note, $M \in \mathcal{A}(\omega) \Longrightarrow M^\vee \in \mathcal{A}(\omega)$. The rest of the proof is as that of Lemma 4.2. □

Theorem 7.3. *Let $(\mathcal{C}, \mathscr{V}, \omega)$ be as in (5) of Definition 7.1. Suppose $\mathcal{E}_\bullet$ is a complex in $Ch^b_\mathcal{A}(\mathscr{V})$. Then, the dual $\mathcal{E}_\bullet^\#$ is also $Ch^b_\mathcal{A}(\mathscr{V})$. Further, there is a canonical isomorphism $\eta : H_{-r}(\mathcal{E}_\bullet^\#) \xrightarrow{\sim} H_{r-d}(\mathcal{E}_\bullet)^\vee$, which is natural with respect to morphisms in $Ch^b_\mathcal{A}(\mathscr{V})$.*

Proof. Same as that of Theorem 4.4. □

Subsequently, we use the standard translation $T = T_s$ in the derived categories.

Theorem 7.4. *Suppose $(\mathcal{C}, \mathscr{V}, \omega)$ be as in (5) of Definition 7.1. Let the other notations below be similar to those in Section 6, adapted to this formal setup. Then,*

1. *The functor $\zeta_0 : \mathcal{A}(\omega) \longrightarrow D^b_{\mathcal{A}(\omega)}(\mathscr{V})$ induces an isomorphism*
$$W(\zeta_0) : W_{St}(\mathcal{A}(\omega)) \xrightarrow{\sim} W^d(D^b_{\mathcal{A}(\omega)}(\mathscr{V}), *, 1, \varpi).$$
2. *The functor $\zeta_1 : \mathcal{A}(\omega) \longrightarrow D^b_{\mathcal{A}(\omega)}(\mathscr{V})$ induces an isomorphism*
$$W^-_{St}(\mathcal{A}(\omega)) \xrightarrow{\sim} W^{d-2}(D^b_{\mathcal{A}(\omega)}(\mathscr{V}), *, 1, \varpi).$$
3. *For $n = d-1, d-3$, we have $W^n(D^b_{\mathcal{A}(\omega)}(\mathscr{V}), *, 1, \varpi) = 0$.*

Further, 4-periodicity determines all the shifted Witt groups $W^n(D^b_{\mathcal{A}(\omega)}(\mathscr{V}), *, 1, \varpi)$.

Proof. Similar to that of Theorem 6.1. □

Theorem 7.5. *Suppose* $(\mathcal{C}, \mathscr{V}, \omega)$ *is as in (5) of Definition 7.1. Then,* $D^b_{\mathcal{A}(\omega)}(\mathcal{A}(\omega))$ *is stable under the duality on* $D^b(\mathcal{A}(\omega))$ *induced by* ${}^\vee : \mathcal{A}(\omega) \longrightarrow \mathcal{A}(\omega)$ (which we continue to denote by ${}^\vee$). *Further, the functors* $\mathcal{A}(\omega) \longrightarrow D^b_{\mathcal{A}(\omega)}(\mathcal{A}(\omega)) \hookrightarrow D^b(\mathcal{A}(\omega))$ *induce the following triangle of isomorphisms*

$$\begin{tikzcd} W(\mathcal{A}(\omega), {}^\vee, \tilde{\varpi}) \arrow[r, "\sim"] \arrow[dr, "\sim"'] & W(D^b_{\mathcal{A}(\omega)}(\mathcal{A}(\omega), {}^\vee, \tilde{\varpi})) \arrow[d, "\wr"] \\ & W(D^b(\mathcal{A}(\omega), {}^\vee, \tilde{\varpi})). \end{tikzcd}$$

The same holds for skew duality also.

Proof. Note that the diagonal isomorphism follows directly from [4, 4.7]. So, we need only to prove that the horizontal map is surjective. The proof would be very similar to that of Theorem 6.2, by an application [12, A.1, A.2]. Some clarification is needed, because $\mathcal{A}(\omega)$ may not have the 2-out-of-3 property, which was used in Theorem 6.2. First, we claim that for complexes $(\mathcal{E}_\bullet, \partial_\bullet)$ in $Ch^b_{\mathcal{A}(\omega)}(\mathcal{A}(\omega))$ all the boundaries $B_i := image(\partial_{i-1}) \subseteq \mathcal{E}_i$ and cycles $Z_i := \ker(\partial_i) \subseteq \mathcal{E}_i$ are in $\mathcal{A}(\omega)$. To see this note that $\mathbb{M}(\mathscr{V})$ is exact and epimorphisms in $\mathbb{M}(\mathscr{V})$ are admissible. Inducting from right, it follows $B_i, Z_i, \frac{\mathcal{E}_i}{B_i}$ are in $\mathbb{M}(\mathscr{V})$. Now, for $X = B_i, Z_i, \frac{\mathcal{E}_i}{B_i}$, by induction from left, it follows $Ext^i(X, \omega) = 0$ for all $i \neq d$, hence are in $\mathcal{A}(\omega)$. This establishes the claim. In fact, proof of [12, A.1, A.2] works whenever $B_i, Z_i, \frac{\mathcal{E}_i}{B_i}$ are in $\mathcal{A}(\omega)$ for all i. This completes the proof.

We comment that one can combine Proposition 3.3 and Theorem 7.3 to give a proof of the duality statement, by constructing a double complex in $D^b(\mathscr{V})$, as in [12, §3]. □

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