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HOMOTOPY OF SECTIONS OF PROJECTIVE MODULES

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Introduction

In the appendix of this paper, Nori discusses the question about sections of real vector bundles over smooth manifolds as follows.

Suppose that V is a smooth real vector bundle of rank n on a smooth manifold A. Let s_0 be a global section of V meeting the zero section of V transversally in the submanifold $B_0 \subset A$ and let B be a (smooth) submanifold of $A \times \mathbb{R}$ that meets $A \times \{0\}$ transversally in B_0. Now s_0 will induce an isomorphism

$$[s_0]: N(A, B_0) \to V|B_0,$$

from the normal bundle $N(A, B_0)$ of B_0 in A to the restriction of V to B_0.

Suppose that

$$\varphi: N(A \times \mathbb{R}, B) \to p_1^*(V)|B$$

is an isomorphism that is compatible with s_0 in the sense that $\varphi|B_0 = [s_0]$. Nori asked: *Can we find a global section s of $p_1^*(V)$ that meets the zero section of $p_1^*(V)$ transversally precisely on B, so that $[s] = \varphi$ and $s|A \times \{0\} = s_0$?*

In the appendix of the paper Nori answers this question affirmatively in the following two cases:

(a) $\dim B \leq n-2 \Leftrightarrow \dim A \leq 2n-3$,

(b) $B = B_0 \times \mathbb{R}$.

Motivated by this discussion, in the appendix of this paper, Nori asks the following algebraic analogue of this question.

Suppose $X = \operatorname{Spec} A$ is a smooth affine variety of dimension n. Let P be a projective A-module of rank r, and $S: P \to I$ a surjective homomorphism of P onto an ideal I of A. Assume that the zero set of I, $V(I) = Y$ is a smooth affine subvariety of dimension $n-r$. Also suppose

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that $Z = V(J)$ is a smooth closed subvariety of $X \times \mathbb{A}^1 = \mathrm{Spec}(A[t])$, where t is a variable, such that Z intersects $X \times 0$ transversally in $Y \times 0$. Also suppose that $\varphi\colon P[t] \to J/J^2$ is a surjective map which is compatible with S (i.e., $\varphi|_{t=0} = [S]$, the isomorphism induced by S from P/IP to I/I^2). The question of Nori is whether there is a surjective map $\psi\colon P[t] \to J$ such that (i) $\psi|_{t=0} = S$ and (ii) $\psi|_Z = \varphi$?

In this paper, we investigate this question for affine algebras (i.e., when A is a noetherian commutative ring). We have an affirmative answer in the following two cases:

(1) $\operatorname{rank} P \geq \dim Y + 3$ and J contains a monic polynomial (see (2.1)),

(2) $J = IA[t]$ is locally complete intersection of height > 2 and I/I^2 is free (see (2.3)).

In the first case (Theorem 2.1), we do not require any smoothness hypothesis. But we have to have the finiteness condition that J contains a monic polynomial. It will be interesting to know if this finiteness condition could be omitted. In the second case, also, it will be interesting to know if the condition that I/I^2 is free can be omitted.

1. Lemma of Quillen

In this section we write down some variants of Quillen's [Q] lemma.

(1.1) Lemma. *Let A be a commutative ring and R be an A-algebra. Suppose that f is an element in A and θ is a unit in $1 + TR_f[T]$, where $R[T]$ is the polynomial ring over R in the variable T. Then there is an integer k, such that for g_1, g_2 in A, whenever $g_1 - g_2$ is in f^kA, there is a unit ψ in $1 + fTR[T]$ such that $\psi_f(T) = \theta(g_1T)\theta(g_2T)^{-1}$.*

The following is an immediate consequence of (1.1).

(1.2) Lemma. *Let $B = A[T]$ be a polynomial ring over a commutative ring A and let N be an A-module and let $M = N \otimes A[T]$. Suppose that s and t are two elements in A such that $(s, t) = A$. Let φ be a unit in $1 + \mathrm{End}(N)_{st}[T]$. Then we can find a unit ψ_1 in $(1 + sT\,\mathrm{End}(N_t)[T])$ and a unit ψ_2 in $(1 + tT\,\mathrm{End}(N_s)[T])$ such that $\varphi = (\psi_1)_s \circ (\psi_2)_t$.*

Obviously, proofs of these two lemmas are exactly the same as in the paper of Quillen [Q].

2. Main results in the affine case

In this section we shall discuss our main results on the question of Nori. Theorem (2.1) is our result for nonextended ideals.

(2.1) Theorem. *Let $R = A[t]$ b*
noetherian ring A, and let I be
polynomial. Suppose that P is a
$\dim R/I + 2$ and suppose that $\mathcal{S}$
$I_0 = \{f(0) | f(x)$ is in $I\}$. Also
surjective map such that $\varphi(0) \equiv \mathcal{S}$
Then there is a surjective map
$\psi(0) = \mathcal{S}$.

Notations. Throughout this p
and use similar obvious notation

To prove (2.1) we need the fol

(2.2) Lemma. *In the setup of*
into I such that $\varphi'(0) = \mathcal{S}$.

Proof. Let $\eta\colon P[t] \to I$ be a
$\eta = \eta_0 + \eta_1 t + \cdots + \eta_k t^k$, wh
We also have $\eta_0 \equiv \mathcal{S}$ modulo
I_0^2. So, we can write $\mathcal{S} - \eta_0 =$
$f_i(t)$, $g_i(t)$ are in I and λ_i i
$\lambda = f_1(t)g_1(t)\lambda_1 + \cdots + f_n(t)g_n$
$\eta_0 + (\mathcal{S} - \eta_0) = \mathcal{S}$ and φ' lift

Proof of Theorem (2.1). Let
nomial, $\dim(A/J) = \dim(R/I)$
by the theorem of Serre, P/J
Hence $P_{1+J} \approx P' \otimes A^2_{1+J}$. So,
$A_{1+s}e_1 \oplus A_{1+s}e_2 \oplus Q$, for some
in P_{1+s}.

Let ϕ' be as in the Lemma

$$tI^2 \operatorname{Hom}(P, R)$$

and since $R_{1+s}e_1$ is a free di
$\phi' + \phi''$ with ϕ'' in $tI^2 \operatorname{Hom}($
a polynomial with leading coe

Let X be the set of all prin
and not containing tI_{1+s}. Cl

$$Q_0$$

Let φ_0 be the restriction
element of $Q_0^* \oplus A_{1+s}[t]$ an

(2.1) Theorem. *Let $R = A[t]$ be a polynomial ring over a commutative noetherian ring A, and let I be an ideal of R that contains a monic polynomial. Suppose that P is a projective A-module with* $\operatorname{rank} P = r \geq \dim R/I + 2$ *and suppose that $\mathcal{S}: P \to I_0$ is a surjective map, where $I_0 = \{f(0) | f(x) \text{ is in } I\}$. Also suppose that $\varphi: P \otimes A[t] \to I/I^2$ is a surjective map such that $\varphi(0) \equiv \mathcal{S}$ modulo I_0^2.*

Then there is a surjective map $\varphi: P \otimes A[t] \to I$ such that ψ lifts φ and $\psi(0) = \mathcal{S}$.

Notations. Throughout this paper, we shall denote $P \otimes A[t]$ by $P[t]$ and use similar obvious notations.

To prove (2.1) we need the following lemma.

(2.2) Lemma. *In the setup of* (2.1) *we can find a lift $\varphi': [t] \to I$ of φ into I such that $\varphi'(0) = \mathcal{S}$.*

Proof. Let $\eta: P[t] \to I$ be an lift of φ into I. Then we can write $\eta = \eta_0 + \eta_1 t + \cdots + \eta_k t^k$, where $\eta_0, \eta_1, \cdots, \eta_k$ are in $\operatorname{Hom}(P, A)$. We also have $\eta_0 \equiv \mathcal{S}$ modulo I_0^2. If follows that $\mathcal{S} - \eta_0$ maps into I_0^2. So, we can write $\mathcal{S} - \eta_0 = f_1(0)g_1(0)\lambda_1 + \cdots + f_n(0)g_n(0)\lambda_n$ where $f_i(t)$, $g_i(t)$ are in I and λ_i is in $\operatorname{Hom}(P, A)$ for $i = 1$ to n. Let $\lambda = f_1(t)g_1(t)\lambda_1 + \cdots + f_n(t)g_n(t)\lambda_n$ and let $\varphi' = \eta + \lambda$. Then $\varphi'(0) = \eta_0 + (\mathcal{S} - \eta_0) = \mathcal{S}$ and φ' lifts φ. This completes the proof of (2.2)

Proof of Theorem (2.1). Let $J = I \cap A$. Since I contains a monic polynomial, $\dim(A/J) = \dim(R/I)$. Also since $\operatorname{rank}(P/JP) \geq \dim(A/J) + 2$, by the theorem of Serre, P/JP has a free direct summand of rank two. Hence $P_{1+J} \approx P' \otimes A^2_{1+J}$. So, we can find an s in J such that $P_{1+s} = A_{1+s}e_1 \oplus A_{1+s}e_2 \oplus Q$, for some submodule Q of P_{1+s} and elements e_1, e_2 in P_{1+s}.

Let ϕ' be as in the Lemma (2.2). Since

$$tI^2 \operatorname{Hom}(P, R)_{1+s} = tI^2_{1+s} \operatorname{Hom}(P_{1+s}, R_{1+s})$$

and since $R_{1+s}e_1$ is a free direct summand of $P[t]_{1+s}$, changing ϕ' by $\phi' + \phi''$ with ϕ'' in $tI^2 \operatorname{Hom}(P, R)$, we can assume that $\phi'_{1+s}(e_1) = f_1$ is a polynomial with leading coefficient, a unit in A_{1+s} (we say f_1 is monic).

Let X be the set of all prime ideals in $\operatorname{spec}(A_{1+s}[t])$, containing (J, f_1) and not containing tI_{1+s}. Clearly, $\dim X < \operatorname{rank} Q_0$, where

$$Q_0 = (A_{1+s}e_2 \oplus Q)[t].$$

Let φ_0 be the restriction of φ'_{1+s} to Q_0. On X, (φ_0, t) is a basic element of $Q_0^* \oplus A_{1+s}[t]$ and $I^2 Q_0^*$ generate Q_0^*. Hence, we can find φ'_0

in $tI^2_{1+s}Q^*_0$ such that $\varphi''_0 = \varphi_0 + \varphi'_0$ is a basic element in Q^*_0 on X (see [EE]).

Define $\psi : P_{1+s}[t] \to I_{1+s}$ such that φ restriction of ψ to Q_0 is φ''_0 and $\psi(e_1) = f_1$. Note that $\psi(0) = \mathscr{S}$ and ψ is a lift of φ_{1+s}.

Claim that $\psi(P_{1+s}[t]) + JR_{1+s} = I'$ and I_{1+s} have the same radical. To see this let $\mathscr{Y}$ be a prime ideal in $\operatorname{spec}(R_{1+s})$ containing I' and not containing I. If t is not in $\mathscr{Y}$ then $\mathscr{Y}$ is in X. Hence $\varphi''_0(Q_0)$ is not contained in $\mathscr{Y}$, which is a contradiction. On the other hand if t is in $\mathscr{Y}$, then $I \subseteq (I_0, t) = (\mathscr{S}(P_{1+s}), t) = (\psi(P_{1+s}[t], t) \subseteq \mathscr{Y}$, which is again a contradiction. This establishes the claim.

Now it follows that $\psi_{1+J} : P[t]_{1+J} \to I_{1+J}$ is a surjective map. Because if $\operatorname{image}(\psi_{1+J})$ is contained in a maximal ideal M, then as f_1 is in M, J is also contained in M. Hence I is also contained in M. Now surjectivity follows from the fact that

$$\operatorname{image}(\psi_{1+J}) + I^2_{1+J} = I_{1+J}.$$

So, after modifying s, we can assume (1) s in J, (2) $P_{1+s}[t] = R_{1+s}e_1 \oplus R_{1+s}e_2 \oplus Q[t]$, and (3) there is a surjective map $\psi_1 : P_{1+s}[t] \to I_{1+s}$ such that $\psi_1(0) = \mathscr{S}$ and ψ_1 lifts φ_{1+s}.

Now let $\psi_2 : P_s[t] \to I_s$ be the extension of $\mathscr{S} : P_s \to I_{0s}$.

Consider the two exact sequences

$$0 \to K_1 \to P_{s(1+s)}[t] \xrightarrow{(\psi_1)_s} I_{s(1+s)} \to 0,$$
$$0 \to K_2 \to P_{s(1+s)}[t] \xrightarrow{(\psi_2)_{1+s}} I_{s(1+s)} \to 0.$$

where K_1 is the kernel of $(\psi_1)_s$ and K_2 is the kernel of $(\psi_2)_{1+s}$. Also note that K_1 and K_2 are projective and K_2 is extended. In fact, since $\psi_1(e_1) = f_1$ is monic, by the theorem of Horrocks [H], K_1 is locally extended and hence by Quillen's theorem [Q], K_1 is extended from $A_{s(1+s)}$.

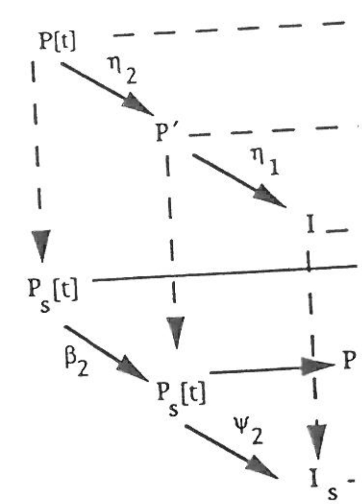
Let "$-$" denote modulo t. Since $\overline{\psi}_1 = \mathscr{S} = \overline{\psi}_2$, we have $\overline{K}_1 \approx \overline{K}_2$. Hence there is an isomorphism $\alpha_0 : \overline{K}_1 \to \overline{K}_2$ such that the diagram of exact sequence

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \overline{K}_1 & \longrightarrow & P_{s(1+s)} & \xrightarrow{\overline{\psi}_1} & \overline{I}_{s(1+s)} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_0} & & \| & & \| & & \\
0 & \longrightarrow & \overline{K}_2 & \longrightarrow & P_{s(1+s)} & \xrightarrow{\overline{\psi}_2} & \overline{I}_{s(1+s)} & \longrightarrow & 0
\end{array}$$

is commutative. Now extend α_0 to
α will induce an isomorphism β:
diagram

$$\begin{array}{ccccc}
0 & \longrightarrow & K_1 & \longrightarrow & P_{s(1+s} \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow \\
0 & \longrightarrow & K_2 & \longrightarrow & P_{s(1+}
\end{array}$$

is commutative. Since $\overline{\alpha} = \alpha_0$, it f
By Quillen's lemma (1.2), $\beta =$
$(1+ts\,\mathrm{End}(P_{1+s})[t])$ and β_2 is a unit
$= (\psi_1\beta_1)_s$.
Now we consider the fibre produ
Here P' is the fibre product of
are given by the properties of fibre
Let $\eta = \eta_1\eta_2 : P[t] \to I$. Then,
surjective, η is also surjective.
Since $\beta_2(0) = \mathrm{Id}_{P_s}$ and $\beta_1(0) =$
Figure 2 (see next page).



is commutative. Now extend α_0 to an isomorphism $\alpha: K_1 \xrightarrow{\sim} K_2$. Then α will induce an isomorphism $\beta: P_{s(1+s)}[t] \to P_{s(1+s)}[t]$, such that the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_1 & \longrightarrow & P_{s(1+s)}[t] & \xrightarrow{\psi_1} & I_{s(1+s)} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \| & & \\
0 & \longrightarrow & K_2 & \longrightarrow & P_{s(1+s)}[t] & \xrightarrow{\psi_2} & I_{s(1+s)} & \longrightarrow & 0
\end{array}$$

is commutative. Since $\overline{\alpha} = \alpha_0$, it follows that $\overline{\beta} = \mathrm{Id}$.

By Quillen's lemma (1.2), $\beta = (\beta_2)_{1+s}(\beta_1^{-1})_s$ where β_1 is a unit in $(1+ts\,\mathrm{End}(P_{1+s})[t])$ and β_2 is a unit in $(1+t\,\mathrm{End}(P_s)[t])$. Hence $(\psi_2\beta_2)_{1+s} = (\psi_1\beta_1)_s$.

Now we consider the fibre product diagram, as in Figure 1.

Here P' is the fibre product of $P_s[t]$ and $P_{1+s}[t]$ via β^{-1}, and η_1, η_2 are given by the properties of fibre product diagrams.

Let $\eta = \eta_1\eta_2: P[t] \to I$. Then, since η_2 is an isomorphism and η_1 is surjective, η is also surjective.

Since $\beta_2(0) = \mathrm{Id}_{P_s}$ and $\beta_1(0) = \mathrm{Id}_{P_{1+s}}$, it follows that $\eta(0)$ is given by Figure 2 (see next page).

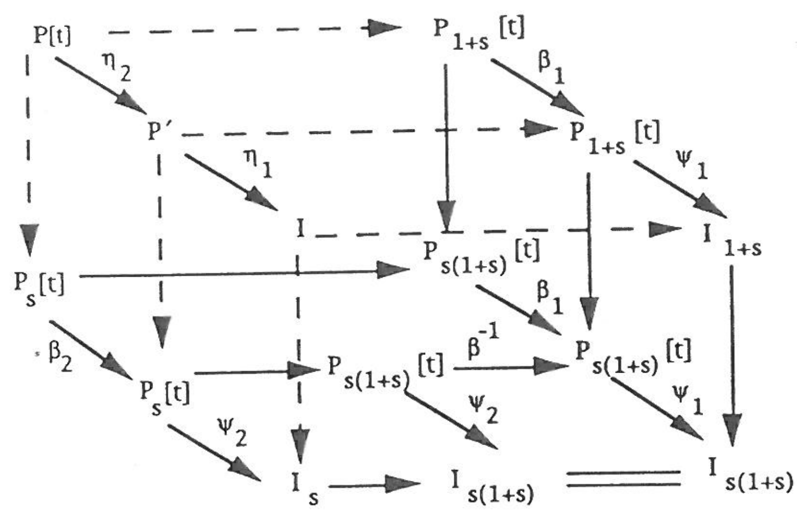


FIGURE 1

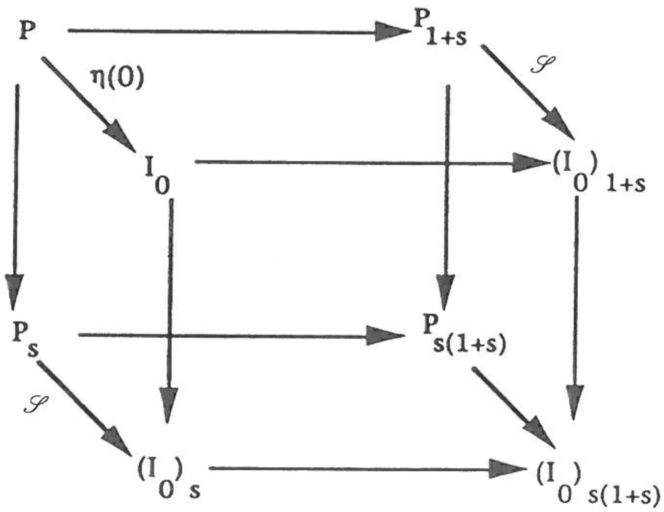


FIGURE 2

Hence $\eta(0) = \mathcal{S}$. Again since s is in I, $\beta_1 \equiv \mathrm{Id}$ modulo I. Hence $\eta \equiv \psi_1$ modulo I. Therefore η is also a lift of φ. This completes the proof of (2.1).

(2.3) Theorem. *Let $R = A[t]$ be a polynomial ring over an affine algebra A over a field k and let I_0 be a smooth and locally complete intersection ideal of height $r > 2$, in A with I_0/I_0^2 free. Write $I = I_0R$ and suppose P is a projective A-module of rank $r = \operatorname{height} I_0$. Let $\mathcal{S}: P \to I_0$ be a surjective map and let $\varphi: P[t] \to I/I^2$ be a surjective map such that $\varphi(0) \equiv \mathcal{S}$ modulo I_0^2. Then there is a surjective map $\psi: P[t] \to I$ such that $\psi(0) = \mathcal{S}$ and ψ lifts φ.*

Proof. Let $\varphi': P[t] \to I$ be the extension of $\mathcal{S}$. Let "—" denote modulo I. Then $\beta = (\overline{\varphi}')^{-1}\overline{\varphi}: P[t]/IP[t] \to P[t]/IP[t]$ is an isomorphism. But $P[t]/IP[t] \approx (P/I_0P) \otimes R$ since $\varphi(0) \equiv \mathcal{S}$ modulo I^2 and $\varphi'(0) \equiv \mathcal{S}$; it follows that $\beta \equiv \mathrm{Id}$ modulo t. Since $I_0/I_0^2 \approx P/I_0P$ is free, by the theorem of Vorst [V, Theorem (3.3)], β is an elementary transformation. Now by [BR], β can be lifted to an isomorphism $\gamma: P[t] \to P[t]$. We can also assume that $\gamma(0) = \mathrm{Id}_P$. Now let $\psi = \varphi' \circ \gamma$. Then $\psi \equiv \varphi'\beta \equiv \varphi$ modulo I, i.e., ψ is a lift of φ. Also note that $\psi(0) = \varphi'(0)\gamma(0) = \mathcal{S}$. This completes the proof of (2.3).

3. Appendix: Homotopy (
(by Madhav V. Nori,

Let V be a smooth real vect
manifold A. If s_0 is a global se
of V transversally in the submani
morphism

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Now let B be a (smooth) subm
transversally in B_0 and let

$$\varphi: N(A \times \mathbb{R}$$

be an isomorphism, so that $\varphi|B_0$
tion:

Is there a global section s of
$p_1^*(V)$ transversally precisely on
Sufficient conditions for an affi
given easily by using obstruction
The Topology of Fibre Bundles.

One first obtains a closed tub
sects $A \times \{0\}$ in a closed tubula
of $p_1^*(V)|W$ that vanishes precis

(a) $[s'] = \varphi$, and
(b) the restrictions of s' and
other.

We thus get a section on $W \cup$
$A \times \{0\} - \operatorname{Int} W$, and which we
on $A \times \mathbb{R} - \operatorname{Int} W$. By obstructi

$$H^i(A \times \mathbb{R} - \operatorname{Int} W$$

for all $i \geq n$ and for all loc
ensure such an extension. I
$H^i(A \times \mathbb{R}, W \cup A \times \{0\}; L)$ and
ogy for the triple $(A \times \mathbb{R}, W \cup$
with

$$H^{i-1}(W \cup A \times \{0\}, A \times$$

These groups vanish for all

3. Appendix: Homotopy of sections of vector bundles

(by Madhav V. Nori, University of Chicago)

Let V be a smooth real vector bundle of rank n on a smooth manifold A. If s_0 is a global section of V meeting the zero section of V transversally in the submanifold $B_0 \subset A$, we have an induced isomorphism

$$[s_0]\colon N(A, B_0) \to V|B_0,$$

where $N(A, B_0)$ is the normal bundle of B_0 in A.

Now let B be a (smooth) submanifold of $A \times \mathbb{R}$ that meets $A \times \{0\}$ transversally in B_0 and let

$$\varphi\colon N(A \times \mathbb{R}, B) \to p_1^*(V)|B$$

be an isomorphism, so that $\varphi|B_0 = [s_0]$. There is then the natural question:

Is there a global section s of $p_1^*(V)$ that meets the zero section of $p_1^*(V)$ transversally precisely on B, so that $[s] = \varphi$ and $s|A \times \{0\} = s_0$?

Sufficient conditions for an affirmative answer to this question may be given easily by using obstruction theory, as outlined in Steenrod's book, *The Topology of Fibre Bundles*.

One first obtains a closed tubular neighbourhood W of B that intersects $A \times \{0\}$ in a closed tubular neighbourhood of B_0, and a section s' of $p_1^*(V)|W$ that vanishes precisely on B so that

(a) $[s'] = \varphi$, and

(b) the restrictions of s' and s_0 to $A \times \{0\} \cap W$ coincide with each other.

We thus get a section on $W \cup A \times \{0\}$, which is *nonvanishing* on $W \cup A \times \{0\} - \operatorname{Int} W$, and which we need to extend to a nonvanishing section on $A \times \mathbb{R} - \operatorname{Int} W$. By obstruction theory, the vanishing of

$$H^i(A \times \mathbb{R} - \operatorname{Int} W, W \cup A \times \{0\} - \operatorname{Int} W; L)$$

for all $i \geq n$ and for all local systems L on $A \times \mathbb{R}$, is sufficient to ensure such an extension. By excision, these groups coincide with $H^i(A \times \mathbb{R}, W \cup A \times \{0\}; L)$ and from the long exact sequence of cohomology for the triple $(A \times \mathbb{R}, W \cup A \times \{0\}, A \times \{0\})$, these groups coincide with

$$\begin{aligned} H^{i-1}(W \cup A \times \{0\}, A \times \{0\}; L) &= H^{i-1}(W, W \cap A \times \{0\}; L) \\ &= H^{i-1}(B, B_0; L). \end{aligned}$$

These groups vanish for all $i \geq n$ if

(a) $\dim B \leq n-2 \Leftrightarrow \dim A \leq 2n-3$, or if
(b) $B = B_0 \times \mathbb{R}$.
So, if (a) or (b) holds, the global section s of $p_1^*(V)$ does indeed exist.

Acknowledgments

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