Ideal theoretic approach to Algebraic ObstructionTheory

Satya Mandal, U. Kansas
K-theory working group, ICTP, Italy

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Correspondence

Theorem ([Swan 1962])

Suppose M is a (compact connected) Hausdorff topological space. Then the association

$$\mathcal{E} \to \Gamma(\mathcal{E})$$

is an equivalence of catagories, between the category $\mathcal{V}(M)$ of vector bundles on M and the $\mathcal{P}(C(M))$ of finitely generated projective modules over the ring of real valued continuous functions $C(M)$.

The Stage

- This theorem sets the stage for research in projective modules over noetherian commutative rings A to follow the lead of the theory of vector bundles.

Background

- The obstruction theory in topology is classical.
- The advent of obstruction theory in algebra is a more recent phenomenon.
- The germ of an algebraic obstruction theory was given by Madhav Nori, around 1990.

The Theme

- Suppose A is a commutative ring of dimension n. Given a projective $A-$module P of rank r, question is whether we can define an obstruction group $E(P)$ and an obstruction invariant $e(P) \in E(P)$ such that

$$e(P) = 0 \iff P \approx Q \oplus A.$$

Serre's Splitting Theorem

- In this lecture A will always denote a noetherian commutative ring with $\dim A = d$.
- Again, following the lead of corresponding theorem in topology, we have the following.

Theorem ([Serre1957])

If P is a projective $A-$module of rank $r > d$, then $P \approx Q \oplus A$ for some Q.

- So, we would study projective modules P of rank $r \leq d$.

Two Approches

There are two approches to algebraic Obstruction theory.

- The ideal theoratic approach: Madhav Nori provided the germ of an algebraic obstruction theory around 1990 ([MS, Ma1, BS2]). The obstruction groups are generated by ideals and local orientations.
- $K-$Theoretic apporach: Subsequently, Barge and Morel ([BM]) proposed a $K-$theoretic approach to the same in 2000. This approah was given a complete shape by Jean Fasel ([Fasel1, Fasel2, FS]).

The Status of these two approaches

- After Nori provided a sense of direction, there was a flurry of activities over the last two decades.
  - The activities remained mainly focused within the top rank case (i.e. when $rank(P) = d$). The theory seems complete, in this case.
  - However, it fails to be functorial, due to the lack of progress in other cases.
  - This theory applies to noetherian commutative rings, without any smoothness hypothesis.

The Status of these two approaches

- The $K-$theoretic approach seems more complete and functorial. It applies mostly when the ring A is regular.
- Two approaches are yet to be fully reconciled. Perhaps, this is because the theory is not developed enough in ideal theoretic approach.
- In this talk, we will mainly discuss the ideal theoretic approach. I am hopeful, others will speak on the $K-$thoeretic approach.

The Predecessor

- Nori's introduction of the program preceded by the work of M. P. Murthy and N. Mohan Kuman ([MkM, Mk2, Murthy 1994, MM]) on Algebraic obstruction theory for affine algebras A over algebraically closed fields.
- For such a ring A, with $\dim A = d$ the Chow group $CH^d(A)$ is the obstrction group. For a projective $A-$module P with rank d the obstraction class was the Chern class $C^d(P) \in CH^d(A)$.

The Predecessor: Continued

- Murthy proved ([Murthy 1994]), for P as above
$$P = Q \oplus A \Longleftrightarrow C^n(P) = 0.$$
- Note the theorem works only in the top rank case.
- There are examples ([Mk1]) stably free non-free projective $A-$modules Q with $rank(Q) = n < d$. So, vanishing of the top Chern class $C^n(Q) = 0$ would not guaranty that $Q = Q_0 \oplus A$.

The Definition: Overview

Notations: A will always denote a noetherian commutative ring, $\dim A = d$ and L will denote a projective $A-$module of rank one.

- For integers $0 \leq n \leq d$, an obstruction group $E^n(A, L)$ was defined.
- Let P be a projective $A-$module of rank d with $\det(P) = L$. Assume $\mathbb{Q} \subseteq A$. Given an isomorphism $\chi : L \xrightarrow{\sim} \wedge^d P$ an obstruction class

$$e(P, \chi) \in E^d(A, L) \quad \textit{was defined}.$$

- Such an $\chi : L \xrightarrow{\sim} \wedge^d P$ is called an orientation of P.

Continued

- In fact, such orientation $\chi : L \xrightarrow{\sim} \wedge^d P$ induces an isomorphism

$$\varphi_\chi : E^d(A, L) \xrightarrow{\sim} E^d(A, \wedge^d P) \quad and \; \varphi_\chi(e(P, \chi)) = e(P, id)$$

- I like to denote

$$e(P) := e(P, id) \in E^d(A, \wedge^d P).$$

Formal Definitions

In this section (in next several frames) we define Euler (obstruction) class groups of commutative noetherian rings.
Definitions.

- We write $F_r = L \oplus A^{r-1}$.
- For an $A-$module M, the group of transvections of M will be denoted by $\mathcal{E}l(M)$ (see [Ma3] for a definition).

Local Orientation

A **local $L-$orientation**, of codimension r, is a pair (I, ω), where

1. I is an ideal of A of height r and
2. ω is an equivalence class of surjective homomorphisms $\omega : F_r/IF_r \twoheadrightarrow I/I^2$.
3. The equivalence is defined by $\mathcal{E}I(F_r/IF_r)-$maps. Sometimes, we denote the equivalence class of ω, by ω.
4. Let $G^r(A, L)$ denote the free abelian group generated by all local orientations (I, ω) of codimension r such that $Spec(A/I)$ is connected.

Global Orientations

1. Suppose I is an ideal of height r and $\omega : F_r/IF_r \twoheadrightarrow I/I^2$ is a local $L-$orientation.
2. By [BhatwaSri2000], there is a unique decomposition
$$I = I_1 \cap I_2 \cap \cdots \cap I_k \quad \ni \quad \forall i \neq j \quad I_i + I_j = A$$
and $Spec(A/I_i)$ is connected.
3. Then ω naturally induces local $L-$orientations $\omega_i : F_r/I_iF_r \twoheadrightarrow I_i/I_i^2$. Denote
$$(I, \omega) := \sum (I_i, \omega_i) \in G^r(A, L).$$

Global Orientations: Continued

1. A local $L-$orientation (I,ω), of codimension r is said to be a Global orientation if there is a surjective lift Ω of ω as follows:

$$\begin{array}{ccc} F_r & \overset{\Omega}{\dashrightarrow\!\!\!\rightarrow} & I \\ \downarrow & & \downarrow \\ F_r/IF_r & \underset{\omega}{\twoheadrightarrow} & I/I^2 \end{array}.$$

2. Let $\mathcal{R}^r(A,L)$ be the subgroup of $G^r(A,L)$, generated by global $L-$orientations.

The Euler class Groups

Now define the **Euler class group** of codimension r cycles as

$$E^r(A, L) = \frac{G^r(A, L)}{\mathcal{R}^r(A.L)}.$$

The Obstruction Classes

- Suppose P a projective $A-$module of rank r.
- In the ideal theoretic approach, there is no satifactory definition of an obstruction (to be called Euler class) class $e(P)$ when $rank(P) = r < d = \dim A$.
- However, a satisfactory definiton of $e(P)$ is available when $rank(P) = d = \dim A$.
- In fact, the theory is fairly complete when $rank(P) = d = \dim A$.

Definition: The Euler Classes

Now we assume that $\mathbb{Q} \subseteq A$.

- Let P be a projective $A-$module of rank d, and $\det(P) \approx L$.
- Let $\chi : L \xrightarrow{\sim} \wedge^d P$ be an orientation.
- Let $f : P \twoheadrightarrow I$ be a surjective homomorphism, where I is an ideal of height d.

Continued: The Euler Classes

- Pick an isomorhism $\gamma : F/IF \xrightarrow{\sim} P/IP$ that is compatible with the orientation $\chi : L \xrightarrow{\sim} \wedge^d P$.
- define ω by the commutative diagram

$$\begin{array}{ccccc} P & \longrightarrow & P/IP & \xleftarrow[\gamma]{\sim} & F/IF \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \overline{f}} & \swarrow{\scriptstyle \omega} & \\ I & \longrightarrow & I/I^2 & & \end{array} \qquad \textit{where} \qquad F = L \oplus A^{d-1}.$$

Continued: The Euler Classes

- Define the Euler class

$$e(P, \chi) = (I, \omega) \in E^d(A, L)$$

- In particular, define

$$e(P) := e(P, id) \in E^d(A, \wedge^d P)$$

- This version of the definition of $E^d(A, L)$ and $e(P, \chi)$ are due to Bhatwadekar and Sridharan ([BhatwaSri2000]).

The Obstruction Theorem

Bhatwadekar and Sridharan proved Nori's Obstruction Conjecture.

Theorem ([BhatwaSri2000])

- *Suppose A is a noetherian commutative ring with* $\dim A = d \geq 2$. *Assume $\mathbb{Q} \subseteq A$.*
- *Suppose P is a projective $A-$module d,*

$$e(P) = 0 \in E^d\left(A, \wedge^d P\right) \Longleftrightarrow P = Q \oplus A.$$

The Real Manifold

- Let $A = \mathbb{R}[x_1, x_2, \ldots, x_n]$ be finitrly generated algebra over the reals $\mathbb{R}$.
- Write $A = \frac{\mathbb{R}[X_1, X_2, \ldots, X_n]}{I}$ where I is an ideal of the polynomial ring $\mathbb{R}[X_1, X_2, \ldots, X_n]$.
- Let M be the set of real points $v \in \mathbb{R}^n$ such that $f(v) = 0$ for all $f \in I$.
- Assume is A is smooth. Then $M \subseteq \mathbb{R}^n$ is a smooth maifold with $\dim M = \dim A = d$. (Implicit function theorem.)

The Structure Theorem

Theorem ([BhatwadekarDasMandal])

Let $X = Spec(A)$ be a smooth real affine variety with $\dim A = d \geq 2$ and $M = M(A)$ be the corresponding manifold.

- *Write $\mathbb{R}(X) = S^{-1}A$, where S is the set of all $f \in A$ that does not vanish at any real point X. (All complex points are killed.)*
- *$C_1, \cdots, C_t$ be the compact connected components of M.*

Theorem: Continued

- $K = \wedge^n(\Omega_{A/\mathbb{R}})$ be the canonical module of A and K_{C_i} be the induced line bundle on C_i.
- Let L be a projective $\mathbb{R}(X)$-module of rank 1 and L_{C_i} be the induced line bundle on C_i.
- Assume that

$$L_{C_i} \simeq K_{C_i} \text{ for } 1 \leq i \leq r \quad \text{and} \quad L_{C_i} \not\simeq K_{C_i} \text{ for } r+1 \leq i \leq t.$$

Then,

$$E^d(\mathbb{R}(X), L) = \mathbb{Z}^r \bigoplus \left(\frac{\mathbb{Z}}{(2)}\right)^{t-r}$$

Topological Obstructions ([Steenrod1951])

- Suppose M is a real smooth manifold with $\dim M = d \geq 2$ and $\mathcal{L}$ is a line bundle over M. Then, for $0 \leq n \leq d$, there are obstruction groups $\mathcal{H}^n(M, \mathcal{L})$.
- If $\mathcal{L}$ is trivial (the orientable case), these groups turn out to be the singular cohomology groups $H^n(M, \mathbb{Z})$. In the non-orientable case, they are the cohomology group $H^n(M, \mathcal{G}_{\mathcal{L}})$ with local coefficients in a bundle of groups.

Topological Obstructions

- For a vector bundle $\mathcal{E}$ on M with rank $r \leq d$, there is an invariant
$$w(\mathcal{E}) \in \mathcal{H}^r\left(M, \wedge^r \mathcal{E}\right).$$
- If $\mathcal{E}$ has a never-vanishing section, then $w(\mathcal{E}) = 0$.
- For rank $r = d$, conversely,

$$w(\mathcal{E}) = 0 \Longrightarrow \quad \mathcal{E} = \mathcal{F} \oplus \mathcal{R}.$$

Algebra and topology

With notaions as in the above theorem, the obstructions in topology and algebra relates as follows.

Theorem ([MandalSheu3])

- *There is a canonical homomorphism* $\epsilon : E(A, L) \to \mathcal{H}^d(M, \mathcal{L}^*)$.
- *In fact,* ϵ, *factors through an isomorphism*

$$E(X(\mathbb{R}), L \otimes X(\mathbb{R})) \xrightarrow{\sim} \mathcal{H}^d(M, \mathcal{L}^*) \quad \textit{where} \quad S \quad \textit{is}$$

- For a projective $A-$module P of rank d, we have

$$\epsilon(e(P)) = w(\mathcal{E}^*) \quad \textit{where} \quad \mathcal{E} \quad \textit{is} \quad \textit{the vector bundle}$$

on M with the module of sections $= P \otimes C(M)$.

The Homomorphism ϵ: Orientable case

We will define ϵ in the orientable case.

- Assume M is orientable.
- Let $C_1, \ldots, C_r$ be the compact connected components of M. Then, the topological obstruction group $\mathcal{H}^d(M, \mathcal{R}) = H^d(M, \mathbb{Z}) = \oplus_{i=1}^r H^d(C_i) = \mathbb{Z}^r$.
- One can prove, in this case, $E^d(A, A)$ is generated by local orientations (m, ω), where $m \in Max(A)$.
- We will give $\epsilon(m, \omega) \in H^d(M, \mathbb{Z})$.

Orientable case

- Suppose (m, ω) is a generator of $\mathcal{G}(A)$, where m is a maximal ideal of A.
- Let v be the point (real or complex) corresponding to m.
- Define

$$\epsilon(m, \omega) = 0 \qquad \textit{if} \quad v \notin \cup C_i$$

Orientable case

- Now suppose, $v \in C_i$, and ω is given by $f_1, \ldots, f_d$.

$$So, \qquad m = (f_1, \ldots, f_d) + m^2.$$

Note $(f_1, \ldots, f_d)$ has an isolated zero at m. Define

$$\epsilon(m, \omega) = index(f_1, \ldots, f_d) \in \mathbb{Z} = H^d(C_i, \mathbb{Z}).$$

Orientable case

- Suppose (I, ω_I) is a global orientation, where I reduced ideal. So, $I = m_1 \cap \cdots \cap m_r \cap \ldots \cap m_t$, where $m_1, \ldots m_r$ are real maximal ideals and $m_{r+1}, \ldots, m_t$ are complex maximal ideal.
- So, there is a a surjective lift of $A^d \twoheadrightarrow I$ of ω_I.
- Then, $\epsilon(I, \omega_I)$ is the topological Euler class of the trivial bundle of rank d. So, $\epsilon(I, \omega_I) = 0$.
- So, ϵ factors through a homomorphism

$$\epsilon : E^d(A, A) \to H^d(M, \mathbb{Z}). \qquad \blacksquare$$

Non-Orientable case

The definition of the homomorphism is similar in the non-orientable case. The index is defined only "modulo 2".

On the real Sphere $\mathbb{S}^n$

- As before, let

$$A_n = \frac{\mathbb{R}[X_0, \ldots, X_n]}{(X_0^2 + \cdots + X_n^2 - 1)} = \mathbb{R}[x_0, x_1, \ldots, x_n]$$

  be the algebraic coordinate ring of $\mathbb{S}^n$ with $n \geq 2$.
- We have $E^n(A_n, A_n) = \mathbb{Z}$.
- The tangent bundle T_n is defined by

$$0 \longrightarrow T_n \longrightarrow A_n^{n+1} \xrightarrow{(x_0, \ldots, x_n)} A_n \longrightarrow 0.$$

On the real Sphere $\mathbb{S}^n$

- If n is odd, then $e(T_n) = 0$.
- If n is even, then $e(T_n) = \pm 2$. *This is a fully algebraic proof that T_n does not have a free direct summand. This result corresponds to the topological result that the tangent bundle on an even dimensional sphere, does not have a no-where vanishing section.*

Vanishing of Chern Classes

For real smooth varieties, deficiencies of the top Chern class, as obstruction class, was considered.

Theorem ([BhatwadekarDasMandal])

Let $X = \mathrm{Spec}(A)$ be a smooth affine variety of dimension $d \geq 2$. over the field $\mathbb{R}$ of real numbers. Let $K = \wedge^d(\Omega_{A/\mathbb{R}})$ denote the canonical module. Let P be a projective A-module of rank d and let $\wedge^d(P) = L$. Let $M(A)$ denote the real manifold of A. Assume that the top Chern class $C^d(P) = 0 \in CH^d(X)$.

Continued

Then $P \simeq A \oplus Q$ in the following cases:

1. $M(A)$ has no compact connected component.
2. For every compact connected component C of the manifoled $M(A)$, $L_C \not\simeq K_C$ where K_C and L_C denote restriction of (induced) line bundles on $M(A)$ to C.
3. n is odd.

Continued

Moreover, if d is even and L is a rank 1 projective A-module such that there exists a compact connected component C of $M(A)$ with the property that $L_C \simeq K_C$, then there exists a projective A-module P of rank d such that $P \oplus A \simeq L \oplus A^{d-1} \oplus A$ (hence $C^d(P) = 0$) but P does not have a free summand of rank 1.

Preview: Low co-dimension case

- The definition of $E^r(A, L)$ was given above, for integers $0 \leq r \leq d = \dim A$.
- For co-dimension $r < d$, theory is not as satisfactory,in the ideal theoretic approach. However, $K-$theoretic approach seems very complete.
- Among the deficiencies, is the failure to give a definition of the obstruction classes $e(P)$ for projective $A-$modules of rank $r < d$.

Whitney class homomorphism

Theorem ([MandalYang 2010])

Suppose P is a projective A−module of rank $r \leq d = \dim A$ *Let* L, L' *is a projective A−module of rank one. Let Q be a projective A−module rank n and* $\chi : L \xrightarrow{\sim} \wedge^n Q$ *be an orientation. Then,*

- *a Whitney class homomorphism was defined:*

$$w(Q, \chi) : E^{d-r}(A, L) \to E^d(A, LL').$$

- *If* $Q \approx Q_0 \oplus A$, *then* $w(Q, \chi) = 0$.

the Definition of $w(Q, \chi)$

The homomorphism $w(Q, \chi)$ is given as follows:

- Write $F_k = L \oplus A^{k-1}, F'_k = L' \oplus A^{k-1}$.
- Let I be an ideal of height $d - n$ and

$$\omega : F'_{d-n}/IF'_{d-n} \twoheadrightarrow I/I^2$$

  be local $L'-$orientation.
- There is an ideal $\tilde{I} \subseteq A$ with $height(\tilde{I}) \geq d$ and a surjective homomorphism $\psi : Q/IQ \twoheadrightarrow \tilde{I}/I$.

Continued

- There is an isomorphism $\gamma : F_n/\tilde{I}F_n \stackrel{\sim}{\to} Q/\tilde{I}Q$ that is compatibale with the orientation χ.
- Let $\beta = \bar{\psi}\gamma$ and $\beta' : F_n/\tilde{I}F_n \to \tilde{I}/\tilde{I}^2$ be a lift of β. The following diagram

$$\begin{array}{ccccc} Q/IQ & \twoheadrightarrow & Q/\tilde{I}Q & \xleftarrow[\sim]{\gamma\sim\chi} & F_n/\tilde{I}F_n \\ \downarrow \psi & & \downarrow \bar{\psi} & \swarrow \beta & \downarrow \beta' \\ \tilde{I}/I & \longrightarrow & \tilde{I}/(I+\tilde{I}^2) & \xleftarrow[f]{} & \tilde{I}/\tilde{I}^2 \end{array} \qquad (I)$$

Continued

- Further, ω induces following

$$\begin{array}{ccc} F'_{d-n}/IF'_{d-n} & \stackrel{\omega}{\twoheadrightarrow} & I/I^2 \\ \downarrow & & \downarrow \\ F'_{d-n}/\tilde{I}F'_{d-n} & \stackrel{\bar{\omega}}{\twoheadrightarrow} & (I+\tilde{I}^2)/\tilde{I}^2 \\ & \searrow^{\omega'} & \downarrow \\ & & \tilde{I}/\tilde{I}^2 \end{array} \qquad (II)$$

Continued

- Combining ω', β' we get a surjective homomorphism

$$\delta = \beta' \oplus \omega' : F_n/\tilde{I}F_n \oplus F'_{d-n}/\tilde{I}F'_{d-n} = \frac{F_n \oplus F'_{d-n}}{\tilde{I}\left(F_n \oplus F'_{d-n}\right)} \twoheadrightarrow \tilde{I}/\tilde{I}^2$$

- Now, let $\gamma_0 : \frac{LL' \oplus A^{d-1}}{\tilde{I}\left(LL' \oplus A^{d-1}\right)} \xrightarrow{\sim} \frac{F_n \oplus F'_{d-n}}{\tilde{I}\left(F_n \oplus F'_{d-n}\right)}$ be an isomorphism that is consistent with the natural isomorphism χ_0:

$$\begin{array}{ccc} LL' & \xrightarrow{\sim} & \wedge^d\left(F_n \oplus F'_{d-n}\right) \\ & \searrow^{\sim} & \uparrow \wr\, \chi_0 \\ & & \wedge^d\left(LL' \oplus A^{d-1}\right) \end{array}.$$

Continued

- Let $\Delta = \delta\gamma_0 = (\beta', \omega')\gamma_0$. So, the diagram

$$\begin{array}{ccc} \dfrac{F_n \oplus F'_{d-n}}{\tilde{I}\left(F_n \oplus F'_{d-n}\right)} & \overset{\delta}{\twoheadrightarrow} & \tilde{I}/\tilde{I}^2 \\ \chi_0 \sim \gamma_0 \uparrow \wr & \nearrow_{\Delta} & \\ \dfrac{LL' \oplus A^{d-1}}{\tilde{I}\left(LL' \oplus A^{d-1}\right)} & & \end{array}$$

  commutes.
- Finally, the association

$$(I, \omega) \mapsto \left(\tilde{I}, \Delta\right) \in E^d(A, LL')$$

The Multiplicative Structure

Theorem ([MandalYang 2010])

- *A multiplicative structure:*

$$\bigoplus_{r=0}^{d} E^r(A,A) \times \bigoplus_{r=0}^{d} E^r(A,A) \xrightarrow{\ \cap\ } \bigoplus_{r=0}^{d} E^r(A,A)$$

is defined, in a natural way.

Continued

- Suppose I ideal of height r and J is an ideal of height s.

$$\omega : F/IF \twoheadrightarrow I/I^2 \quad and \quad \omega' : F'/JF' \twoheadrightarrow J/J^2$$

  are two local orientations, where $F = A^r, F' = A^s$.
- Then, ω, ω' induces a surjective homomorphism

$$\eta : \frac{F \oplus F'}{(I+J)(F \oplus F')} \twoheadrightarrow \frac{(I+J)}{(I+J)^2}$$

- If $height(I+J) \geq r+s$, the intersection is defined by

$$(I, \omega) \cap (J, \omega') := (I+J, \eta) \in E^{r+s}(A, L).$$

The restriction map

Definition ([Mandal Yang 2012]): Let A be a noetherian commutative ring with $\dim A = d$ and $J \subseteq A$ be an ideal. Let L be a rank one projective $A-$module. For integers n, with $2n \geq d+3$, there is a group homomorphism

$$\rho = \rho_J : E^n(A, L) \to E^n\left(\frac{A}{J}, \frac{L}{JL}\right)$$

defined as follows:

Continued

- Write $F = L \oplus A^{n-1}$. Let $\omega : F/IF \twoheadrightarrow I/I^2$ be local $L-$orientation.
- Find an ideal I_1 and a local orientation $\omega_1 : F/I_1F \twoheadrightarrow I_1/I_1^2$ such that $(I, \omega) = (I_1, \omega_1)$ and *height* $\left(\frac{I_1+J}{J}\right) \geq n$.
- Then, ω_1 induces an orientation β as follows:

$$\begin{array}{ccc} F/I_1F & \xrightarrow{\omega_1} & I_1/I_1^2 \\ \downarrow & & \downarrow \\ \frac{F}{(I_1+J)F} & \xrightarrow[\beta]{} & \frac{I_1+J}{I_1^2+J} \end{array}.$$

Continued

- Define

$$\rho(I,\omega) = \left(\frac{I_1+J}{J}, \beta\right) \in E^n\left(\frac{A}{J}, \frac{L}{JL}\right).$$

Some Pull Back

Definition([Mandal Yang 2012]) Let $f : R \to A$ be a ring homomorphisms.

- Let $n \geq 1$ be a fixed integer.
- Let L be a rank one projective $R-$module and $L' = L \bigotimes A$.
- Assume f is flat or more generally, that for any ideal I of R, which is locally generated by n elements and $height(I) = n$, we have $height(IA) \geq n$.

Continued

- Then, there is a homomorphism

$$f^* : E^n(R, L) \to E^n(A, L') \qquad \textit{defined as follows :}$$

Continued

- Write $F = L \oplus R^{n-1}, F' = F \otimes A$.
- Let I be an ideal of R of height n and $\omega : F/IF \twoheadrightarrow I/I^2$ be a local $L-$orientation.
- Then ω induces a local $L'-$orientation ω' by the diagram:

$$\begin{array}{ccc} F/IF & \xrightarrow{\ \omega\ } & I/I^2 \\ \downarrow & & \downarrow \\ \frac{F \otimes A}{I(F \otimes A)} & \xrightarrow[\omega']{} & \frac{IA}{I^2 A}. \end{array}$$

Continued

- The association $(I, \omega) \mapsto (J, \omega') \in E^n(A, L')$ defines the pull back homomorphism

$$f^* : E^n(R, L) \longrightarrow E^n(A, L')$$

The Relative groups

Definition([Mandal Yang 2012]) Suppose $\mathcal{I}$ is an ideal of A.

- Let $D(A, \mathcal{I})$ be defined by the fiber product diagram:

$$\begin{array}{ccc} D(A,\mathcal{I}) & \xrightarrow{p_1} & A \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle q} \\ A & \xrightarrow[q]{} & \frac{A}{\mathcal{I}} \end{array}$$

- Define the relative Euler class group

$$E^n(A, \mathcal{I}, L) = Kernel\left(p_1^*\right).$$

Excision Theorems

Theorem ([Mandal Yang 2012])

Let $A, L, \mathcal{I}, p_1, p_2, q$ *be as above and* $\dim A = d$*. For integers* n *, assume* $2n \geq d+3$*.*

- *Then, the following*

$$E^n(A, \mathcal{I}, L) \xrightarrow{p_2^*} E^n(A, L) \xrightarrow{q^*} E^n(\tfrac{A}{\mathcal{I}}, \tfrac{L}{\mathcal{I}L})$$

is an exact sequence.

Excision: Continued

Theorem ([Mandal Yang 2012])

- *Further, assume* $q : A \twoheadrightarrow \frac{A}{\mathcal{I}}$ *has a splitting* β *and* $L = L_0 \otimes_\beta A$ *for some* $\frac{A}{\mathcal{I}}$*−module. Then the following*

$$0 \longrightarrow E^n(A, \mathcal{I}, L) \xrightarrow{p_2^*} E^n(A, L) \xrightarrow{q^*} E^n(\frac{A}{\mathcal{I}}, L_0)$$

is an exact sequence.

Excision: Continued

Theorem ([Mandal Yang 2012])

- *Further, if β^* is defined, then the following*

$$0 \longrightarrow E^n(A, \mathcal{I}, L) \xrightarrow{p_2^*} E^n(A, L) \xrightarrow{q^*} E^n\left(\frac{A}{\mathcal{I}}, L_0\right) \longrightarrow 0$$

is exact.

Polynomial Rings

Corollary ([Mandal Yang 2012])

- *Let R be a commutative ring with* $\dim R = d$.
- *Let $A = R[X]$ and $B = R[X, X^{-1}]$.*
- *Let L_0 be a projective $R-$module of rank one.*
- *Write $L = L_0 \otimes A, \ L' = L_0 \otimes B$.*
- *Assume that $2n \geq d + 4$.*

Continued

Then,

- The sequence,

$$0 \longrightarrow E^n(A,(X),L) \longrightarrow E^n(A,L) \longrightarrow E^n(R,L_0) \longrightarrow 0$$

  is a split exact sequence.

- The sequence,

$$0 \longrightarrow E^n(B,(X-1),L') \longrightarrow E^n(B,L') \longrightarrow E^n(R,L_0) \longrightarrow$$

  is a split exact sequence.

Continued

Then,

- Further, if R is a regular domain that is essentially of the finite type over an infinite field k, then

$$\rho_X : E^n(A, L) \xrightarrow{\sim} E^n(R, L_0)$$

is an isomorphism. In particular, the relative group

$$E^n(A, (X), L) = 0.$$

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