K-theory of the categories of (perfect) $C\mathbb{M}$- modules

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Prelude

- I hope to be able to show,
  how much **commutative algebra** is in there to **cultivate and harvest**, in **Algebraic K-Theory**.

Prelude

- Algebraic K-Theory may mean different things to **different cohort**.
- Most of them are doing Motivic ($\mathbb{A}^1$) Homotopy. They call that Algebraic K-Theory.
- I talk about Quillen K-Theory.
- Then this is that good old Classocal K-Theory. $K_0(A)$, $K_1(A)$, $K_2(A)$ for a commutative ring A.

Basic Notations

- Throughout, X will denote a quasi projective scheme over a noetherian affine scheme $\operatorname{Spec}(A)$.
  You can assume $X = \operatorname{Spec}(A)$.
- We use the usual notations

$$\begin{cases} QCoh(X) = Category\ of\ quasi\ coherent\ modules\ on\ X \\ Coh(X) = Category\ of\ coherent\ modules\ on\ X \\ \mathscr{V}(X) = Category\ of\ locally\ free\ modules\ on\ X \end{cases}$$

(Perfect) $C\mathbb{M}(X)$ Modules

- Recall, $grade(M) = \min\left\{r : \mathcal{E}xt^r\left(M, \mathcal{O}_X\right) \neq 0\right\}$.
  If X is Cohen-Macaulay then

$$grade(M) = co\dim\left(\mathrm{Supp}\left(M\right)\right) = height(ann(M))$$

- For a closed subset $Z \subseteq X$,

$$grade(Z, X) := grade\left(\mathcal{O}_Z\right) \ \textit{with any subscheme structure } \mathcal{O}_Z$$

(Perfect) $C\mathbb{M}(X)$ Modules

- A module $M \in Coh(X)$ will be called **$C\mathbb{M}$-module** if

$$grade(M) = \dim_{\mathscr{V}(X)} M =: k$$

- They have only one non vanishing $\mathcal{E}xt^{k}(M, \mathcal{O}_X) \neq 0$.
- A $C\mathbb{M}(X)$-module is also called a **Perfect** module. There is a limited amount of literature on these.

Examples

- Suppose $\mathscr{I} \subseteq \mathcal{O}_X$ is a locally complete intersection ideal with $height(\mathscr{I}) = k$. Then

$$M = \frac{\mathcal{O}_X}{\mathscr{I}} \quad \text{is a } C\mathbb{M}(X)-\text{module.}$$

Notations

Let $k \in \mathbb{Z}$ $k \geq 0$. Denote the full subcategories

$$\begin{cases} \mathbb{M}(X) = \{M \in Coh(X) : \dim_{\mathscr{V}(X)}(M) < \infty\} \\ \mathbb{M}^k(X) = \{M \in \mathbb{M}(X) : grade(M) \geq k\} \\ C\mathbb{M}^k(X) = \left\{M \in \mathbb{M}(X) : grade(M) = \dim_{\mathscr{V}(X)}(M) = k\right\} \end{cases}$$

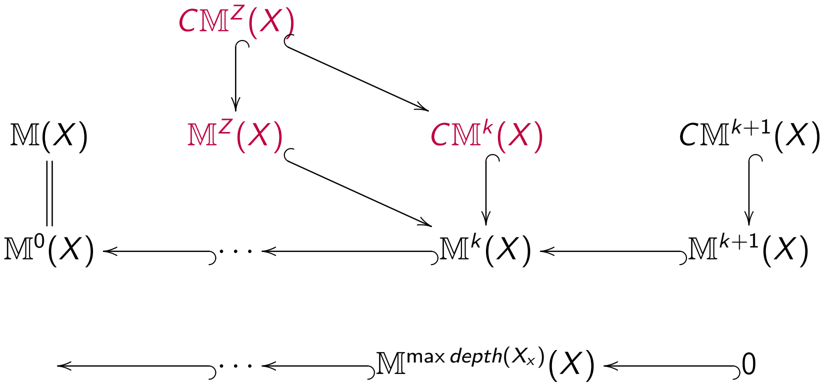
Notations

Let $Z \subseteq X$ be closed. Denote the full subcategories

$$\begin{cases} Coh^Z(X) = \{M \in Coh(X) : \mathrm{Supp}\,(M) \subseteq Z\} \\ \mathbb{M}^Z(X) = \{M \in \mathbb{M}(X) : \mathrm{Supp}\,(M) \subseteq Z\} \\ C\mathbb{M}^Z(X) = \left\{M \in \mathbb{M}^Z(X) : \dim_{\mathscr{V}(X)}(M) = grade\,(\mathcal{O}_Z)\right\} \\ \mathrm{D}^b(\mathscr{E}) = \text{Bounded Derived Category of } \mathscr{E} \end{cases}$$

Filtration and subcategories

With $k = grade(Z, X)$, we have the diagram of subcategories:



Derived Equivalences

In the following (two) lines, all the horizontal functors are **equivalences of derived categories**.

$$
\begin{array}{ccccccc}
D^b(C\mathbb{M}^k(X)) & \xrightarrow[\zeta]{\sim} & D^b(\mathbb{M}^k(X)) & \xrightarrow[\iota]{\sim} & \mathscr{D}^k(\mathbb{M}(X)) & \xleftarrow[\iota']{\sim} & \mathscr{D}^k(\mathscr{V}(X)) \\
\uparrow & & & & \uparrow & & \downarrow \\
\textit{Derived Cat} & & & & \textit{Derived SubCat} & & D^b(\mathscr{V}(X)) \\
\downarrow & & & & \downarrow & & \uparrow \\
D^b(C\mathbb{M}^Z(X)) & \xrightarrow[\zeta]{\sim} & D^b(\mathbb{M}^Z(X)) & \xrightarrow[\iota]{\sim} & \mathscr{D}^Z(\mathbb{M}(X)) & \xleftarrow[\iota']{\sim} & \mathscr{D}^Z(\mathscr{V}(X))
\end{array}
$$

K-Theory and Commutative Algebra

- K-Theory is an **invariant of such** Derived categories.
- Main ingredient to prove these Derived equivalences comes from the some commutative algebra, as follows.

Commutative Algebra: Theorem:

X be quasi projective, $Z \subseteq X$ closed, $grade(Z, X) = k$. Let

$$\cdots \longrightarrow \mathscr{F}_k \longrightarrow \cdots \longrightarrow \mathscr{F}_1 \longrightarrow \mathscr{F}_0 \longrightarrow \mathscr{F}_{-1}$$

be a complex in $Coh(X) \ni \mathcal{H}_i(\mathscr{F}_\bullet) \in Coh^Z(X)$, $\forall 0 \leq i \leq k$. Then there are maps of complexes $\nu_\bullet : \mathscr{E}_\bullet \longrightarrow \mathscr{F}_\bullet$ as follows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{E}_k & \longrightarrow \cdots \longrightarrow & \mathscr{E}_1 & \longrightarrow & \mathscr{E}_0 & \longrightarrow & 0 \\ & & \downarrow \nu_k & & \downarrow \nu_1 & & \downarrow \nu_0 & & \\ \cdots & \longrightarrow & \mathscr{F}_k & \longrightarrow \cdots \longrightarrow & \mathscr{F}_1 & \longrightarrow & \mathscr{F}_0 & \longrightarrow & \mathscr{F}_{-1} \end{array}$$

Continued: Theorem:

such that

1. $\mathscr{E}_i \in \mathscr{V}(X)\ \forall i$, and $\mathscr{E}_i = 0$ unless $0 \leq i \leq k$.
2. The map $\mathcal{H}_0(\nu_\bullet) : \mathcal{H}_0(\mathscr{E}_\bullet) \twoheadrightarrow \mathcal{H}_0(\mathscr{F}_\bullet)$ is surjective.
3. $\mathcal{H}_0(\mathscr{E}_\bullet) \in C\mathbb{M}^k(X)$ and $\mathcal{H}_i(\mathscr{E}_\bullet) = 0\ \forall i \neq 0$.

Commutative Algebra: Comments:

- $\mathscr{E}_\bullet$ would direct sum of certain Koszul complexes.
- Method of proofs goes back to a unpublished paper of Hans-Bjørn Foxby, executed for finite length homologies. Same was use in the paper of Roberts-Srinivas.
- In our case, we embed $X \subseteq Y = \mathrm{Proj}(S)$. The proof essentially reduces to the case, when $X = Y = \mathrm{Proj}(S)$. Then (1) **Prime avoidance works**, (2) There is a correspondence between the **Graded S-modules** and **Quasi coherent sheaves** on Y.

K-Theory Spaces

Applications To K − *Theory*

K-Theory Spaces

- Define the K-Theory space of a (small) exact category $\mathscr{E}$,

$$\mathsf{K}(\mathscr{E}) = \Omega\,|N_\bullet(\mathbb{Q}\mathscr{E})|$$

$$\begin{cases} \mathbb{Q}\mathscr{E} = \textit{the } \mathbb{Q} \textit{ category of } \mathscr{E} \\ N_\bullet(\mathbb{Q}\mathscr{E}) = \text{the Nerve of } \mathbb{Q}\mathscr{E}; \text{ a simplicial set} \\ |N_\bullet(\mathbb{Q}\mathscr{E})| = \textit{the Geometric realization}, \\ \text{AKA the classifying space of } \mathbb{Q}\mathscr{E} \\ \Omega\,|N_\bullet(\mathbb{Q}\mathscr{E})| = \text{the loop space } (\textit{space of all loops}) \end{cases}$$

K-Groups

- The K-groups are defined as $\mathrm{K}_n(\mathscr{E}) = \pi_n(\mathrm{K}(X))$
- The K-theory spectra is also defined,
  to obtain negative K-groups,
  which we postpone or skip.

As Functors

- Let CatExact denote the category of (small) exact categories, and Exact Functors.
- The association

$$\begin{cases} \mathscr{E} \mapsto \mathrm{K}(\mathscr{E}) \\ F \mapsto \mathrm{K}(F) \end{cases} \quad \begin{cases} \text{is a functor} \quad \underline{\mathrm{CatExact}} \longrightarrow \underline{\mathrm{Top}} \\ \textit{In fact}, \ \mathrm{K}(\mathscr{E}) \textit{ is a CW complex.} \end{cases}$$

- The association

$$\begin{cases} \mathscr{E} \mapsto \mathrm{K}_n(\mathscr{E}) \\ F \mapsto \mathrm{K}_n(F) \end{cases} \quad \begin{cases} \text{is a sequence of functors} \\ \underline{\mathrm{CatExact}} \longrightarrow \underline{\mathrm{Ab}} \end{cases}$$

Primary Interest

X is quasi projective/ $\mathrm{Spec}\,(A)$, $Z \subseteq X$ is closed, $U = X - Z$.

- The K-Theory of $Coh(X)$ and $\mathscr{V}(X)$ are of interest.
- Relationships between the K-theories,
  $\mathrm{K}\,(Coh(X))$, $\mathrm{K}\,(Coh(U))$, $\mathrm{K}\,(Coh(Z))$, and
  $\mathrm{K}\,(\mathscr{V}(X))$, $\mathrm{K}\,(\mathscr{V}(U))$, $\mathrm{K}\,(\mathscr{V}(Z))$ are of interest.
  Clarifications are needed, what kind of
  scheme structure we impose on Z, if any.

Primary Interest

- Quillen's results on $\mathrm{K}(Coh(X))$, are most up to date. So, rest of the talk, we focus on $\mathrm{K}(\mathscr{V}(X))$.
- There is also interest in **Grothendieck-Witt theory**, by incorporating dualities, when available. For example, $\mathscr{V}(X)$ has a natural duality $\mathcal{E} \mapsto \mathcal{H}om(\mathcal{E}, \mathcal{O}_X)$. For exact categories $(\mathscr{E}, ^\vee)$ with a duality, one associates a Grothendieck-Witt space $GW(\mathscr{E}, ^\vee)$. Theory works fairly similar to K-Theory. I will skip them today.

Primary Interest

- It follows from the resolution theorem that

$$\mathsf{K}\left(\mathscr{V}(X)\right) \xrightarrow{\sim} \mathsf{K}\left(\mathbb{M}(X)\right) \quad \text{is a homotopy equivalence.}$$

Consequently,

$$\mathsf{K}_n\left(\mathscr{V}(X)\right) \xrightarrow{\sim} \mathsf{K}_n\left(\mathbb{M}(X)\right) \quad \text{is isomorphism, } \forall \text{ n} \geq 0.$$

Exercise: Prove $K_0\left((\mathscr{V}(X)\right) \cong K_0\left(\mathbb{M}(X)\right)$.

- So, our focus is now on $\mathbb{M}(X)$.

The Homotopy Fiber

As before X is quasi projective, $Z \subseteq X$ is closed, $U = X - Z$.

- Note U has a natural subscheme structure. Consider the map of K-Theory spaces $\mathrm{K}(\mathscr{V}(X)) \xrightarrow{\varepsilon} \mathrm{K}(\mathscr{V}(U))$
- Topologically, there is a Homotopy Fiber:

$$\mathscr{F}(\varepsilon) \overset{\iota}{\dashrightarrow} \mathrm{K}(\mathscr{V}(X)) \xrightarrow{\varepsilon} \mathrm{K}(\mathscr{V}(U))$$

The Triangle Fiber

- This leads to a long exact sequence of Homotopy groups

$$\cdots \longrightarrow \pi_n(\mathscr{F}(\varepsilon)) \longrightarrow \mathrm{K}_n(\mathscr{V}(X)) \xrightarrow{\varepsilon} \mathrm{K}_n(\mathscr{V}(U))$$

$$\longrightarrow \pi_{n-1}(\mathscr{F}(\varepsilon)) \longrightarrow \cdots$$

The Triangles

- They are like Triangles of spaces:

$$
\begin{array}{ccc}
 & \mathscr{F}(\varepsilon) & \\
\swarrow^{\iota} & & \nwarrow^{\Omega K \mapsto \mathscr{F}(\varepsilon)} \\
K(\mathscr{V}(X)) & \xrightarrow[\varepsilon]{} & K(\mathscr{V}(U))
\end{array}
$$

$$
\begin{array}{ccc}
 & \pi_\bullet(\mathscr{F}(\varepsilon)) & \\
\swarrow & & \nwarrow^{-1} \\
K_\bullet(\mathscr{V}(X)) & \xrightarrow[\varepsilon]{} & K_\bullet(\mathscr{V}(U))
\end{array}
$$

The Description

- So, the question is what kind of Algebraic description can we give for $\mathscr{F}(\varepsilon)$, which should be basically depend on Z.
- **Theorem.** We have

$$\begin{array}{ll} \mathsf{K}\left(C\mathbb{M}^Z(X)\right) \xrightarrow{\sim} \mathscr{F}(\varepsilon) & \text{is a homotopy equivalence} \\ \mathsf{K}_n\left(C\mathbb{M}^Z(X)\right) \xrightarrow{\sim} \pi_n\left(\mathscr{F}(\varepsilon)\right) & \text{is an isomorphism } \forall \text{n} \end{array}$$

The Exact sequence

Consequently, there is a long exact sequence of $\mathbb{K}$-groups

$$
\begin{array}{ccccccc}
\cdots & & \cdots & & \cdots & & \\
\mathrm{K}_1\left(C\mathbb{M}^Z(X)\right) & \longrightarrow & \mathrm{K}_1\left(\mathscr{V}(X)\right) & \xrightarrow{\varepsilon} & \mathrm{K}_1\left(\mathscr{V}(U)\right) & \longrightarrow & \\
\mathrm{K}_0\left(C\mathbb{M}^Z(X)\right) & \longrightarrow & \mathrm{K}_0\left(\mathscr{V}(X)\right) & \xrightarrow{\varepsilon} & \mathrm{K}_0\left(\mathscr{V}(U)\right) & \longrightarrow & \\
\mathbb{K}_{-1}\left(C\mathbb{M}^Z(X)\right) & \longrightarrow & \mathbb{K}_{-1}\left(\mathscr{V}(X)\right) & \xrightarrow{\varepsilon} & \mathbb{K}_{-1}\left(\mathscr{V}(U)\right) & \longrightarrow & \\
\cdots & & \cdots & & \cdots & &
\end{array}
$$

Known Results

- The results above is an improvement of a result of Quillen, stated when $co\dim(Z) = 1$. This was written down in a paper Daniel Grayson.
- In between, a description of the homotopy fiber $\mathscr{F}(\varepsilon)$ was given by Thomason. This was stated in terms of K-Theory of chain complexes of respective exact categories.

Jargon of Spectra

- There is a zig-zag sequence

$$\mathbb{K}\left(C\mathbb{M}^{Z}(X)\right) \longrightarrow \mathbb{K}\left(\mathscr{V}(X)\right) \longrightarrow \mathbb{K}\left(\mathscr{V}(U)\right)$$

that is a homotopy fibration of $\mathbb{K}$-Theory spectra.

Jargon of Spectra : Delooping

- A spectra is a sequence $\mathcal{X} := \{X_0, X_1, \ldots\}$ of pointed topological spaces, together with homotopy equivalences

$$\Omega X_n \xrightarrow[\sim]{\sigma_n} X_{n+1} \quad \text{to be called the Bonding maps.}$$

- Define

$$\pi_k(\mathcal{X}) := \begin{cases} \pi_k(X_0) & k \geq 0 \\ \pi_0(X_{-k}) = \cdots = \pi_{-k}(X_0) & k \leq -1 \end{cases}$$

- We do everything algebraically or formally.

Two Step Proof:

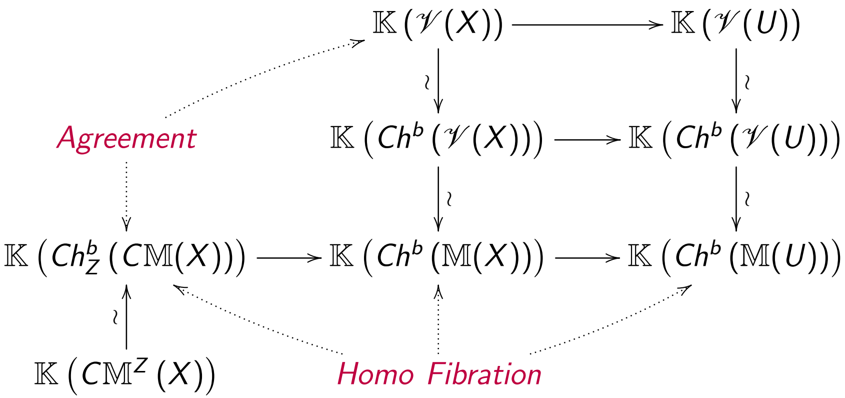
Two main Hypotheses needs to be checked are

- There is an equivalence $\mathrm{D}^b\left(C\mathbb{M}^Z(X)\right) \xrightarrow{\sim} \mathscr{D}^Z\left(\mathscr{V}(X)\right)$.
- The zig-zag sequence

$$\begin{array}{ccccc} \mathrm{D}^b\left(C\mathbb{M}^Z(X)\right) & & & & \\ \downarrow \wr & \searrow & & & \\ \mathscr{D}^Z\left(\mathscr{V}(X)\right) & \longrightarrow & \mathrm{D}^b\left(\mathscr{V}(X)\right) & \longrightarrow & \mathrm{D}^b\left(\mathscr{V}(U)\right) \end{array}$$

is exact up to direct summand (*up to factor*).

Two Step Proof:



Recall

- We recall the Derived equivalence

$$\mathrm{D}^b(C\mathbb{M}^k(X)) \xrightarrow[\zeta]{\sim} \mathrm{D}^b(\mathbb{M}^k(X)) \xrightarrow[\iota]{\sim} \mathscr{D}^k(\mathbb{M}(X)) \xleftarrow[\iota']{\sim} \mathscr{D}^k(\mathscr{V}(X))$$

- For any fixed integer $k \geq 0$, we run the same program, as above

Homotopy Fibration: Codimension k

- Assume X is Cohen Macaulay. For $x \in X$, denote $X_x = \mathrm{Spec}\left(\mathcal{O}_{X,x}\right)$ abd $X^{(k)}$ denote the set of codimension k points.
  The sequence

$$\mathbb{K}\left(C\mathbb{M}^{k+1}(X)\right) \longrightarrow \mathbb{K}\left(C\mathbb{M}^{k}(X)\right) \longrightarrow \coprod_{x \in X^{(k)}} \mathbb{K}\left(C\mathbb{M}^{k}(X_x)\right)$$

  is a Homotopy Fibration of $\mathbb{K}$-Theory spectra.

The exact sequence: Co dimension k

- This leads to an exact sequence

$$\cdots \longrightarrow \bigoplus_{x\in X^{(k)}} \mathbb{K}_{n-1}\left(C\mathbb{M}^k(X_x)\right) \longrightarrow$$

$$\mathbb{K}_n\left(C\mathbb{M}^{k+1}(X)\right) \longrightarrow \mathbb{K}_n\left(C\mathbb{M}^k(X)\right) \longrightarrow \bigoplus_{x\in X^{(k)}} \mathbb{K}_n\left(C\mathbb{M}^k(X_x)\right) \longrightarrow$$

$$\mathbb{K}_{n-1}\left(C\mathbb{M}^{k+1}(X)\right) \longrightarrow \cdots$$

Gersten Complex : Co dimension k

Combine three of the above

$$
\begin{array}{ccccc}
\oplus_{x \in X^{(k-1)}} \mathbb{K}_{n+1}\left(C\mathbb{M}^{k-1}(X_x)\right) & & & & \mathbb{K}_{n-1}\left(C\mathbb{M}^{k+2}(X)\right) \\
\downarrow & \searrow & & & \downarrow \\
\mathbb{K}_n\left(C\mathbb{M}^{k}(X)\right) & \longrightarrow & \oplus_{x \in X^{(k)}} \mathbb{K}_{n}\left(C\mathbb{M}^{k}(X_x)\right) & \longrightarrow & \mathbb{K}_{n-1}\left(C\mathbb{M}^{k+1}(X)\right) \\
\downarrow & & & \searrow & \downarrow \\
\mathbb{K}_n\left(C\mathbb{M}^{k-1}(X)\right) & & & & \oplus_{x \in X^{(k+1)}} \mathbb{K}_{n-1}\left(C\mathbb{M}^{k+1}(X_x)\right)
\end{array}
$$

For $x \in X^{(k)}$, $C\mathbb{M}^k(X_x)$ is the category of modules of finite length and finite projective dimension.

Gersten Complex : Co dimension k

- The broken diagonal arrows form the Gersten complex.
- There are other versions of this complex, even without the hypothesis that X is quasi-projective.
- Original Gersten complex, in Quillen's paper, deals with G-Theory (i.e. of $Coh(X)$). They terminate at the G_0-term $\bigoplus_{x \in X^{(n)}} G_0(X_x)$.

Gersten Complex : Co dimension k

$$0 \longrightarrow \mathbb{K}_n(X) \longrightarrow \bigoplus_{x\in X^{(0)}} \mathbb{K}_n(\mathbb{M}^0(X_x)) \longrightarrow \bigoplus_{x\in X^{(1)}} K_{n-1}(\mathbb{M}^1(X_x)) \longrightarrow \cdots \longrightarrow \bigoplus_{x\in X^{(d-1)}} K_{n-d+1}(\mathbb{M}^{d-1}(X_x)) \longrightarrow \bigoplus_{x\in X^{(d)}} \mathbb{K}_{n-d}(\mathbb{M}^d(X_x)) \longrightarrow 0$$

Epilogue

- Given exact categories $(\mathscr{E}, ^\vee)$ with duality, one defines a $\mathbb{G}W(\mathscr{E}, ^\vee)$ space (spectra) and extend all of the above.
- Note $C\mathbb{M}^k(X)$ has a natural duality $M \mapsto \mathscr{E}xt^n(M, \mathcal{O}_X)$. This also restricts to $C\mathbb{M}^Z(X)$ if $k = grade(Z, X)$.
- There would be a sequence $\mathbb{G}W^r(\mathscr{E}, ^\vee)$ of $\mathbb{G}W$ spaces (spectra), with periodicity four. You have two because, given a duality, there is a skew duality.