Journal of Algebra 540 (2019) 168–205

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

The monoid structure on homotopy obstructions

Satya Mandal [*,1], Bibekananda Mishra

University of Kansas, Lawrence, KS 66045, USA

A R T I C L E   I N F O

Article history:
Received 24 November 2018
Available online 6 September 2019
Communicated by Steven Dale Cutkosky

Keywords:
Commutative algebra
Projective modules
Complete intersections

See two the subsequent papers!

A B S T R A C T

Let A be a commutative noetherian ring, containing a field k, with $1/2 \in k$, $\dim A = d$, and let P be a projective A-module, with $rank(P) = n$. Let $\mathcal{LO}(P)$ denote the set of all pairs (I, ω), where I is an ideal of A and $\omega : P \twoheadrightarrow I/I^2$ is a surjective map. The homotopy relations on $\mathcal{LO}(P)$, induced by $\mathcal{LO}(P[T])$, leads to a set $\pi_0(\mathcal{LO}(P))$ of equivalence classes in $\mathcal{LO}(P)$. There are two distinguished elements $\mathbf{e}_0, \mathbf{e}_1 \in \pi_0(\mathcal{LO}(P))$, respectively, the images of $(0, 0)$ and $(A, 0)$. Define the obstruction class

$$\varepsilon(P) = \mathbf{e}_0 \in \pi_0(\mathcal{LO}(P)),$$

to be called the **(Nori) homotopy class** of P. The following results are under suitable smoothness or regularity hypotheses. We prove, if $2n \geq d + 2$, then $\pi_0(\mathcal{LO}(P))$ has a natural structure of a monoid, which is a group if $P \cong Q \oplus A$. When $2n \geq d + 3$, we prove

$$P \cong Q \oplus A \Longleftrightarrow \varepsilon(P) = \mathbf{e}_1 \quad \text{(“the additive zero”)}.$$

Further, we give a definition of a Euler class group $E(P)$. Under suitable smoothness hypotheses, we prove, if $P \cong Q \oplus A$ and $2n \geq d + 3$, then there is natural isomorphism $E(P) \xrightarrow{\sim} \pi_0(\mathcal{LO}(P))$ of groups.

* Corresponding author.
E-mail addresses: mandal@ku.edu (S. Mandal), bibekanandamishra@ku.edu (B. Mishra).

[1] Partially supported by a General Research Grant (No. 2301857) from University of Kansas.

https://doi.org/10.1016/j.jalgebra.2019.08.022
0021-8693/

1. Introduction

Throughout this article, unless further qualifications are added, A will denote a noetherian commutative ring, with $\dim A = d$. Also, P will denote a projective A-module with $rank(P) = n$.

This article is a continuation of the study of Homotopy obstructions of projective modules, that was started in [15]. It was pointed out in [15] that, the study of the Homotopy obstructions of projective modules, evolved out of some germs of ideas, in two components, given by Madhav V. Nori (around 1990), through some verbal and informal communications, and was referred to as the "Homotopy Program". The readers would be very well advised to familiarize themselves with the introduction of [MM]. We would try to avoid any repetition, and pick up from where we left in [15]. We make additional introductory comments here only to reestablish the context. One of the two components of these germs, was the Homotopy Question. The following is a statement of the same from [13], which would almost certainly be an adaptation by the respective author [13], of the more precise formulation communicated by Nori.

Question 1.1 *(Homotopy Question).* Suppose $X = \operatorname{Spec}(A)$ is a smooth affine variety, with $\dim X = d$. Let P be a projective A-module of rank n and $f_0 : P \twoheadrightarrow I$ be a surjective homomorphism, onto an ideal I of A. Assume $Y = V(I)$ is smooth with $\dim Y = d - n$. Also suppose $Z = V(J) \subseteq \operatorname{Spec}(A[T]) = X \times \mathbb{A}^1$ is a smooth subscheme, such that Z intersects $X \times 0$ transversally in $Y \times 0$. Now, suppose that $\varphi : P[T] \twoheadrightarrow \frac{J}{J^2}$ is a surjective map such that $\varphi_{|T=0} = f_0 \otimes \frac{A}{I}$. Then the question is, whether there is a surjective map $F : P[T] \twoheadrightarrow J$ such that (i) $F_{|T=0} = f_0$ and (ii) $F_{|Z} = \varphi$. Assume $2n \geq d + 3$.

The statement of Question 1.1, is a simple translation of the theorem of Nori [13, §3 Appendix], on smooth vector bundles V over smooth manifolds M, using a vector bundle to projective module dictionary. The Question 1.1, as stated, would fail to have an affirmative answer, without the regularity hypothesis [5, Example 6.4]. Even when A is regular, without the condition $2n \geq d + 3$, the question would not have an affirmative answer (see [5, Example 3.15]). However, existing results (see [13,5,3]) indicate that with suitable hypotheses the regularity and/or transversality hypotheses may be spared. Up to date, the best affirmative result (assumes $2n \geq d + 3$) on (1.1) is due to Bhatwadekar and Keshari [3], preceded by [21,13,20,5].

While the Homotopy Question (1.1) always had the flavor of being central to the Homotopy Program, it was never articulated as such. In fact, it was never well understood by the researchers how or why so? This article clarifies and establishes the centrality of the Homotopy Question (1.1). The other half of these two pillars in this program is the definitions of Euler class groups. Followed by the outline given by Nori, for integers $0 \leq n \leq d$ and line bundles L, definitions of Euler class groups $E^n(A, L)$ were given in [7,6,18]. In fact, Nori originally outlined a definition of $E^d(A, A)$, when A is smooth (see [21]). For any projective A-module P, with $rank(P) = d$, an Euler class $e(P) \in E^d(A, \wedge^d P)$ was defined and it was proved [7] that

$$e(P) = 0 \Longleftrightarrow P \cong Q \oplus A.$$

When $rank(P) \leq d - 1$, a desire to define a similar obstruction class $e(P)$, in some appropriate obstruction group or set seemed too ambitious. We accomplish this goal, under additional conditions (see Corollary 4.6), by understanding the implicit Homotopy relations in the statement of the Homotopy Question 1.1. We introduce the following notations:

$$\begin{cases} \mathcal{L}O(P) = \left\{(I,\omega) : \omega : P \twoheadrightarrow \frac{I}{I^2}, \text{ is a surjective map, where } I \text{ is an ideal}\right\} \\ \mathcal{L}O^n(P) = \{(I,\omega) \in \mathcal{L}O(P) : height(I) = n\} \\ \mathcal{L}O^n_c(P) = \{(I,\omega) \in \mathcal{L}O(P) : height(I) = n, \text{ and } V(I) \text{ is connected}\} \end{cases}$$

There is a (chain) homotopy relation ingrained in the statement of (1.1), by substituting $T = 0, 1$, on the set $\mathcal{L}O(P)$. The set of equivalence classes would be denoted by $\pi_0(\mathcal{L}O(P))$. In $\mathcal{L}O(P)$, there are two distinguished elements $(0,0), (A,0) \in \mathcal{L}O(P)$, and their images in $\pi_0(\mathcal{L}O(P))$ are denoted, respectively, by $\mathbf{e}_0$ and $\mathbf{e}_1$. Define the obstruction class

$$\varepsilon(P) := \mathbf{e}_0 \in \pi_0(\mathcal{L}O(P)), \tag{1}$$

to be called the **(Nori) Homotopy class** of P. We give a summary of the main results in this article, before making further introductory remarks. Let A and P be as above.

1. (See Corollary 4.6.) Suppose A is essentially smooth, over an infinite perfect field k. Assume $2n \geq d + 3$, with $1/2 \in k$. Then, we prove

$$P \cong Q \oplus A \Longleftrightarrow \varepsilon(P) = \mathbf{e}_1 \quad \text{("the additive zero"; see (3))}.$$

2. (See Theorem 4.3.) Suppose A is essentially smooth, over an infinite perfect field k, with $1/2 \in k$. Assume $2n \geq d+3$. Let $(I,\omega) \in \mathcal{L}O^n(P)$ and let $[(I,\omega)] \in \pi_0(\mathcal{L}O(P))$ be its image. Then, $\omega : P \twoheadrightarrow \frac{I}{I^2}$ lifts

$$\text{to a surjective map} \quad \Omega : P \twoheadrightarrow I \quad \Longleftrightarrow \quad \varepsilon(P) = [(I,\omega)] \in \pi_0(\mathcal{L}O(P)).$$

3. (See Theorem 6.5.) Assume A is a regular ring, containing a field k, with $1/2 \in k$. Assume $2n \geq d + 2$. Then, we prove that $\pi_0(\mathcal{L}O(P))$ has a natural structure of an abelian monoid. In this additive structure, $\mathbf{e}_1 \in \pi_0(\mathcal{L}O(P))$ is the identity. For $(I,\omega_1), (J,\omega_2) \in \mathcal{L}O^n(P)$, if $I + J = A$, the sum in $\pi_0(\mathcal{L}O(P))$ is given by

$$[(I,\omega_1)] + [(J,\omega_2)] = [(IJ, \omega_1 \star \omega_2)]$$

where $\omega_1 \star \omega_2 : P \twoheadrightarrow \frac{IJ}{(IJ)^2}$ is obtained by combining ω_1 and ω_2, using Chinese remainder theorem.
Further, if $P = Q \oplus A$, then $\mathbf{e}_0 = \mathbf{e}_1$ and $\pi_0(\mathcal{L}O(P))$ has a structure of a group.

4. To further establish centrality of the Homotopy Question (1.1) in this program, define Euler class group

$$E(P) := \frac{\mathbb{Z}(\mathcal{L}O^n_c(P))}{\mathscr{R}(P)}$$

where $\mathscr{R}(P) \subseteq \mathbb{Z}(\mathcal{L}O(P))$ is the subgroup generated by the global orientations, namely, those $(I, \omega) \in \mathcal{L}O^n(P)$ such that, ω lifts to a surjective map $P \twoheadrightarrow I$ (*here* (I, ω) *is considered as an element in* $\mathbb{Z}(\mathcal{L}O^n_c(P))$*, by decomposing* I *in to connected components*).
   (a) (See Definition 7.2.) Assume A is a regular ring, containing a field k, with $1/2 \in k$. Assume $2n \geq d + 2$ and $P = Q \oplus A$. Then, we prove that there is a natural surjective group homomorphism

$$\varphi : E(P) \twoheadrightarrow \pi_0 \left(\mathcal{L}O(P)\right)$$

   (b) (See Theorem 7.3.) Further, assume A is essentially smooth over an infinite perfect field k and $1/2 \in k$. If $2n \geq d+3$, then we prove that the homomorphism φ is an isomorphism.
   (c) (See Theorem 7.6.) Assume A is a noetherian commutative ring (without any regularity hypothesis), $P = Q \oplus A$ and $2n \geq d+3$. Let $(I, \omega) \in \mathcal{L}O^n(P)$. Assume its image

$$\overline{(I, \omega)} = 0 \in E(P).$$

   Then, ω lifts to a surjective map $\Omega : P \twoheadrightarrow I$.
   (d) (Corollary 7.9.) In Section 7.2, we exploit the work of N. Mohan Kumar and M.P. Murthy [25,24], when the base field k is algebraically closed, and $P \cong Q \oplus A$, with $rank(P) = n = d$. If A is smooth and $1/2 \in k$, we prove $\pi_0 \left(\mathcal{L}O(P)\right) \cong CH^d(A)$, where $CH^d(A)$ denotes the Chow group of zero cycles.

The desire to define an obstruction class, for P to split off a free direct summand, is age old and might have been considered too bold. However, we are able to give such a definition (1) of an obstruction class $\varepsilon(P)$, and the result in item 1 (Corollary 4.6) establishes the splitting property. The result in item 2 (Theorem 4.3) was the main objective of the Homotopy Question (1.1), in such a homotopy obstruction theory set up. The structure of $\pi_0 \left(\mathcal{L}O(P)\right)$ has been an open problem since the inception of the Homotopy Program, while the exact nature of the structure to expect was not clear. In item 3 (Theorem 6.5) we settle this issue, by proving that the homotopy obstruction set $\pi_0 \left(\mathcal{L}O(P)\right)$ has structure of a monoid. The definition, in item 4, of Euler class group

$E(P)$ is new. In deed, for a line bundle L, $E(L \oplus A^{n-1})$ coincides with $E^n(A, L)$, as defined in [7,6,18]. Further, under suitable smoothness conditions, the results in item 4 (see §7 for more details), establish a relationship, as in (4a), (4b), between homotopy obstructions $\pi_0(\mathcal{LO}(P))$ and the Euler class group $E(P)$, which ties together the two components of the germs of ideas originally given by Nori (around 1990). When $n = d$, k is algebraically closed, $P = Q \oplus A$, under suitable other conditions, we establish that $\pi_0(\mathcal{LO}(P))$ coincides with the Chow group $CH^d(A)$ of zero cycles.

While we described our results above, in terms of $\pi_0(\mathcal{LO}(P))$, there are three other descriptions of $\pi_0(\mathcal{LO}(P))$ available in §2. We use these descriptions of $\pi_0(\mathcal{LO}(P))$ interchangeably. Consider the notations:

$$\begin{cases} \mathcal{Q}(P) = \{(f, s) \in P^* \oplus A : s(1-s) \in f(P)\} \\ \widetilde{\mathcal{Q}}(P) = \{(f, p, s) \in P^* \oplus P \oplus A : f(p) + s(s-1) = 0\} \\ \widetilde{\mathcal{Q}}'(P) = \{(f, p, z) \in P^* \oplus P \oplus A : f(p) + z^2 = 1\} \end{cases}$$

Given a polynomial extension $A \hookrightarrow A[T]$, substituting $T = 0, 1$, we have two set theoretic maps, in each case

$$\begin{cases} \mathcal{Q}(P) \xleftarrow{T=0} \mathcal{Q}(P[T]) \xrightarrow{T=1} \mathcal{Q}(P) \\ \widetilde{\mathcal{Q}}(P) \xleftarrow{T=0} \widetilde{\mathcal{Q}}(P[T]) \xrightarrow{T=1} \widetilde{\mathcal{Q}}(P) \\ \widetilde{\mathcal{Q}}'(P) \xleftarrow{T=0} \widetilde{\mathcal{Q}}'(P[T]) \xrightarrow{T=1} \widetilde{\mathcal{Q}}'(P) \end{cases}$$

These lead to chain homotopy relations and accordingly, $\pi_0(\mathcal{Q}(P))$, $\pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$, $\pi_0\left(\widetilde{\mathcal{Q}}'(P)\right)$ are defined. If and when $1/2 \in A$ (*which we often assume*), there is a bijection

$$\widetilde{\mathcal{Q}}(P) \xrightarrow{\sim} \widetilde{\mathcal{Q}}'(P), \text{ which induces a bijection } \quad \pi_0\left(\widetilde{\mathcal{Q}}(P)\right) \xrightarrow{\sim} \pi_0\left(\widetilde{\mathcal{Q}}'(P)\right).$$

In Section 2, we establish the following commutative diagram of natural bijections:

$$\begin{tikzcd} \pi_0\left(\widetilde{\mathcal{Q}}(P)\right) \arrow[r, two heads, "\overline{\nu}", "\sim"'] \arrow[d, "\overline{\eta}"', "\wr"] & \pi_0(\mathcal{Q}(P)) \arrow[dl, "\overline{\eta}'", "\sim"'] \\ \pi_0(\mathcal{LO}(P)) & \end{tikzcd}$$

We comment on the use of the phrase "Homotopy Program". Perhaps, the phrase was first used by Mandal, in a conversation with Nori to describe this whole set of problems. Among what were encapsulated in the program are the following:

1. (**Part 1**) A coherent theory of obstructions, based on homotopy was expected. It was also expected that these homotopy obstructions would come together with the concept of Euler class groups.
2. (**Part 2**) The theory should reconcile with the $\mathbb{A}^1$-homotopy approach (*also known as Motivic or Chow-Witt group approach* [2,23]).
3. (**Part 3**) When A is a real smooth affine algebra, this algebraic homotopy obstruction theory should also reconcile with the Topological counter part, in the sense analogous to [17].

Results in this article addresses Part 1 of this program, in a comprehensive manner. In deed, a coherent theory of homotopy obstructions is established, as was expected. Note that the theory is not expected to behave too well for the lower half of the range of $n = rank(P)$. When P is not free, the definition of the Euler class group $E(P)$ is new, which was needed to bring the two components of the Homotopy Program together. The destination of the road map that emerged out of the introduction of the Homotopy Question (1.1) was not very well understood. This article clarifies and brings us to that destination. This completes the Part 1 of the program. This was accomplished entirely by the methods of commutative algebra, which was possible due to the strength of the Homotopy Question.

In a sense, Part 2 of the program was resolved in [1,15] fairly satisfactorily, by settling the problem of Fabien Morel [23, pp. 13] affirmatively. It was established that, under suitable regularity conditions, $\widetilde{CH}^d(A) \cong \pi_0\left(\widetilde{\mathcal{Q}}(A^d)\right)$, where $\widetilde{CH}^n(A)$ denotes the Chow-Witt group of A, for $n \geq 2$, introduced in [2].

For clarity, let us specify that, for integers $n \geq 2$, four invariants of A has been discussed as obstruction houses. Namely, they are the Euler class groups $E^n(A, A)$, the (naive) homotopy groups $\pi_0\left(\widetilde{\mathcal{Q}}(A^n)\right)$, the $\mathbb{A}^1$-homotopy groups and the Chow-Witt groups $\widetilde{CH}^n(A)$. For our purpose, $\pi_0\left(\widetilde{\mathcal{Q}}(A^n)\right)$ coincides with $\mathbb{A}^1$-homotopy groups (see [23, Remark 8.10]). We avoid discussions on complexities of various comparison theorems between naive homotopy and Chow-Witt groups [23,1]. However, for each projective A-module P, this article introduces two more invariants of P, namely the (Nori) Homotopy monoid $\pi_0(\mathcal{LO}(P)) \cong \pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$ and the Euler class group $E(P)$. Newer questions (A.6) have been raised, what would be an appropriate $\mathbb{A}^1$-homotopy or Chow-Witt interpretations for $\pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$, analogous to the original question of Morel [23, p. 13]. While such an $\mathbb{A}^1$-homotopy interpretation would be of its own interest, this may become useful for further study of the structure of these monoids $\pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$, like finite generation and others. The Part 3 of the program would have to be addressed subsequently. It appears, there is no well formulated or well studied topological counter part to these monoids $\pi_0(\mathcal{LO}(P))$, in the literature.

We contrast Euler class groups $E(A^n) = E^n(A, A)$, Chow Witt groups $\widetilde{CH}^n(A)$, and Homotopy obstructions $\pi_0(\mathcal{LO}(P))$, along with the history. The goal, indeed a desire, that eventually emerged out of the introduction of the Homotopy Question (1.1), along with the definition of $E^n(A, A) = E(A^n)$, was to define an obstruction class $\varepsilon(P)$ in a suitable obstruction set (desirably a group), for P to split off a free direct summand. However, the study of Euler class groups $E(A^n)$ stole most of the attention, while the study of the implications of the Homotopy Question (1.1) was largely left ignored. The $\mathbb{A}^1$-Homotopy approach to Euler class groups emerged out of the introduction of Chow-Witt groups $\widetilde{CH}^n(A)$ in 2000 [2], followed by the book of Fabien Morel [23], in 2012. Morel defined a surjective map $E^n(A, A) \twoheadrightarrow \widetilde{CH}^n(A)$, and indicated that this map could be an isomorphism [23, pp. 13], when $n = \dim A$. Morel did not indicate that some variation of his $\mathbb{A}^1$-Homotopy approach, may lead to a definition of an obstruction class $\varepsilon(P)$, to split off a free direct summand. Both $E(A^n)$ and $\widetilde{CH}^n(A)$ are invariants of A. They are not precise enough to house such an obstruction class $\varepsilon(P)$. Even the Euler class groups $E(P)$ defined in this article (4) are not large enough to house such an obstruction $\varepsilon(P)$. Nori provided the precise insight (1.1), exactly what would work (around 1990). This article brings Nori's insight to the fullest fruition and establishes that the Homotopy obstruction set $\pi_0(\mathcal{LO}(P))$ is the appropriate set to house such an obstruction $\varepsilon(P)$.

We comment on the organization of this article. First and foremost, it is best that the reader is familiar with the introduction of [15]. In section 2, we lay out the basic definitions and the foundation of this article. In this section, we define the Homotopy obstruction set $\pi_0(\mathcal{LO}(P))$, and give three other descriptions of the same, as mentioned above. In section 3 we prove that the chain homotopy relations on $\widetilde{\mathcal{Q}}'(P)$, is indeed an equivalence relation, under further regularity hypotheses. In section 4, we prove our main results on lifting and splitting, which are independent of the additive structure on $\pi_0(\mathcal{LO}(P))$. In section 5, we define the involution map $\Gamma : \pi_0\left(\widetilde{\mathcal{Q}}(P)\right) \xrightarrow{\sim} \pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$, which may be thought of as a substitute for the additive-inverse map, without any regard to the existence of any additive structure on $\pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$. In section 6, we establish the monoid structure on $\pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$. In section 7, we define the Euler class group $E(P)$, and compare it with the homotopy obstruction monoid $\pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$, as well with Chow group of zero cycles. In the Appendix A, we define $\pi_0\left(\widetilde{\mathcal{Q}}(P)\right) : \underline{Sch}_A \longrightarrow \underline{Sets}$, as pre-sheaf, and raise the question (A.6) of its motivic interpretation.

Acknowledgment

The authors would like to thank Madhav V. Nori for his academic support, guidance and for sharing his invaluable insight on this program, over a long period of time. Thanks are also due to Marco Schlichting for clarifying some of the results in [23].

2. Foundation of homotopy obstructions

In this section, we establish some notations and, for a projective module P, over a noetherian ring A, give several descriptions of the homotopy pre-sheaves.

Notations 2.1. Throughout, A will denote a commutative noetherian ring with $\dim A = d$ and k will denote a field. Often, but not always, we will assume $1/2 \in A$ and/or $k \subseteq A$.

For A-modules M, N, we denote $M[T] := M \otimes A[T]$ and $M^* = Hom(M, A)$. For $f \in Hom(M, N)$, denote $f[T] := f \otimes 1 \in Hom(M[T], N[T])$. Homomorphisms $f : M \longrightarrow \frac{I}{I^2}$ would be identified with the induced maps $\frac{M}{IM} \longrightarrow \frac{I}{I^2}$.

For surjective homomorphisms $\omega_1 : M \twoheadrightarrow \frac{I_1}{I_1^2}$, $\omega_2 : M \twoheadrightarrow \frac{I_2}{I_2^2}$, where I_1, I_2 are two ideals, with $I_1 + I_2 = A$, $\omega_1 \star \omega_2 : M \twoheadrightarrow \frac{I_1 I_2}{(I_1 I_2)^2}$ will denote the unique surjective map induced by ω_1, ω_2.

For a projective A-module P, $\mathbb{Q}(P) = (\mathbb{Q}(P), q)$ will denote the quadratic space $\mathbb{H}(P) \perp A$, where $\mathbb{H}(P) = P^* \oplus P$ is the hyperbolic space. So, $P^* \oplus P \oplus A$ is the underlying projective module of $\mathbb{Q}(P)$ and, for $(f, p, s) \in P^* \oplus P \oplus A$, $q(f, p, s) = f(p) + s^2$.

The category of (noetherian) schemes over $\mathrm{Spec}\,(A)$ will be donated by $\underline{Sch}_A$. The category of sets will be denoted by $\underline{Sets}$. Given a pre-sheaf $\mathcal{F} : \underline{Sch}_A \to \underline{Sets}$, and a scheme $X \in \underline{Sch}_A$, define $\pi_0(\mathcal{F})(X)$ by the pushout

$$\begin{tikzcd} \mathcal{F}(X \times \mathbb{A}^1) \arrow[r, "T=0"] \arrow[d, "T=1"'] & \mathcal{F}(X) \arrow[d] \\ \mathcal{F}(X) \arrow[r] & \pi_0(\mathcal{F})(X) \end{tikzcd} \quad \text{in } \underline{Sets} \tag{2}$$

So, $X \mapsto \pi_0(\mathcal{F})(X)$ is also a pre-sheaf on $\underline{Sch}_A$. For an affine scheme $X = \mathrm{Spec}\,(B) \in \underline{Sch}_A$ and a pre-sheaf $\mathcal{F}$, as above, we write $\mathcal{F}(B) := \mathcal{F}(\mathrm{Spec}\,(B))$ and $\pi_0(\mathcal{F})(B) := \pi_0(\mathcal{F})(\mathrm{Spec}\,(B))$.

Given a projective A-module P, we define a homotopy obstruction set $\pi_0(\mathcal{LO}(P))$ and establish various other descriptions of the same. These are analogous to similar obstruction sets available in the literature, when $P = A^n$ is free.

Definition 2.2. Let A be a noetherian commutative ring, $X = \mathrm{Spec}\,(A)$ and P be a projective A-module. By a **local P-orientation**, we mean a pair (I, ω) where I is an ideal of A and $\omega : P \twoheadrightarrow \frac{I}{I^2}$ is a surjective homomorphism, which is identified with the surjective homomorphism $\frac{P}{IP} \twoheadrightarrow \frac{I}{I^2}$, induced by ω. A local P-orientation will simply be referred to as a **local orientation**, when P is understood. Denote

$$\begin{cases} \mathcal{LO}(P) = \{(I, \omega) : (I, \omega) \text{ is a local } P \text{ orientation}\} \\ \mathcal{Q}(P) = \{(f, s) \in P^* \oplus A : s(1-s) \in f(P)\} \\ \widetilde{\mathcal{Q}}(P) = \{(f, p, s) \in P^* \oplus P \oplus A : f(p) + s(s-1) = 0\} \\ \widetilde{\mathcal{Q}}'(P) = \{(f, p, z) \in P^* \oplus P \oplus A : f(p) + z^2 = 1\} \end{cases} \tag{3}$$

There is a commutative diagram of set theoretic maps, denoted as follows:

$$\begin{tikzcd} \widetilde{\mathcal{Q}}(P) \arrow[r, two heads, "\nu"] \arrow[d, "\eta"'] & \mathcal{Q}(P) \arrow[dl, "\eta'"] \\ \mathcal{LO}(P) & \end{tikzcd} \quad \text{where, for } (f,p,s) \in \widetilde{\mathcal{Q}}(P), \quad \nu(f,p,s) = (f,s) \tag{4}$$

and $\eta'(f,s) = \eta(f,p,s) = (I,\omega)$, where $I = f(P) + As$ and $\omega : P \twoheadrightarrow \frac{I}{I^2}$ is the homomorphism induced by f. These maps η, η', ν are surjective. If and when $1/2 \in A$ (*which we often assume*), there is also a bijection

$$\kappa : \widetilde{\mathcal{Q}}(P) \xrightarrow{\sim} \widetilde{\mathcal{Q}}'(P) \quad \text{sending} \quad (f,p,s) \mapsto (2f, 2p, 2s-1) \tag{5}$$

Now, suppose P is a fixed projective A-module, and schemes $Y \in \underline{Sch}_A$, with $\pi : Y \to \mathrm{Spec}\,(A)$. Then, $\mathcal{LO}(\pi^*P)$, $\widetilde{\mathcal{Q}}(\pi^*P)$, $\mathcal{Q}(\pi^*P)$, $\widetilde{\mathcal{Q}}'(\pi^*P)$ are likewise defined (3). The associations $Y \mapsto \mathcal{LO}(\pi^*P)$, $Y \mapsto \mathcal{Q}(\pi^*P)$, $Y \mapsto \widetilde{\mathcal{Q}}(\pi^*P)$, $Y \mapsto \widetilde{\mathcal{Q}}'(\pi^*P)$ are pre-sheaves on $\underline{Sch}_A$. However, the pre sheaf nature of $Y \mapsto \mathcal{LO}(\pi^*P)$ requires some clarification. For example, for a ring homomorphism $\beta : A \longrightarrow B$, and $(I,\omega) \in \mathcal{LO}(P)$ is sent to $(\beta(P)B, \omega') \in \mathcal{LO}(P \otimes B)$, where $\omega' : P \otimes B \longrightarrow \frac{\beta(I)B}{\beta(I^2)B}$ is induced by ω.

By the pushout diagram (2), applied to these pre-sheaves, the Homotopy obstructions pre-sheaves

$$Y \mapsto \begin{cases} \pi_0\left(\mathcal{LO}(P)\right)(Y), \\ \pi_0\left(\mathcal{Q}(P)\right)(Y) \\ \pi_0\left(\widetilde{\mathcal{Q}}(P)\right)(Y), \\ \pi_0\left(\widetilde{\mathcal{Q}}'(P)\right)(Y). \end{cases} \qquad \text{are defined.} \tag{6}$$

For historical reasons, we explicitly define the Homotopy obstruction set $\pi_0\left(\mathcal{LO}(P)\right)$, by the pushout diagrams, in $\underline{Sets}$, as follows:

$$\begin{tikzcd} \mathcal{LO}(P[T]) \arrow[r, "T=0"] \arrow[d, "T=1"'] & \mathcal{LO}(P) \arrow[d] \\ \mathcal{LO}(P) \arrow[r] & \pi_0\left(\mathcal{LO}(P)\right) \end{tikzcd} \qquad \text{in} \quad \underline{Sets}. \tag{7}$$

In deed, $\pi_0\left(\mathcal{LO}(P)\right)$ was the Homotopy obstruction explicitly envisioned by Nori (see [13]).

For the convenience of our discussions, we make the following notational adjustment.

Notations 2.3. Until Appendix A, we would only be interested in the value of the above homotopy pre sheaves (6), when $Y = \mathrm{Spec}\,(A)$. To simplify notations, we will make the following notational adjustment:

$$\begin{cases} \pi_0\left(\widetilde{\mathcal{Q}}(P)\right) := \pi_0\left(\widetilde{\mathcal{Q}}(P)\right)(A) \\ \pi_0\left(\widetilde{\mathcal{Q}}'(P)\right) := \pi_0\left(\widetilde{\mathcal{Q}}'(P)\right)(A) \\ \pi_0\left(\mathcal{LO}(P)\right) := \pi_0\left(\mathcal{LO}(P)\right)(A) \end{cases}$$

Technically, as well, this adjustment would not make any difference. We would prove subsequently that all these sets are isomorphic, when $1/2 \in A$. This set $\pi_0\left(\mathcal{LO}(P)\right)$ would be referred to as **Homotopy obstruction Set** of P.

We record, the following basic lemma

Lemma 2.4. *Use the notations as above (2.3) and assume $1/2 \in A$. Then, the bijection κ, induces an isomorphism*

$$\overline{\kappa} : \pi_0\left(\widetilde{\mathcal{Q}}(P)\right) \xrightarrow{\sim} \pi_0\left(\widetilde{\mathcal{Q}}'(P)\right)$$

Further, the maps η, ν, η' (in diagram (4)) induce set theoretic maps, as denoted in the commutative diagram of maps of pre-sheaves:

$$\begin{tikzcd} \pi_0\left(\widetilde{\mathcal{Q}}(P)\right) \arrow[r, two heads, "\overline{\nu}"] \arrow[d, two heads, "\overline{\eta}"'] & \pi_0\left(\mathcal{Q}(P)\right) \arrow[dl, two heads, "\overline{\eta}'"] \\ \pi_0\left(\mathcal{LO}(P)\right) & \end{tikzcd}$$

Proof. It follows from definition of pushout. $\square$

We proceed to prove that, the above is a commutative triangle of bijections:

$$\begin{tikzcd} \pi_0\left(\widetilde{\mathcal{Q}}(P)\right) \arrow[r, two heads, "\overline{\nu}", "\sim"'] \arrow[d, two heads, "\overline{\eta}"', "\wr"] & \pi_0\left(\mathcal{Q}(P)\right) \arrow[dl, two heads, "\sim"', "\overline{\eta}'"] \\ \pi_0\left(\mathcal{LO}(P)\right) & \end{tikzcd} \tag{8}$$

We fix notations, for $(f, p, s) \in \widetilde{\mathcal{Q}}(P)$, its equivalence class in $\pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$ will be denoted by $[(f, p, s)]$ and similar notations will be used for $(f, s) \in \mathcal{Q}(P)$ and $(I, \omega) \in \mathcal{LO}(P)$. Note, given $(I, \omega) \in \mathcal{LO}(P)$, ω lifts to a homomorphism f, as follows:

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[dr, "\omega"'] & I \arrow[d] \\ & \frac{I}{I^2} \end{tikzcd} \tag{9}$$

By Nakayama's lemma there is an element $s \in I$ such that $(1-s)I \subseteq f(P)$. Consequently, $(f,s) \in \mathcal{Q}(P)$ and $I = (f(P), s)$. This association would not be unique. Such a pair $(f,s) \in \mathcal{Q}(P)$ will be referred to as a lift of (I,ω) in $\mathcal{Q}(P)$. Now define the map:

$$\chi : \mathcal{LO}(P) \longrightarrow \pi_0\left(\mathcal{Q}(P)\right) \quad \text{by} \quad \chi\left(I,\omega\right) = [(f,s)] \in \pi_0\left(\mathcal{Q}(P)\right) \tag{10}$$

where $(f,s) \in \mathcal{Q}(P)$ is any lift of (I,ω) in $\mathcal{Q}(P)$, (*as in diagram (9)*) and $[(f,s)]$ is its equivalence class. In several lemmas, we establish that χ is well defined.

Lemma 2.5. *Use the notations as in (2.3). Let $(I,\omega_I) \in \mathcal{LO}(P)$ and $(f,s) \in \mathcal{Q}(P)$ be a lift, as in diagram (9). Further, assume that $t(1-t) \in f(P)$, with $I = (f(P),s) = (f(P),t)$. Then*

$$[(f,s)] = [(f,t)] \in \pi_0\left(\mathcal{Q}(P)\right).$$

Proof. First note, $(1-s)I \subseteq f(P)$ and $(1-t)I \subseteq f(P)$. Write $I[T] = IA[T]$. So,

$$I[T] = f(P)A[T] + sA[T] = f(P)A[T] + tA[T].$$

Let $S(T) = t + T(s-t)$. Clearly, $S(T) \in I[T]$. Further,

Claim. $(1 - S(T))I[T] \subseteq f(P)A[T]$.

We have $(1-S(T))I[T] = (1-S(T))\left(f(P)A[T] + sA[T]\right)$. So, we only need to prove that $(1-S(T))s \in f(P)A[T]$. But

$$(1-S(T))s = (1-t)s - T(s-t)s = (1-t)s - T[(s-1)s + (1-t)s] \in f(P)A[T]$$

So, the claim is established. Therefore, $(1-S(T))S(T) \in f(P)A[T]$. Denote $f[T] := f \otimes 1_{A[T]}$. Then, $f[T] : P[T] \twoheadrightarrow f(P)A[T]$ is a surjection. Clearly, $(f[T], S(T)) \in \mathcal{Q}(P[T])$. Now, $(f[T],S(T))_{T=0} = (f,t)$ and $(f[T],S(T))_{T=1} = (f,s)$. The proof is complete. $\square$

Lemma 2.6. *Use the notations as in (2.3). Suppose $(I,\omega) \in \mathcal{LO}(P)$ and f,g be two lifts of ω as follows:*

$$\begin{tikzcd} P \arrow[r, two heads, "f"] \arrow[dr, "\omega"'] & f(P) \arrow[d, two heads] \\ & \frac{I}{I^2} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} P \arrow[r, two heads, "g"] \arrow[dr, "\omega"'] & g(P) \arrow[d, two heads] \\ & \frac{I}{I^2} \end{tikzcd}$$

$$\ni \quad I = (f(P), s) = (g(P), t) \text{ and } s(1-s) \in f(P),\ t(1-t) \in g(P).$$

Then

$$[(f, s)] = [(g, t)] \in \pi_0 (\mathcal{Q}(P))$$

Proof. Note, $(g - f)(P) \subseteq I^2$. Let $F = f[T] + T(g[T] - f[T]) \in P[T]^*$. It is obvious that

$$I[T] = F(P[T]) + I[T]^2$$

For completeness, we give a proof.

$$\forall\ x \in I, x = (1-s)x + sx = f(p) + sx \quad \text{where} \quad p \in P,\ sx \in I^2$$

So,

$$(\text{modulo } I[T]^2) \quad x \equiv f(p) \equiv F[T](p).$$

So,

$$\exists \quad S(T) \in I[T] \ \ni (1 - S(T))I[T] \subseteq F[T](P[T])$$

So, $(F[T], S(T)) \in \mathcal{Q}(P[T])$. Therefore,

$$[(f, S(0))] = [(F(0), S(0))] = [(F(1), S(1))] = [(g, S(1))]$$

Now, the proof is complete by (2.5). □

Theorem 2.7. *Use the notations as in (2.3). Let $(I, \omega) \in \mathcal{LO}(P)$. Then, $\chi(I, \omega)$ as defined in equation (10), is well defined.*

Proof. Follows from Lemma 2.6. □

Now, we prove that $\overline{\nu}$ is a bijection, as follows.

Theorem 2.8. *Use the notations as in (2.3). Then, the map*

$$\overline{\nu} : \pi_0 \left(\widetilde{\mathcal{Q}}(P) \right) \twoheadrightarrow \pi_0 (\mathcal{Q}(P)) \qquad \text{is a bijection.}$$

Proof. Define a map $\Psi_0 : \mathcal{Q}(P) \to \pi_0 \left(\widetilde{\mathcal{Q}}(P) \right)$ as follows: Given $(f, s) \in \mathcal{Q}(P)$, $\exists\ p \in P \ \ni\ f(p) = s(1-s)$. Define

$$\Psi_0(f, s) := [(f, p, s)] \in \pi_0 \left(\widetilde{\mathcal{Q}}(P) \right).$$

We show that this association is a well defined map. To show this, suppose there is another $q \in P$ such that $f(q) = s(1-s)$. Note $f(p-q) = 0$. So, $f[T](p + T(q-p)) = f(p) + Tf(q-p) = f(p) + 0 = s(1-s)$. Therefore,

$$H(T) := (f[T], p + T(q-p), s) \in \widetilde{\mathcal{Q}}(P[T])$$

and, hence

$$H(0) = (f, p, s) \sim H(1) = (f, q, s).$$

This establishes that Ψ_0 is well defined. Now, we show that Ψ_0 is homotopy invariant. To see this, suppose $H(T) = (F, S(T)) \in \mathcal{Q}(P[T])$. Then, $S(T)(1 - S(T)) = F(p(T))$, for some $p(T) \in P[T]$. Write $\widetilde{H} = (F, p(T), S(T)) \in \widetilde{\mathcal{Q}}(P[T])$. So,

$$\Psi_0(F(0), s(0)) = [\widetilde{H}(0)] = [\widetilde{H}(1)] = \Psi_0(F(1), S(1))$$

This establishes that Ψ_0 factors through a map

$$\Psi : \pi_0 (\mathcal{Q}(P)) \to \pi_0 \left(\widetilde{\mathcal{Q}}(P)\right).$$

It is easy to check that $\overline{\nu}$ and Ψ are inverse of each other. The proof is complete. $\square$

Lemma 2.9. *Use the notations as in (2.3). Then, the map $\chi : \mathcal{LO}(P) \longrightarrow \pi_0 (\mathcal{Q}(P))$ (see (10)) induces a well defined map $\overline{\chi} : \pi_0 (\mathcal{LO}(P)) \longrightarrow \pi_0 (\mathcal{Q}(P))$, which is the inverse of the map $\overline{\eta}' : \pi_0 (\mathcal{Q}(P)) \longrightarrow \pi_0 (\mathcal{LO}(P))$.*

Consequently, all the maps $\overline{\eta}$, $\overline{\eta}'$, $\overline{\nu}$ in diagram (8), are bijections.

Proof. The latter statement follows from the first one. Given a homotopy $H(T) \in \mathcal{LO}(P[T])$, it lifts to a homotopy $\widetilde{H}(T) = (F(T), S(T)) \in \mathcal{Q}(P[T])$. So, $\chi(H(0)) = [(F(0), S(0))] = [(F(1), S(1))] = \chi(H(1))$. So, χ is homotopy invariant, hence $\overline{\chi}$ is well defined. It is easy to see that this induced map is the inverse of $\overline{\eta}'$. The proof is complete. $\square$

Corollary 2.10. *Use the notations as in (2.3). Recall the notation*

$$Q_{2n}(A) = \left\{(x_1, \ldots, x_n; y_1, \ldots, y_n; z) \in A^{2n+1} : \sum x_i y_i = z(1-z)\right\}.$$

If $P = A^n = \oplus Ae_i$ is free, then $Q_{2n}(A) \cong \widetilde{\mathcal{Q}}(P)$ is a bijection. This bijection induces a bijection $\pi_0 (Q_{2n}(A)) \cong \pi_0 \left(\widetilde{\mathcal{Q}}(P)\right)$.

Before we proceed, we introduce the following notions.

Notations 2.11. Suppose A is a commutative noetherian ring, with $\dim A = d$ and P is a projective A-module, with $rank(P) = n$. Denote $\zeta = \overline{\nu}^{-1}\chi : \mathcal{LO}(P) \longrightarrow \pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$ and $\zeta_0 : \widetilde{\mathcal{Q}}(P) \longrightarrow \pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$. So, we have a commutative diagram:

$$\begin{tikzcd}
\widetilde{\mathcal{Q}}(P) \arrow[d, "\eta"'] \arrow[dr, "\zeta_0"] & \\
\mathcal{LO}(P) \arrow[r, "\zeta"'] & \pi_0\left(\widetilde{\mathcal{Q}}(P)\right)
\end{tikzcd}$$

Remark 2.12. The equation $\sum_{i=1}^n X_iY_i + Z(Z-1) = 0$ would be the main motivation behind the definition of $\widetilde{\mathcal{Q}}(P)$. For a field k and the ring $\mathscr{B}(k) = \frac{k[X_1,\ldots,X_n;Y_1,\ldots,Y_n;Z]}{(\sum_{i=1}^n X_iY_i+Z(Z-1))}$, Swan [31] computed the total Chow ring $CH(\mathscr{B}(k))$ of $\mathscr{B}(k)$. This ring $\mathscr{B}(k)$ is sometimes referred to as the **universal ring**, for complete intersections. Using the structure of the Chow ring $CH(\mathscr{B}(k))$, together with Riemann-Roch Theorem, Mohan Kumar and Nori [26] proved that the ideal $I = (X_1,\ldots,X_n,Z)\mathscr{B}(k) \subseteq \mathscr{B}(k)$, is not image of a projective $\mathscr{B}(k)$-module of rank n.

In the more recent past, the notation $Q_{2n} := \mathrm{Spec}\,(\mathscr{B}(k))$ has been somewhat standard in the literature of the motivic approach to Euler class theory.

In fact, sometimes it would be convenient to work with $\widetilde{\mathcal{Q}}(P)$ than $\mathcal{LO}(P)$. This is due to the fact that, when $1/2 \in A$, $\widetilde{\mathcal{Q}}(P) \cong \widetilde{\mathcal{Q}}'(P)$, which has a nice quadratic structure that we can exploit (see §3).

3. Homotopy equivalence

In this section, we prove the following key homotopy theorem.

Theorem 3.1. *Let A be a regular ring over a field k, with $1/2 \in k$. Let P be a projective A-module, with $rank(P) = n \geq 2$, and $(\mathbb{Q}(P), q) = \mathbb{H}(P) \perp A$ (see 2.1). Recall $\widetilde{\mathcal{Q}}'(P) \subseteq \mathbb{Q}(P) = P^* \oplus P \oplus A$. Suppose $H(T) \in \widetilde{\mathcal{Q}}'(P[T])$. Then, there is an orthogonal transformation $\sigma(T) \in O(\mathbb{Q}(P[T]), q)$, such that*

$$H(T) = \sigma(T)(H(0)) \qquad \text{and} \qquad \sigma(0) = 1.$$

Proof. Let $H(T) = (f(T), p(T), s(T)) \in \widetilde{\mathcal{Q}}'(P[T])$ be a homotopy, as above. So, $H(0) \in \widetilde{\mathcal{Q}}'(P)$. Then,

$$A[T]H(T) \cong A[T]H(0) \cong (A[T], q_0) \qquad \text{are isometric,}$$

where q_0 is the trivial quadratic space of rank one. The bilinear inner product in $\mathbb{Q}(P)$ will be denoted $\langle -, - \rangle$. We have the following split exact sequences of quadratic spaces:

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] & \mathbb{Q}(P[T]) \arrow[r, "\langle H(T), - \rangle"] & A[T] \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & K_0 \arrow[r] & \mathbb{Q}(P) \arrow[r, "\langle H(0), - \rangle"'] & A \arrow[r] & 0 \end{tikzcd}$$

Therefore, $K = (A[T]H(T))^{\perp}$, $K_0 = (AH(0))^{\perp}$ are orthogonal complements. Write $\overline{K} := K \otimes \frac{A[T]}{(T)}$. Note, for $\wp \in \operatorname{Spec}(A)$, $\mathbb{Q}(P)_{\wp} \cong (A, q_{2n+1})$, where $q_{2n+1} = \sum_{i=1}^{n} X_i Y_i + Z^2$. So, $\overline{K}_{\wp} \cong (K_0)_{\wp}$ are isometric. It is standard (see [15, Lemma 4.1]), that $(K_0)_{\wp} = (A_{\wp} H(0))^{\perp} \cong (A, q_{2n})_{\wp}$ where $q_{2n} = \sum_{i=1}^{n} X_i Y_i$. In other words, $\overline{K}$ is locally trivial. By the Quadratic version [15, Theorem 3.5] of Lindel's theorem [11], there is an isometry $\tau : K \xrightarrow{\sim} \overline{K} \otimes A[T]$. Further, it follows $\overline{K} = (AH(0))^{\perp} \cong K_0$. Therefore, there is an isometry $\sigma_0 : \overline{K} \xrightarrow{\sim} K_0$, which extends to an isometry $\sigma_0 \otimes 1 : \overline{K} \otimes A[T] \xrightarrow{\sim} K_0 \otimes A[T]$. Then, $\sigma_1 := (\sigma_0 \otimes 1)\tau : K \xrightarrow{\sim} K_0 \otimes A[T]$ is an isometry. Finally, note

$$(A[T]H(T), q_{|A[T]H(T)}) \cong (A[T], q_0) \cong (A[T]H(0), q_{|A[T]H(0)}).$$

Now, consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d, "\sigma_1"'] & \mathbb{Q}(P[T]) \arrow[r, "\langle H(T), - \rangle"] \arrow[d, dashed, "\sigma(T)"] & A[T] \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & K_0 \otimes A[T] \arrow[r] & \mathbb{Q}(P[T]) \arrow[r, "\langle H(0), - \rangle"'] & A[T] \arrow[r] & 0 \end{tikzcd} \tag{11}$$

of quadratic spaces. In this diagram, the horizontal lines are split exact sequences of quadratic spaces. Hence, there is an isometry $\sigma(T) \in O\left(\mathbb{Q}(P[T]), q\right)$, such that the diagram commutes. That means, for all $\mathbf{v} \in \mathbb{Q}(P[T])$, we have $\langle H(T), \mathbf{v} \rangle = \langle H(0), \sigma(T)\mathbf{v} \rangle$. Replacing $\sigma(T)$ by $\sigma(T)^{-1}$, we have $\sigma(T)H(0) = H(T)$. So, we have $\sigma(0)H(0) = H(0)$. Again, by replacing $\sigma(T)$ by $\sigma(T)\sigma(0)^{-1}$, we have $\sigma(0) = 1$. The proof is complete. $\square$

The following Corollary would be of some importance for our future discussions.

Corollary 3.2. *Let A be a regular ring over a field k, with $1/2 \in k$. Let P be a projective A-module, with rank$(P) = n \geq 2$, and $(\mathbb{Q}(P), q) = \mathbb{H}(P) \perp A$. Let $\mathbf{u}, \mathbf{v} \in \widetilde{\mathcal{Q}}'(P)$ be such that $[\mathbf{u}] = [\mathbf{v}] \in \pi_0\left(\widetilde{\mathcal{Q}}'(P)\right)$. Then, there is a homotopy $H(T) \in \widetilde{\mathcal{Q}}'(P[T])$ such that $H(0) = \mathbf{u}$ and $H(1) = \mathbf{v}$. Equivalently, for $\mathbf{u}, \mathbf{v} \in \widetilde{\mathcal{Q}}(P)$ if $\zeta_0(\mathbf{u}) = \zeta_0(\mathbf{v}) \in \pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$, then there is a homotopy $H(T) \in \widetilde{\mathcal{Q}}(P[T])$ such that $H(0) = \mathbf{u}$ and $H(1) = \mathbf{v}$.*

Proof. Suppose $\mathbf{u}, \mathbf{v} \in \widetilde{\mathcal{Q}}(P)$ such that $[\mathbf{u}] = [\mathbf{v}] \in \pi_0\left(\widetilde{\mathcal{Q}}'(P)\right)$. Then, there is a sequence of homotopies $H_1(T), \ldots, H_m(T) \in \widetilde{\mathcal{Q}}'(P[T])$ such that $\mathbf{u} =: \mathbf{u}_0 := H_1(0)$, $\mathbf{u}_m := H_m(1) = \mathbf{v}$ and $\forall\ i = 1, \ldots, m-1$, we have $\mathbf{u}_i := H_i(1) = H_{i+1}(0)$. By Theorem 3.1, for $i = 1, \ldots, m$ there are orthogonal matrices $\sigma_i(T) \in O(\mathbb{Q}(P[T]), q)$ such that $\sigma_i(0) = 1$ and $H_i(T) = \sigma_i(T)H_i(0) = \sigma_i(T)\mathbf{u}_{i-1}$. Therefore, $\mathbf{u}_i = H_i(1) = \sigma_i(1)\mathbf{u}_{i-1}$.

Write $H(T) = \sigma_m(T)\cdots\sigma_1(T)\mathbf{u}_0$. Then, $H(T) \in \widetilde{\mathcal{Q}}'(P[T])$ and $H(0) = \mathbf{u}_0$ and $H(1) = \mathbf{u}_m$. This establishes first part of the statement on $\pi_0\left(\widetilde{\mathcal{Q}}'(P)\right)$. The latter assertion on $\pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$ follows from the former, by the bijective correspondences $\widetilde{\mathcal{Q}}'(P) \xrightarrow{\sim} \widetilde{\mathcal{Q}}(P)$ and $\widetilde{\mathcal{Q}}'(P[T]) \xrightarrow{\sim} \widetilde{\mathcal{Q}}(P[T])$. This completes the proof. $\square$

Remark 3.3. Another way to state (3.2) would be that the homotopy relation on $\widetilde{\mathcal{Q}}(P)$ is actually an equivalence relation.

In a slightly more formal language, the above is summarized as follows.

Theorem 3.4. *Let A be a regular ring over a field k, with $1/2 \in k$. Let P be a projective A-module, with $rank(P) = n \geq 2$, and $(\mathbb{Q}(P), q) = \mathbb{H}(P) \perp A$. For, $\sigma(T) \in O(\mathbb{Q}(P[T]), q)$ and $\mathbf{u} \in \widetilde{\mathcal{Q}}'(P)$, define the* (left) *action $\sigma(T)\mathbf{u} := \sigma(1)\mathbf{u} \in \widetilde{\mathcal{Q}}'(P)$. Denote $O(\mathbb{Q}(P[T]), q, T) = \{\sigma(T) \in O(\mathbb{Q}(P[T]), q) : \sigma(0) = 1\}$. Then, the map*

$$\frac{\widetilde{\mathcal{Q}}'(P)}{O(\mathbb{Q}(P), q, T)} \longrightarrow \pi_0\left(\widetilde{\mathcal{Q}}'(P)\right) \quad \text{is a bijection.}$$

Proof. Similar to the proof of (3.2). $\square$

4. Homotopy triviality and lifting

In this section, under further smoothness conditions, we establish that for $(I, \omega_I) \in \mathcal{LO}(P)$, the triviality of $\zeta(I, \omega_I)$ implies that ω_I lifts to a surjective map $P \twoheadrightarrow I$. We start this section with the following notations and definitions.

Definition 4.1. Suppose A is a commutative noetherian ring, with $\dim A = d$ and P is a projective A-module, with $rank(P) = n$. There are two distinguished points in $\widetilde{\mathcal{Q}}(P)$, namely:

$$\mathbf{0} := (0,0,0) \in \widetilde{\mathcal{Q}}(P), \quad \mathbf{1} := (0,0,1) \in \widetilde{\mathcal{Q}}(P)$$

$$\text{We denote} \quad \mathbf{e}_0 = \zeta_0(\mathbf{0}) \in \pi_0\left(\widetilde{\mathcal{Q}}(P)\right), \quad \text{and} \quad \mathbf{e}_1 = \zeta_0(\mathbf{1}) \in \pi_0\left(\widetilde{\mathcal{Q}}(P)\right).$$

Use the same notations $\mathbf{e}_0,\ \mathbf{e}_1 \in \pi_0(\mathcal{LO}(P)) \cong \pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$, to denote their respective images. Define the obstruction class

$$\varepsilon(P) := \mathbf{e}_0 \in \pi_0(\mathcal{LO}(P)) \cong \pi_0\left(\widetilde{\mathcal{Q}}(P)\right).$$

In the light of (1.1), $\varepsilon(P)$ will be referred to as **(Nori) Homotopy Class** of P, which may sometimes be shortened. Note, for any $f \in P^*$ and $p \in P$, $\varepsilon(P) := \mathbf{e}_0 = \zeta_0(f,0,0) = \zeta_0(0,p,0) \in \pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$.

We record the following obvious observation.

Lemma 4.2. *Suppose A is a commutative noetherian ring with $\dim A = d$ and P is a projective A-module. Let $p \in P$ and $f \in P^*$ be such that $f(p) = 1$ (i.e. $P \cong Q \oplus A$). Let*

$$\mathbf{0} = (0,0,0), \ \mathbf{u} = (f,0,0), \ \mathbf{1} = (0,0,1) \in \widetilde{\mathcal{Q}}(P).$$

Then, $\zeta_0(\mathbf{0}) = \zeta_0(\mathbf{u}) = \zeta_0(\mathbf{1}) \in \pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$. In other words,

$$\varepsilon(P) = \mathbf{e}_0 = \mathbf{e}_1.$$

Proof. The first equality is obvious and was mentioned above (4.1). To prove the second equality, write $H(T) = ((1-T)f, Tp, T)$. Then, $(1-T)f(Tp) = T(1-T)$. So, $H(T) \in \widetilde{\mathcal{Q}}(P[T])$. We have $H(0) = \mathbf{u}$ and $H(1) = (0,p,1)$.

Now write $G(T) = (0,(1-T)p,1)) \in \widetilde{\mathcal{Q}}(P[T])$. Then, $G(0) = (0,p,1)$ and $G(1) = (0,0,1)$. The proof is complete. $\square$

The following is the main result in this section.

Theorem 4.3. *Suppose A is an essentially smooth ring over an infinite perfect field k, with $1/2 \in k$ and $\dim A = d$. Let P be a projective A-module with $rank(P) = n$, with $2n \geq d+3$. Suppose $(I, \omega_I) \in \mathcal{LO}(P)$, with $height(I) \geq n$. Then, ω_I lifts to a surjective map $P \twoheadrightarrow I$ if and only if $\varepsilon(P) = \zeta(I, \omega_I)$.*

Proof. Suppose ω_I lifts to a surjective map $f : P \twoheadrightarrow I$. Write $H(T) = (f(T),0,0) \in \widetilde{\mathcal{Q}}(P[T])$. Then, $\zeta(I,\omega_I) = \zeta_0(H(1)) = \zeta_0(H(0)) = \zeta_0(\mathbf{0}) = \varepsilon(P)$.

Conversely, suppose $\zeta(I,\omega_I) = \zeta_0(\mathbf{0})$. For notational convenience, fix $f_0 \in P^*$, and let $\mathbf{v}_0 = (f_0,0,0) \in \widetilde{\mathcal{Q}}(P)$. Then, $\zeta(I,\omega_I) = \zeta_0(\mathbf{0}) = \zeta_0(\mathbf{v}_0)$. There is an element $\mathbf{u} = (f_1,p_1,s_1) \in \widetilde{\mathcal{Q}}(P)$ such that $\eta(\mathbf{u}) = (I,\omega_I)$. By Moving Lemma argument 4.5 (below), we can assume that $height(f_0(P)) \geq n$ and $height(f_1(P)) \geq n$. We have, $\zeta_0(\mathbf{u}) = \zeta_0(\mathbf{v}_0)$. By (3.2), there is a homotopy $H(T) = (f(T), p(T), S(T)) \in \widetilde{\mathcal{Q}}(P[T])$ such that $H(0) = \mathbf{v}_0$ and $H(1) = \mathbf{u}$. Write $\eta(H(T)) = (J,\Omega)$. We would apply [3, Theorem 4.13], for which we would need $height(J) \geq n$. So, we modify $H(T)$, as follows. Denote $Z(T) = 1 - S(T)$. Write $\mathscr{P} = \{\wp \in \operatorname{Spec}(A[T]) : height(\wp) \leq n-1, T(1-T)Z(T) \notin \wp\}$. For $\wp \in \mathscr{P}$, let $\delta(\wp)$ be the maximum of the length of chains in $\mathscr{P}$, ending at $\wp$. Then $\delta : \mathscr{P} \longrightarrow \mathbb{N}$ is a generalized dimension functions (*consult* [12, pp. 36-37]). Note, $\forall\ \wp \in \mathscr{P}$, we have $\delta(\wp) \leq n-1$. Now, $(f(T), T(1-T)Z(T)^2) \in P[T]^* \oplus A[T]$ is basic on $\mathscr{P}$. So, there is an element $g(T) \in P[T]^*$ such that $F(T) = f(T) + T(1-T)Z(T)^2 g(T)$ is basic on $\mathscr{P}$. It follows, $F(0) = f(0)$ and $F(1) = f(1)$.

We have $Z(T)(1 - Z(T)) = (1 - S(T))S(T) =$

$$f(T)(p(T)) = F(T)(p(T)) - T(1-T)Z(T)^2 g(T)(p(T))$$

Write $\mathcal{J} = (f(T)(P[T]), Z(T))$. Then $\mathcal{J} = (F(T)(P[T]), Z(T))$. Write $M = \frac{\mathcal{J}}{F(T)(P[T])}$. Let $p_1, \ldots, p_m$ be a set of generators of P. So, $\mathcal{J}$ is generated by $f(T)(p_1), \ldots, f(T)(p_m)$, $Z(T)$. Use "overline" to denote the images in M and repeat the proof of Nakayama's Lemma, as follows:

$$\begin{pmatrix} \overline{f(T)(p_1)} \\ \overline{f(T)(p_2)} \\ \cdots \\ \overline{f(T)(p_m)} \\ \overline{Z(T)} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 0 & -T(1-T)Z(T)g(T)(p_1) \\ 0 & 0 & \cdots & 0 & -T(1-T)Z(T)g(T)(p_2) \\ 0 & 0 & \cdots & 0 & \cdots \\ 0 & 0 & \cdots & 0 & -T(1-T)Z(T)g(T)(p_m) \\ 0 & 0 & \cdots & 0 & Z(T) - T(1-T)Z(T)g(T)(p(T)) \end{pmatrix} \begin{pmatrix} \overline{f(T)(p_1)} \\ \overline{f(T)(p_2)} \\ \cdots \\ \overline{f(T)(p_m)} \\ \overline{Z(T)} \end{pmatrix}$$

So,

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & T(1-T)Z(T)g(T)(p_1) \\ 0 & 1 & \cdots & 0 & T(1-T)Z(T)g(T)(p_2) \\ 0 & 0 & \cdots & 0 & \cdots \\ 0 & 0 & \cdots & 1 & T(1-T)Z(T)g(T)(p_m) \\ 0 & 0 & \cdots & 0 & 1 - Z(T) + T(1-T)Z(T)g(T)(p(T)) \end{pmatrix} \begin{pmatrix} \overline{f(T)(p_1)} \\ \overline{f(T)(p_2)} \\ \cdots \\ \overline{f(T)(p_m)} \\ \overline{Z(T)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 0 \end{pmatrix}$$

With $Z'(T) = Z(T) - T(1-T)Z(T)g(T)(p(T))$, the determinant of this matrix is $1 - Z'(T)$. It follows, $(1 - Z'(T))\mathcal{J} \subseteq F(T)(P[T])$. So, $(1 - Z'(T))Z'(T) = F(T)(q(T))$ for some $q(T) \in P[T]$. Note, $Z'(0) = Z(0)$ and $Z'(1) = Z(1)$. Therefore, $(F(T), q(T), Z'(T)) \in \widetilde{\mathcal{Q}}(P[T])$. Also, with $S'(T) = 1 - Z'(T)$, $(F(T), q(T), S'(T)) \in \widetilde{\mathcal{Q}}(P[T])$. We have, $S'(T)(1 - S'(T)) = (1 - Z'(T))Z'(T) = F(T)(q(T))$ $S'(0) = 1 - Z'(0) = 1 - Z(0) = S(0) = 0$ and $S'(1) = 1 - Z'(1) = 1 - Z(1) = S(1)$.

Write $\mathcal{H}(T) = (F(T), q(T), S'(T))$ and $\eta(\mathcal{H}(T)) = (J', \Omega')$. It is clear $\mathcal{H}(0) = (f_0, q(0), 0)$, $\mathcal{H}(1) = (f_1, q(1), S(1))$. So, $\eta(\mathcal{H}(0)) = \eta(\mathbf{v}_0)$ and $\eta(\mathcal{H}(1)) = \eta(\mathbf{u}) = (I, \omega_I)$.

We have $J' = (F(T)(P[T]), S'(T))$. We claim that $height(J') \geq n$. To see this, let $J' \subseteq \wp \in \operatorname{Spec}(A[T])$. If $T \in \wp$, then $I_0 := f_0(P) \subseteq \wp$ and hence $height(\wp) \geq n$. Likewise, if $1 - T \in \wp$, then $I_1 := f_1(P) \subseteq \wp$ and hence $height(\wp) \geq n$. So, we assume $T(1-T) \notin \wp$. If $Z(T) \in \wp$, then $\mathcal{J} = (F(T)(P[T]), Z'(T)) = (F(T)(P[T]), Z(T)) \subseteq \wp$, which is impossible because $S'(T) \in \wp$. So, $T(1-T)Z(T) \notin \wp$. Since F is basic on $\mathscr{P}$, $height(\wp) \geq n$. This establishes the claim.

So, $\mathcal{H}(T) = (F(T), q(T), S'(T)) \in \widetilde{\mathcal{Q}}(P[T])$ is such that $\eta(\mathcal{H}(0)) = (I_0, \omega_{I_0})$, $\eta(\mathcal{H}(1)) = (I, \omega_I)$ and with $\eta(\mathcal{H}(T)) = (J', \Omega')$, we have $height(J') \geq n$. If $T \in \wp \in Ass\left(\frac{A[T]}{J'}\right)$ then $(J'(0), T) = (I_0, T) \subseteq \wp$. Then, $height(\wp) \geq n + 1$. This is impossible because $A[T]$ is regular (Cohen-Macaulay) and J' is local complete intersection ideal. Hence,

$$
\begin{tikzcd}
\frac{A[T]}{TJ'} \arrow[r, "T=0"] \arrow[d] & A \arrow[d] \\
\frac{A[T]}{J'} \arrow[r] & \frac{A}{J'(0)}
\end{tikzcd}
$$

is a patching diagram (see (4.4) below). So, the map $\Omega' : P[T] \twoheadrightarrow \frac{J'}{(J')^2}$ and $f_0 : P \twoheadrightarrow I_0$ combines to give a surjective maps $\phi : P[T] \twoheadrightarrow \frac{J'}{T(J')^2}$. Now, by [3, Theorem 4.13], there is a surjective homomorphism $\varphi : P[T] \twoheadrightarrow J'$ such that $\varphi(0) = f_0$ and $\varphi \otimes \frac{A[T]}{J'} = \Omega'$. Now, it follows that $\varphi(1)$ is a lift of ω_I. This completes the proof. $\square$

We used the following lemma above, while it needs a proof. The standard references for Patching diagrams are [22,28,27]. We will be specific in the following statement, because the literature does not seem complete regarding definitions of Patching diagrams of modules that are not projective.

Lemma 4.4. *Let R be a noetherian commutative ring and $A = R[T]$. Let I be a locally complete intersection ideal of A with height$(I) = r$. Assume $T : \frac{A}{I} \hookrightarrow \frac{A}{I}$ is injective (i.e. $T \notin \wp \in Ass\left(\frac{A}{I}\right)$). Then,*

$$
\begin{tikzcd}
\frac{I}{TI^2} \arrow[r] \arrow[d] & \frac{I}{TI} \arrow[d] \\
\frac{I}{I^2} \arrow[r] & \frac{I}{I^2+TI}
\end{tikzcd}
$$

is a Patching diagram, in the sense that it is a Cartesian square. Further,

1. $\frac{I}{TI} \xrightarrow{\sim} I(0)$.
2. $\frac{I}{I^2+TI} \xrightarrow{\sim} \frac{I(0)}{I(0)^2}$.

Proof. The patching diagram follows, because $I^2 \cap (TI) = TI^2$.

To see this, first we have $TI^2 \subseteq I^2 \cap (TI)$. Suppose $f \in I^2 \cap (TI)$. Then, $f = Tg$ with $g \in I$. Now, consider the map

$$T : \frac{I}{I^2} \longrightarrow \frac{I}{I^2}$$

Since $\frac{I}{I^2}$ is projective $\frac{A}{I}$-module and $T : \frac{A}{I} \hookrightarrow \frac{A}{I}$ is injective, T is also injective on $\frac{I}{I^2}$. So, $g \in I^2$. So, $f = Tg \in TI^2$.

Now, we prove $\frac{I}{TI} \xrightarrow{\sim} I(0)$. Obviously, the map is surjective. Suppose $f(T) \in I$ and $f(0) = 0$. Then, $f = Tg$. Since T is non zero divisor on $\frac{A}{I}$, $g \in I$. So, $f \in TI$.

Finally, we prove $\frac{I}{I^2+TI} \xrightarrow{\sim} \frac{I(0)}{I(0)^2}$. Again, the map is on to. Suppose $f(T) \in I$ and $f(0) \in I(0)^2$. Then, $f(0) = \sum f_i(0)g_i(0)$. Then, $f - \sum f_i g_i \in (T) \cap I = TI$ (by the above, if we like). So, $f \in I^2 + TI$. $\square$

We close this section with the following "moving lemma argument", which is fairly standard. A number of variations of the same (4.5) would be among the frequently used tools for the rest of our discussions.

Lemma 4.5 *(Moving Lemma). Suppose A is a commutative noetherian ring with* $\dim A = d$ *and P is a projective A-module with rank$(P) = n$. Assume $2n \geq d + 1$. Let $K \subseteq A$ be an ideal with height$(K) \geq n$ and $(I, \omega_I) \in \mathcal{LO}(P)$. Then, there is an element $\mathbf{v} = (f, p, s) \in \widetilde{\mathcal{Q}}(P)$ such that $\eta(\mathbf{v}) = (I, \omega_I)$. Further, with $J = f(P) + A(1 - s)$, we have height$(J) \geq n$ and $J + K = A$.*

Proof. Let $f_0 : P \twoheadrightarrow I$ be any lift of ω_I. Then, $I = f_0(P) + I^2$. By Nakayama's Lemma, there is an element $t \in I$, such that $(1-t)I \subseteq f_0(P)$. Therefore, $t(1-t) = f_0(p_0)$ for some $p_0 \in P$. (*Readers are referred to [12] regarding generalities on Basic Element Theory and generalized dimension functions.*) Write

$$\mathscr{P} = \{\wp \in \operatorname{Spec}(A) : t \notin \wp, \text{ and either } K \subseteq \wp \text{ or } height(\wp) \leq n - 1\}$$

There is a generalized dimension function (see [12]) $\delta : \mathscr{P} \longrightarrow \mathbb{N}$, such that $\delta(\wp) \leq n - 1 \ \forall \ \wp \in \mathscr{P}$. Now $(f_0, t^2) \in P^* \oplus A$ is basic on $\mathscr{P}$. So, there is an element $g \in P^*$ such that $f := f_0 + t^2 g$ is basic on $\mathscr{P}$. It follows, $f(P) + At = f_0(P) + At = I$ and $I = f(P) + I^2$. By Nakayama's Lemma, there is an element $s \in I$, such that $(1 - s)I \subseteq f(P)$ and hence $f(p) = s(1 - s)$, for some $p \in P$, Hence, $I = (f(P), s)$. Now, write $J = f(P) + A(1 - s)$. For $J \subseteq \wp \in \operatorname{Spec}(A)$, $s \notin \wp$ and hence $t \notin \wp$. Since, f is basic on $\mathscr{P}$, $height(\wp) \geq n$. This establishes, $height(J) \geq n$.

Now suppose $J + K \subseteq \wp \in \operatorname{Spec}(A)$. By the same argument above, $t \notin \wp$. Hence, $\wp \in \mathscr{P}$. This is impossible, because f is basic on $\mathscr{P}$. So, $J + K = A$. Now, $\mathbf{v} = (f, p, s) \in \widetilde{\mathcal{Q}}(P)$, satisfies the requirement. $\square$

The following is a converse of Lemma 4.2.

Corollary 4.6. *Suppose A is an essentially smooth ring over an infinite perfect field k, with $1/2 \in k$ and* $\dim A = d$*. Let P be a projective A-module with rank$(P) = n$. Assume $2n \geq d + 3$. Then,*

$$\varepsilon(P) = \mathbf{e}_1 \quad \Longleftrightarrow \quad P \cong Q \oplus A$$

for some projective A-module Q.

Proof. Suppose $P \cong Q \oplus A$. Then, by (4.2), $\varepsilon(P) = \mathbf{e}_0 = \mathbf{e}_1$. Conversely, suppose $\varepsilon(P) = \mathbf{e}_0 = \mathbf{e}_1$. Fix $f_0 \in P^*$ such that $height(f_0(P)) = n$. Then, $\zeta_0(f_0, 0, 0) = \mathbf{e}_0 = \mathbf{e}_1$.

Then, it follows from Theorem 4.3 that $\eta(0,0,1)$ lifts to a surjective map $P \twoheadrightarrow A$. This completes the proof. $\square$

5. The involution

In this section, we introduce an involution map $\Gamma : \pi_0\left(\widetilde{\mathcal{Q}}(P)\right) \longrightarrow \pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$. This can be thought of as a substitute to additive inverse map, without any regard to existence of an addition.

Definition 5.1. Suppose A is a commutative noetherian ring and P is a projective A-module, with $rank(P) = n$. For $(f, p, s) \in \widetilde{\mathcal{Q}}(P)$, define $\Gamma(f, p, s) = (f, p, 1 - s)$. This association, $\mathbf{v} \mapsto \Gamma(\mathbf{v})$, establishes a bijective correspondence

$$\Gamma : \widetilde{\mathcal{Q}}(P) \xrightarrow{\sim} \widetilde{\mathcal{Q}}(P), \quad \text{such that} \quad \Gamma^2 = 1.$$

We would say that Γ is an involution on $\widetilde{\mathcal{Q}}(P)$, which will be a key instrument in the subsequent discussions. (This notation Γ will be among the standard notations throughout this article.)

We record the following obvious lemma.

Lemma 5.2. *Suppose A is a commutative noetherian ring and P is a projective A-module, with $rank(P) = n$ and $\Gamma : \widetilde{\mathcal{Q}}(P) \xrightarrow{\sim} \widetilde{\mathcal{Q}}(P)$ is the involution. Let $\mathbf{v} = (f, p, s) \in \widetilde{\mathcal{Q}}(P)$ and denote $\eta(\mathbf{v}) = (I, \omega_I)$ and $\eta(\Gamma(\mathbf{v})) = (J, \omega_J)$. Then,*

1. *$I \cap J = f(P)$.*
2. *For $H(T) \in \widetilde{\mathcal{Q}}(P[T])$, we have $\Gamma(H(T))_{T=t} = \Gamma(H(t))$.*
3. *Therefore, $\forall\ \mathbf{v}, \mathbf{w} \in \widetilde{\mathcal{Q}}(P) \qquad \zeta_0(\mathbf{v}) = \zeta_0(\mathbf{w}) \Longleftrightarrow \zeta_0(\Gamma(\mathbf{v})) = \zeta_0(\Gamma(\mathbf{w}))$.*

In deed, Γ factors through an involution on $\pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$, as follows.

Corollary 5.3. *Suppose A is a commutative noetherian ring and P is a projective A-module, with $rank(P) = n$. Then, the involution $\Gamma : \widetilde{\mathcal{Q}}(P) \xrightarrow{\sim} \widetilde{\mathcal{Q}}(P)$ induces a bijective map $\widetilde{\Gamma} : \pi_0\left(\widetilde{\mathcal{Q}}(P)\right) \xrightarrow{\sim} \pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$, such that $\widetilde{\Gamma}^2 = 1$ and $\zeta_0\Gamma = \widetilde{\Gamma}\zeta_0$. We say $\widetilde{\Gamma}$ is an involution.* (The notation $\widetilde{\Gamma}$ will also be among our standard notations throughout this article.)

Proof. First, consider the map $\zeta_0\Gamma : \widetilde{\mathcal{Q}}(P) \longrightarrow \pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$. For, $H(T) \in \widetilde{\mathcal{Q}}(P[T])$, we have $\zeta_0\Gamma(H(0)) = \zeta_0\Gamma(H(1))$. Therefore, $\zeta_0\Gamma$ is homotopy invariant. Hence, it induces a well defined map $\widetilde{\Gamma} : \pi_0\left(\widetilde{\mathcal{Q}}(P)\right) \xrightarrow{\sim} \pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$. Clearly, $\widetilde{\Gamma}^2 = 1$ and $\widetilde{\Gamma}$ is a bijection. The proof is complete. $\square$

The following is a way to compute the involution.

Corollary 5.4. *Suppose A is a commutative noetherian ring and P is a projective A-module, with $rank(P) = n$. Suppose $(I, \omega) \in \mathcal{LO}(P)$. For any $\mathbf{v} = (f, p, s) \in \widetilde{\mathcal{Q}}(P)$ with $\eta(\mathbf{v}) = (I, \omega)$, write $\eta(\Gamma(\mathbf{v})) = (J, \omega_J)$. Then,*

$$\widetilde{\Gamma}(\zeta(I, \omega)) = \zeta(J, \omega_J) \in \pi_0\left(\widetilde{\mathcal{Q}}(P)\right).$$

Proof. Obvious. $\square$

The following is another version of the Moving Lemma 4.5.

Lemma 5.5 *(Moving Representation). Suppose A is a commutative noetherian ring, with* $\dim A = d$. *Let P be a projective A-module, with $rank(P) = n$ and $2n \geq d + 1$. Let $x \in \pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$ and let $K \subseteq A$ be an ideal with $height(K) \geq n$. Then, there is a local P-orientation $(J, \omega_J) \in \mathcal{LO}(P)$ such that $x = \zeta(J, \omega_J)$, $height(J) \geq n$ and $J + K = A$.*

Proof. Let $x = \zeta(I, \omega_I)$. First, $\eta(\mathbf{u}) = (I, \omega_I)$ for some $\mathbf{u} \in \widetilde{\mathcal{Q}}(P)$. Denote $(I_0, \omega_{I_0}) := \eta(\Gamma(\mathbf{u}))$. Then, $\tilde{\Gamma}(x) = \zeta(I_0, \omega_{I_0})$.

Now, we apply Moving Lemma 4.5, to (I_0, ω_{I_0}) and K. There is $\mathbf{v} \in \widetilde{\mathcal{Q}}(P)$, such that $\eta(\mathbf{v}) = (I_0, \omega_{I_0})$, and with $\eta(\Gamma(\mathbf{v})) = (J, \omega_J)$, we have $height(J) \geq n$ and $J + K = A$. Now, $x = \tilde{\Gamma}(\tilde{\Gamma}(x)) = \tilde{\Gamma}(\zeta(I_0, \omega_{I_0})) = \zeta(J, \omega_J)$. The proof is complete. $\square$

6. The monoid structure on $\pi_0\left(\mathcal{LO}(P)\right)$

In this section, we define and establish a natural monoid structure on the homotopy obstruction set $\pi_0\left(\mathcal{LO}(P)\right) \cong \pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$, when $2rank(P) \geq \dim A + 2$ and A is a regular ring over a field k, with $1/2 \in k$. We start with the following basic ingredient of the group structure.

Definition 6.1. Let A be a commutative noetherian ring, with $\dim A = d$, and P be a projective A-module, with $rank(P) = n \geq 2$. Let $(I, \omega_I), (J, \omega_J) \in \mathcal{LO}(P)$ be such that $I + J = A$. Let $\omega := \omega_I \star \omega_J : P \twoheadrightarrow \frac{IJ}{(IJ)^2}$ be the unique surjective map induced by ω_I, ω_J. We define a pseudo-sum

$$(I, \omega_I)\hat{+}(J, \omega_J) := (IJ, \omega) \in \pi_0\left(\mathcal{LO}(P)\right).$$

Note, pseudo-sum commutes.

In the rest of this section, we establish that the pseudo sum respects homotopy, when $2n \geq d + 2$, and A is a regular ring over a field k, with $1/2 \in k$. Consequently, this leads to a addition operation on $\pi_0\left(\mathcal{LO}(P)\right)$. The following is the key lemma.

Lemma 6.2. *Let A be a commutative noetherian ring and P be a projective A-module, with $\dim A = d$, rank$(P) = n$, and $2n \geq d + 2$. Consider a homotopy*

$$H(T) = (f(T), p(T), Z(T)) \in \widetilde{\mathcal{Q}}(P[T]).$$

Write $\eta(H(0)) = (K_0, \omega_{K_0})$ and $\eta(H(1)) = (K_1, \omega_{K_1})$. Further suppose $(J, \omega_J) \in \mathcal{LO}(P)$ such that $K_0 + J = K_1 + J = A$ and height$(J) \geq n$. Then, there is a homotopy $\mathcal{H}(T) \in \widetilde{\mathcal{Q}}(P[T])$ such that $\eta(\mathcal{H}(0)) = (K_0 J, \omega_{K_0 J})$ and $\eta(\mathcal{H}(1)) = (K_1 J, \omega_{K_1 J})$, where, for $i = 0, 1$ $\omega_{K_i J} := \omega_{K_i} \star \omega_J : P \twoheadrightarrow \frac{K_i J}{(K_i J)^2}$. Consequently,

$$(K_0, \omega_{K_0}) \hat{+} (J, \omega_J) = (K_1, \omega_{K_1}) \hat{+} (J, \omega_J) \in \pi_0 \left(\mathcal{LO}(P)\right).$$

Proof. We will write $f = f(T)$, $p = p(T)$ and $Z = Z(T)$. Denote $Y = 1 - Z$ and $\eta(\Gamma(H(T)) = (\mathbb{J}, \omega_{\mathbb{J}})$. Then, $\mathbb{J} = (f(P[T]), Y)$. Write

$$\mathscr{P} = \{\wp \in \operatorname{Spec}(A[T]) : YT(1-T) \notin \wp, J \subseteq \wp\}.$$

There is a generalized dimension function $\delta : \mathscr{P} \longrightarrow \mathbb{N}$ such that $\forall\ \wp \in \mathscr{P}$, $\delta(\wp) \leq \dim\left(\frac{A[T]}{JA[T]}\right) \leq d + 1 - height(J) \leq d + 1 - n \leq n - 1$. Further, $(f, Y^2T(1-T))$ is a basic element in $P[T]^* \oplus A[T]$, on $\mathscr{P}$. Therefore, there is an element $\lambda := \lambda(T) \in P[T]^*$ such that

$$f' = f + Y^2T(1-T)\lambda \text{ is basic on } \mathscr{P}. \quad \text{So,} \quad f'(0) = f(0),\ f'(1) = f(1).$$

We have $\mathbb{J} = (f(P[T]), Y) = (f'(P[T]), Y)$. Further,

$$Z(1-Z) = Y(1-Y) = f(p) = f'(p) - Y^2T(1-T)\lambda(p).$$

So,

$$Y = f'(p) - Y^2T(1-T)\lambda(p) + Y^2$$

Write $M = \frac{\mathbb{J}}{f'(P[T])}$. Let $p_1, \ldots, p_m$ be a set of generators of P. Use "overline" to indicate images in M. We intend to repeat the proof of Nakayama's Lemma and we have

$$\begin{pmatrix} \overline{f(p_1)} \\ \overline{f(p_2)} \\ \cdots \\ \overline{f(p_m)} \\ \overline{Y} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 0 & -\lambda(p_1)YT(1-T) \\ 0 & 0 & \cdots & 0 & -\lambda(p_2)YT(1-T) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & -\lambda(p_m)YT(1-T) \\ 0 & 0 & 0 & 0 & Y - \lambda(p)YT(1-T) \end{pmatrix} \begin{pmatrix} \overline{f(p_1)} \\ \overline{f(p_2)} \\ \cdots \\ \overline{f(p_m)} \\ \overline{Y} \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & \lambda(p_1)YT(1-T) \\ 0 & 1 & \cdots & 0 & \lambda(p_2)YT(1-T) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & \lambda(p_m)YT(1-T) \\ 0 & 0 & 0 & 0 & 1-Y+\lambda(p)YT(1-T) \end{pmatrix} \begin{pmatrix} \overline{f(p_1)} \\ \overline{f(p_2)} \\ \cdots \\ \overline{f(p_m)} \\ \overline{Y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 0 \end{pmatrix}$$

Multiplying by the adjoint matrix and computing the determinant, with $Y' = Y - \lambda(p)YT(1-T)$, we have

$$(1-Y')\mathbb{J} \subseteq f'(P[T]).$$

We have $Y'(0) = Y(0) = 1 - Z(0)$, $Y'(1) = Y(1) = 1 - Z(1)$. Further,

$$Y'(1-Y') = f'(p') \qquad \text{for some} \quad p' \in P[T].$$
$$\text{Therefore} \qquad H'(T) = (f', p', Y') \in \widetilde{\mathcal{Q}}(P[T]).$$

We have

$$\mathbb{J} = (f(P[T]), Y) = (f'(P[T]), Y) = (f'(P[T]), Y').$$

In fact, $\eta(H'(T)) = (\mathbb{J}, \omega_{\mathbb{J}})$ and write $\eta(\Gamma(H'(T))) = (\mathbb{I}, \omega_{\mathbb{I}})$. Claim

$$\mathbb{I} + JA[T] = A[T]. \qquad i.e. \qquad (f'(P[T]), 1-Y') + JA[T] = A[T].$$

To see this, let

$$\mathbb{I} + JA[T] \subseteq \wp \in \operatorname{Spec}(A[T])$$

1. If $Y \in \wp$ then $\mathbb{J} = (f'(P[T]), Y) = (f'(P[T]), Y') \subseteq \wp$. So, $Y' \in \wp$, which is impossible, since $1 - Y' \in \wp$. So, $\wp \in D(Y)$.
2. Since f' is unimodular of $\mathscr{P}$ and $\wp \in D(Y)$, we must have $T(1-T) \in \wp$.
3. Now, $T \in \wp$ implies,

   $$\mathbb{I}(0) + J = (f'(0)(P), 1 - Y'(0)) + J = (f(0)(P), 1 - Y(0)) + J = K_0 + J = A \subseteq \wp$$

   which is impossible.
4. Likewise, $1 - T \in \wp$ implies,

   $$\mathbb{I}(1) + J = (f'(1)(P), 1 - Y'(1)) + J = (f(0)(P), 1 - Y(1)) + J = K_1 + J = A \subseteq \wp.$$

   This is also impossible.

This establishes the claim. Recall, $\omega_{\mathbb{I}} : P[T] \twoheadrightarrow \frac{\mathbb{I}}{\mathbb{I}^2}$ is induced by f'. Extend $\omega_J : A^n \twoheadrightarrow \frac{J}{J^2}$ to a surjective map $\omega_{JA[T]} : A[T]^n \twoheadrightarrow \frac{JA[T]}{J^2A[T]}$. Let

$$\Omega := \omega_{\mathbb{I}} \star \omega_{JA[T]} : P[T] \twoheadrightarrow \frac{J\mathbb{I}}{J^2\mathbb{I}^2} \quad \text{be induced by } \omega_{\mathbb{I}}, \text{ and } \omega_{JA[T]}.$$

Now, there is a homotopy$\mathcal{H}(T) \in \widetilde{\mathcal{Q}}(P[T])$, such that $\eta(\mathcal{H}(T)) = (\mathbb{I}JA[T], \Omega)$. Specializing at $T = 0$ and $T = 1$, we have

$$\eta(\mathcal{H}(0)) = (K_0J, \omega_{K_0J}), \qquad \eta(\mathcal{H}(1)) = (K_1J, \omega_{K_1J}).$$

The proof is complete. $\square$

Now, we define addition on $\pi_0(\mathcal{LO}(P))$.

Definition 6.3. Let A be a regular ring, containing a field k, with $1/2 \in k$, with $\dim A = d$. Let P be a projective A-module, with $rank(P) = n \geq 2$, and $2n \geq d + 2$. Let $x, y \in \pi_0(\mathcal{LO}(P))$. By Moving Lemma 5.5, we can write $x = [(I, \omega_I)]$, $y = [(J, \omega_J)]$, for some $(I, \omega_I), (J, \omega_J) \in \mathcal{LO}(P)$, with $height(IJ) \geq n$, and $I + J = A$. Define

$$x + y := (I, \omega_I)\hat{+}(J, \omega_J) \in \pi_0(\mathcal{LO}(P)) \qquad \text{as defined in (6.1)}.$$

We establish that $x + y$ is well defined (6.4).

Proposition 6.4. *Under the setup and notations, as in (6.3), $x + y$ is well defined.*

Proof. Let $x = [(I_1, \omega_{I_1})], y = [(J_1, \omega_{J_1})] \in \pi_0(\mathcal{LO}(P))$, be another pair of choices, as in (6.3). That means, $height(I_1J_1) \geq n$, $I_1 + J_1 = A$. We prove

$$(I, \omega_I)\hat{+}(J, \omega_J) = (I_1, \omega_{I_1})\hat{+}(J_1, \omega_{J_1}).$$

By Moving Lemma 4.5, there is $(K, \omega_K) \in \mathcal{LO}(P)$ such that $x = [(K, \omega_K)]$, $height(K) \geq n$ and $K + I_1 \cap J_1 = A$.

We have $\mathbf{u}, \mathbf{u}_1 \in \widetilde{\mathcal{Q}}(P)$ such that $\eta(\mathbf{u}) = (I, \omega_I)$, and $\eta(\mathbf{u}_1) = (K, \omega_K)$. Since $x = [(I, \omega_I)] = [(K, \omega_K)] \in \pi_0(\mathcal{LO}(P))$, it follows $\mathbf{u}$, $\mathbf{u}_1$ are equivalent in $\widetilde{\mathcal{Q}}(P)$. By (3.2), there is a homotopy $H(T) \in \widetilde{\mathcal{Q}}(P[T])$ such that $H(0) = \mathbf{u}$, and $H(1) = \mathbf{u}_1$. It follows from Lemma 6.2,

$$(I, \omega_I)\hat{+}(J, \omega_J) = (K, \omega_K)\hat{+}(J, \omega_J) = (J, \omega_J)\hat{+}(K, \omega_K)$$

Likewise, the above is

$$= (J_1, \omega_{J_1})\hat{+}(K, \omega_K) = (K, \omega_K)\hat{+}(J_1, \omega_{J_1}) = (I_1, \omega_{I_1})\hat{+}(J_1, \omega_{J_1})$$

The proof is complete. $\square$

The final statement on the binary structure on $\pi_0(\mathcal{LO}(P)) \cong \pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$, is as follows.

Theorem 6.5. *Suppose A is a regular ring over a field k, with $1/2 \in k$ and $\dim A = d$. Let P be a projective A-module with $rank(P) = n$. Assume $2n \geq d+2$.* (Subsequently, we use the notations in $\pi_0(\mathcal{LO}(P))$ and $\pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$ interchangeably.) *Then, the addition operation on $\pi_0(\mathcal{LO}(P))$, defined in (6.3) has the following properties.*

1. *The addition in $\pi_0(\mathcal{LO}(P))$ is commutative and associative. Further, the image $\mathbf{e}_1 := [(A,0)] \in \pi_0(\mathcal{LO}(P))$, of $(0,0,1) \in \widetilde{\mathcal{Q}}(P)$, acts as the additive identity in $\pi_0(\mathcal{LO}(P))$. In other words, $\pi_0(\mathcal{LO}(P))$ has a structure of an abelian monoid.*
2. *Let $\mathbf{e}_0 := [(0,0)] \in \pi_0(\mathcal{LO}(P))$ be the image of $(0,0,0) \in \widetilde{\mathcal{Q}}(P)$. Then, $x+\widetilde{\Gamma}(x) = \mathbf{e}_0$, $\forall\, x \in \pi_0(\mathcal{LO}(P))$, where $\widetilde{\Gamma}$ is the involution map.*
3. *If $\mathbf{e}_0 = \mathbf{e}_1 \in \pi_0(\mathcal{LO}(P))$, then $\pi_0(\mathcal{LO}(P))$ is an abelian group, under this addition.* (Recall (4.6), if $2n \geq d+3$, and if A is essentially smooth over an infinite perfect field, then $\mathbf{e}_0 = \mathbf{e}_1$ if and only if $P \cong Q \oplus A$.)

Proof. Given $x, y, z \in \pi_0(\mathcal{LO}(P))$, by the Moving Lemma 5.5, we can write

$$x = [(K, \omega_K)],\ y = [(I, \omega_I)],\ z = [(J, \omega_J)] \quad \ni\ K+I = K+J = I+J = A$$

and $height(K) \geq n$, $height(I) \geq n$, $height(J) \geq n$. By definition (6.3),

$$(x+y)+z = ((K,\omega_K)\hat{+}(I,\omega_I))\hat{+}(J,\omega_J) = x+(y+z).$$

$$\text{and} \qquad x+y = (K,\omega_K)\hat{+}(I,\omega_I) = (I,\omega_I)\hat{+}(K,\omega_K) = y+x.$$

So, the associativity and commutativity hold. It is obvious that, for all $x \in \pi_0(\mathcal{LO}(P))$, we have $x + \mathbf{e}_1 = x$. So, $\mathbf{e}_1$ acts as the additive identity. This establishes (1).

Let $x = [(K, \omega_K)] \in \pi_0(\mathcal{LO}(P))$, with $height(K) \geq n$. There is $\mathbf{u} = (f, p, s) \in \widetilde{\mathcal{Q}}(P)$, with $\eta(\mathbf{u}) = (K, \omega_K)$. Write $\eta(\Gamma(\mathbf{u})) = (I_1, \omega_{I_1})$. We can assume $height(I_1) \geq n$. It follows.

$$x + \widetilde{\Gamma}(x) = \zeta_0(f, 0, 0) = \mathbf{e}_0. \qquad \text{This establishes (2).}$$

If $\mathbf{e}_0 = \mathbf{e}_1$, it follows from (2) that, $\pi_0(\mathcal{LO}(P))$ has a group structure. This establishes (3).

This completes the proof. $\square$

Remark 6.6. Use the notation as in (6.5). When $\mathbf{e}_0 \neq \mathbf{e_1}$, the results in (6.5) describe a situation similar to the construction of Witt group, from the monoid of isometry classes of quadratic spaces.

For $x, y \in \pi_0(\mathcal{LO}(P))$ define $x \sim y$ if $x + n\mathbf{e}_0 = y + m\mathbf{e}_0$, for integers $m, n \geq 0$. This is easily checked to be an equivalence relation. Let $\mathbb{E}(\pi_0(\mathcal{LO}(P)))$ be the set of all equivalence classes. Then, $\mathbb{E}\left(\pi_0\left(\mathcal{LO}(P)\right)\right)$ has a structure of an abelian group, induced by the additive structure on $\pi_0\left(\mathcal{LO}(P)\right)$. The natural map

$$\ell : \pi_0\left(\mathcal{LO}(P)\right) \twoheadrightarrow \mathbb{E}\left(\pi_0\left(\mathcal{LO}(P)\right)\right)$$

is a surjective homomorphism of monoids. The identity element of $\mathbb{E}\left(\pi_0\left(\mathcal{LO}(P)\right)\right)$ is $\ell(\mathbf{e}_0) = \ell(\mathbf{e}_1)$. For $x \in \pi_0\left(\mathcal{LO}(P)\right)$, the additive inverse of $\ell(x)$ is $\ell(\widetilde{\Gamma}(x))$.

Clearly, if $\mathbf{e}_0 = \mathbf{e_1}$, then $\mathbb{E}\left(\pi_0\left(\mathcal{LO}(P)\right)\right) = \pi_0\left(\mathcal{LO}(P)\right)$.

7. The Euler class groups

Suppose A is a noetherian commutative ring with $\dim A = d$ and P is a projective A-module, with $rank(P) = n$. In this section, in analogy to the definition of the Euler class groups $E^n(A)$ in [6,18], we define a group $E(P)$, which would also be called the Euler class group of P. Subsequently, we compare $E(P)$ with $\pi_0\left(\mathcal{LO}(P)\right)$. Also, refer to some superfluous aspect of the definitions in [6,18], pointed out in [15]. (*In the sequel, for a set S, the free abelian group generated by S will be denoted by $\mathbb{Z}(S)$*).

Definition 7.1. Suppose A is a noetherian commutative ring, with $\dim A = d$ and P is a projective A-module, with $rank(P) = n \geq 0$. Denote,

$$\begin{cases} \mathcal{LO}^n(P) = \{(I, \omega_I) \in \mathcal{LO}(P) : height(I) = n\}, \\ \mathcal{LO}^n_c(P) = \{(I, \omega_I) \in \mathcal{LO}(P) : V(I) \text{ is connected and } height(I) = n\}. \end{cases}$$

Let $(I, \omega_I) \in \mathcal{LO}^n(P)$ and $I = \cap_{i=1}^m I_i$ be a decomposition, where $V(I_i) \subseteq \operatorname{Spec}(A)$ are connected. The local orientation $(I, \omega_I) \in \mathcal{LO}^n(P)$ induce $(I_i, \omega_{I_i}) \in \mathcal{LO}^n_c(P)$, for $i = 1, \ldots, m$. Denote

$$(I, \omega_I)_{\mathbb{Z}} = \sum_{i=1}^m (I_i, \omega_{I_i}) \in \mathbb{Z}\left(\mathcal{LO}^n_c(P)\right).$$

A local orientation $(I, \omega_I) \in \mathcal{LO}^n(P)$ would be called **global**, if ω_I lifts to a surjective map $P \twoheadrightarrow I$. Let $\mathscr{R}(P)$ denote the subgroup of $\mathbb{Z}\left(\mathcal{LO}^n_c(P)\right)$, generated by the set $\{(I, \omega_I)_{\mathbb{Z}} : (I, \omega_I) \in \mathcal{LO}^n(P), \text{ and is global}\}$.

Define

$$E(P) = \frac{\mathbb{Z}\left(\mathcal{LO}^n_c(P)\right)}{\mathscr{R}(P)} \quad \text{to be called the Euler class group of } P.$$

Images of $(I, \omega_I) \in \mathcal{LO}^n(P)$ in $E(P)$ will be denoted by $\overline{(I, \omega_I)}$, which is same as the image of $(I, \omega_I)_{\mathbb{Z}}$.

Subsequently, we assume $\mathbf{e}_0 = \mathbf{e}_1 \in \pi_0\left(\mathcal{LO}(P)\right)$, and hence $\pi_0\left(\mathcal{LO}(P)\right)$ is a group. In this case, we define a homomorphism $\rho : E(P) \longrightarrow \pi_0\left(\mathcal{LO}(P)\right)$, as follows.

Definition 7.2. Suppose A is a regular ring over a field k, with $1/2 \in k$ and $\dim A = d$, and P is a projective A-module with $rank(P) = n$. Assume $2n \geq d+2$. (*Use the notations in* (6.6)). The restriction β, of the map ζ, to $\mathcal{L}O_c^n(P)$, gives the following commutative diagram:

$$\begin{tikzcd} \mathcal{L}O_c^n(P) \arrow[d, hook] \arrow[dr, "\beta"] & \\ \mathcal{L}O^n(P) \arrow[r, two heads, "\zeta"'] & \pi_0\left(\mathcal{L}O(P)\right) \end{tikzcd}$$

We assume $\mathbf{e}_0 = \mathbf{e}_1$. So, $\pi_0\left(\mathcal{L}O(P)\right)$ has the structure of an abelian group. The map β extends to a group homomorphism $\rho_0 : \mathbb{Z}\left(\mathcal{L}O_c^n(P)\right) \longrightarrow \pi_0\left(\mathcal{L}O(P)\right)$.

Now suppose $(I, \omega_I) \in \mathcal{L}O^n(P)$ is global. Let $f : P \twoheadrightarrow I$ be a lift of ω and $I = \cap_{i=1}^m I_i$ be a decomposition of I in to connected components. Then,

$$(I, \omega)_{\mathbb{Z}} = \sum_{i=1}^m (I_i, \omega_i) \in \mathbb{Z}\left(\mathcal{L}O_c^n(P)\right).$$

We have

$$\rho_0\left((I, \omega)_{\mathbb{Z}}\right) = \sum_{i=1}^m [(I_i, \omega_i)] = [\eta(f, 0, 0))] = \mathbf{e}_0 = \mathbf{e}_1$$

Therefore, ρ_0 factors through a group homomorphism $\rho : E(P) \twoheadrightarrow \pi_0\left(\mathcal{L}O(P)\right)$. In fact, ρ is surjective.

Proof. We only need to give a proof that ρ is surjective. For $x \in \pi_0\left(\mathcal{L}O(P)\right)$, by 5.5, $x = [(I, \omega_I)]$ for some $(I, \omega_I) \in \mathcal{L}O^n(P)$. Let $I = \cap_{i=1}^m I_i$ be a decomposition, with $V(I_i)$ connected and $\omega_i : P \twoheadrightarrow \frac{I_i}{I_i^2}$ be the surjective map induced by ω_I. Then,

$$\rho_0((I, \omega_I)_{\mathbb{Z}}) = \sum_{i=1}^m [(I_i, \omega_i)] = [(I, \omega_I)] \in \pi_0\left(\mathcal{L}O(P)\right).$$

So, ρ_0 is surjective and hence so is ρ. This completes the proof. $\square$

Theorem 7.3. *Suppose k is an infinite perfect field, with $1/2 \in k$ and A is an essentially smooth ring over k, with $\dim A = d$. Suppose P is a projective A-module with $\operatorname{rank}(P) = n$ and $2n \geq d+3$. Assume $P \cong Q \oplus A$. Then, $\pi_0\left(\mathcal{L}O(P)\right)$ is an abelian group and the homomorphism $\rho : E(P) \longrightarrow \pi_0\left(\mathcal{L}O(P)\right)$ is an isomorphism.*

Proof. We only need to prove that ρ is injective. Let $\rho(x) = 0$ for some $x \in E(P)$. We can write $x = \overline{(I, \omega_I)}$, for some $(I, \omega_I) \in \mathcal{L}O^n(P)$. By Lemma 4.2, we have $[(I, \omega_I)] = \mathbf{e}_1 = \mathbf{e}_0$.

It follows from Theorem 4.3 that ω_I lifts to a surjective map $f : P \twoheadrightarrow I$. Therefore, (I, ω_I) is global. Hence $x = \overline{(I, \omega_I)} = 0$. So, ρ is an isomorphism. This completes the proof. $\square$

Corollary 7.4. *Suppose k is an infinite perfect field, with $1/2 \in k$ and A is an essentially smooth ring over k, with $\dim A = d$. Suppose P is a projective A-module with rank$(P) = n$ and $2n \geq d+3$. Assume $P \cong Q \oplus A$. Suppose $(I, \omega_I) \in \mathcal{LO}^n(P)$ and $\overline{(I, \omega_I)} = 0 \in E(P)$. Then, ω_I lifts to a surjective homomorphism $P \twoheadrightarrow I$.*

Proof. It is immediate from Theorem 7.3. $\square$

In fact, a stronger version (7.5) of (7.4) follows, by the same arguments as in [6].

7.1. The vanishing of Euler cycles

We use the notations as in Definition 7.1. An element $x \in E(P)$ is, sometimes, referred to as an Euler cycle. In this subsection, we prove a less restrictive version of Corollary 7.4. We will follow the arguments in the proof of [6, Theorem 4.2], which mainly depends on the availability of Subtraction and Addition Principles. Accordingly, the following is a version of [6, Proposition 3.3].

Proposition 7.5. *Suppose A is a noetherian commutative ring, with $\dim A = d$ and P is a projective A-module, with rank$(P) = n$. Assume $2n \geq d+3$ and $P \cong Q \oplus A$.*

Let $J_0, J_1, J_2, J_3 \subseteq A$ be ideals, with height$(J_i) \geq n$ for $i = 0, 1, 2, 3$, $J_0 + J_1J_2 = A$ and $J_0J_1J_2 + J_3 = A$. Also, let

$$\alpha : P \twoheadrightarrow J_0 \cap J_1, \quad \beta : P \twoheadrightarrow J_0 \cap J_2 \quad \text{be surjective maps } \ni \alpha \otimes \frac{A}{J_0} = \beta \otimes \frac{A}{J_0}.$$

Further, assume that there is a surjective map

$$\gamma : P \twoheadrightarrow J_1 \cap J_3 \quad \ni \quad \gamma \otimes \frac{A}{J_1} = \alpha \otimes \frac{A}{J_1}.$$

Then, there is a surjective map

$$\delta : P \twoheadrightarrow J_2 \cap J_3 \quad \ni \quad \delta \otimes \frac{A}{J_3} = \gamma \otimes \frac{A}{J_3}, \quad \delta \otimes \frac{A}{J_2} = \beta \otimes \frac{A}{J_2}.$$

If $A = R[X]$ is a polynomial ring over a regular ring R, over an infinite field k, same is true, when $2n \geq \dim A + 2$.

Proof. Denote $\omega_0 = \alpha \otimes \frac{A}{J_0} = \beta \otimes \frac{A}{J_0}$, $\omega_1 = \alpha \otimes \frac{A}{J_1} = \gamma \otimes \frac{A}{J_1}$, $\omega_2 = \beta \otimes \frac{A}{J_2}$, $\omega_3 = \gamma \otimes \frac{A}{J_3}$. By Moving Lemma 4.5 there is $\mathbf{u} = (f, p, s) \in \widetilde{\mathcal{Q}}(P)$, such that

$$\eta(\mathbf{u}) = (J_0, \omega_0), \quad \eta(\Gamma(\mathbf{u})) = (J_4, \omega_4), \quad J_1J_2J_3 + J_4 = A, \quad height(J_4) \geq n.$$

As is intended, $f(P) = J_0 \cap J_4$, with $J_0 + J_4 = A$.

Denote $g := f : P \twoheadrightarrow J_0J_4$ be the a surjective map defined by f. Then, $g \otimes \frac{A}{J_0} = \omega_0$. By Addition Principle [3, Theorems 5.6, 5.7], applied to γ and g, there is a surjective map

$$\mu : P \twoheadrightarrow (J_1 \cap J_3) \cap (J_0 \cap J_4) \quad \ni \quad \mu \otimes \frac{A}{J_1 \cap J_3} = \gamma \otimes \frac{A}{J_1 \cap J_3} = \omega_1 \star \omega_3,$$

$$\text{and} \qquad \mu \otimes \frac{A}{J_0 \cap J_4} = g \otimes \frac{A}{J_0 \cap J_4} = \omega_0 \star \omega_4.$$

It follows, $\mu \otimes \frac{A}{J_0 \cap J_1} = \omega_0 \star \omega_1 = \alpha \otimes \frac{A}{J_0 \cap J_1}$. By Subtraction Principle [3, Theorems 3.7, 4.11], applied to μ and α, there is a surjective map $\nu : P \twoheadrightarrow J_3 \cap J_4$ such that $\nu \otimes \frac{A}{J_3 \cap J_4} = \mu \otimes \frac{A}{J_3 \cap J_4} = \omega_3 \star \omega_4$. By Addition Principle [3, Theorems 5.6, 5.7], applied to ν and β, there is a surjective map

$$\lambda : P \twoheadrightarrow (J_0 \cap J_2) \cap (J_3 \cap J_4) \quad \ni \quad \lambda \otimes \frac{A}{J_0 \cap J_2} = \beta \otimes \frac{A}{J_0 \cap J_2} = \omega_0 \star \omega_2$$

$$\text{and,} \qquad \lambda \otimes \frac{A}{J_3 \cap J_4} = \nu \otimes \frac{A}{J_3 \cap J_4} = \omega_3 \star \omega_4.$$

Now apply Subtraction Principle [3, Theorems 3.7, 4.11], to λ and g. There is a surjective map

$$\delta : P \twoheadrightarrow J_2 \cap J_3 \quad \ni \quad \delta \otimes \frac{A}{J_2 \cap J_3} = \lambda \otimes \frac{A}{J_2 \cap J_3} = \omega_2 \star \omega_3.$$

So, $\delta \otimes \frac{A}{J_2} = \omega_2$ and $\delta \otimes \frac{A}{J_3} = \omega_3$. The proof is complete. □

The following is the version of Corollary 7.1.

Theorem 7.6. *Suppose A is a commutative noetherian ring with $\dim A = d$ and P is a projective A-module, with rank$(P) = n$. Assume $2n \geq d + 3$ and $P \cong Q \oplus A$. Let $(J, \omega_J) \in \mathcal{LO}^n(P)$ and $\overline{(J, \omega_J)} = 0 \in E(P)$. Then, ω_J lifts to a surjective map $P \twoheadrightarrow J$.*

If $A = R[X]$ is a polynomial ring over a regular ring R, over an infinite field k, same is true when $2n \geq \dim A + 2$.

Proof. Suppose $(J, \omega_J) \in \mathcal{LO}^n(P)$ and $\overline{(J, \omega_J)} = 0 \in E(P)$. We have a set

$$\{(J_t, \omega_t) : 1 \leq t \leq r + s\}$$

such that

1. *height*$(J_t) = n$.
2. There are surjective maps $\alpha_t : P \twoheadrightarrow J_t$ such that α_t lifts ω_t.

3. And

$$(J,\omega)_{\mathbb{Z}} + \sum_{l=r+1}^{r+s} (J_t,\omega_t)_{\mathbb{Z}} = \sum_{t=1}^{r} (J_t,\omega_t)_{\mathbb{Z}} \quad \text{in} \quad \mathbb{Z}\left(\mathcal{L}O_c^n(P)\right) \tag{12}$$

holds in the free group $\mathbb{Z}\left(\mathcal{L}O_c^n(P)\right)$.

First assume that $J_1, J_2, \ldots, J_r$ are pairwise comaximal. In this case, $J, J_{r+1}, \ldots, J_{r+s}$ are pairwise comaximal. Write

$$J' = \cap_{l=r+1}^{r+s} J_t, \qquad J'' = \cap_{t=1}^{r} J_t. \qquad \text{Then} \quad J \cap J' = J''.$$

Further, by Addition Principle [3, Theorems 5.6, 5.7], there are surjective homomorphisms $\alpha' : P \twoheadrightarrow J'$ and $\alpha'' : P \twoheadrightarrow J''$ such that

$$(J',\omega')_{\mathbb{Z}} = \sum_{t=r+1}^{r+s} (J_t,\omega_t)_{\mathbb{Z}}, \quad (J'',\omega'')_{\mathbb{Z}} = \sum_{t=r+1}^{r+s} (J_t,\omega_t)_{\mathbb{Z}} \quad \text{in} \quad \mathbb{Z}\left(\mathcal{L}O_c^n(P)\right)$$

where $\omega' = \alpha' \otimes A/J'$ and $\omega'' = \alpha'' \otimes A/J''$. So, by Subtraction Principle [3, Theorems 3.7, 4.11], there is a surjective homomorphism $\alpha : P \twoheadrightarrow J$ such that $\alpha \otimes A/J = \alpha'' \otimes A/J = \omega$.

Now, we consider that $J_1, J_2, \ldots, J_r$ are not, necessarily, pairwise comaximal. Given an Equation, as in (12), we would associate an integer $n(\text{Eqn}-12) \geq 0$, as follows. Let S_i be the set of all connected components of J_i and $S = \cup_{i=1}^{r} S_i$. For $K \in S$, let $n(K)+1$ be the cardinality of the set $\{t : K + J_t \neq A\}$. Let $n(\text{Eqn}-12) = \sum_{K\in S} n(K)$. We have $n(\text{Eqn}-12) = 0$ if and only if $J_1, J_2, \ldots, J_r$ are comaximal.

Now, assume $n(\text{Eqn}-12) \geq 1$. Therefor, $n(K) \geq 1$ for some $K \in S$. We can assume $K \in S_1$ and $K + J_2 \neq A$. So, $\exists\ \tilde{K}$ a connected component of J_2 such that $K + \tilde{K} \neq A$.

First, assume $K \neq \tilde{K}$. Both $K, \tilde{K}$ cannot be connected component of J. (*components add up to A.*) Without loss of generality, assume K is not a connected component of J. Using Eqn-12, it follows that there is an integer l, with $r+1 \leq l \leq r+s$, such that (1) K is a connected component of J_l, (2) $\alpha_l \otimes A/K = \alpha_1 \otimes A/K$. Assume $l = r+1$ and denote $\omega_K := \alpha_l \otimes A/K = \alpha_1 \otimes A/K : P \twoheadrightarrow K/K^2$. We write $J_1 = K \cap K_1$ and $J_{r+1} = K \cap K_2$ where $K + K_1 = A = K + K_2$. By Moving Lemma 4.5, applied to $\omega_{K_1} := \alpha_1 \otimes \frac{A}{K_1}$, there is an ideal K_3 such that (3) $height(K_3) \geq n$, (4) K_3 is comaximal to $J, J_j, \forall 1 \leq j \leq r+s$, (5) there is a surjective map $\beta : P \twoheadrightarrow K_3 \cap K_1$ such that $\alpha_1 \otimes A/K_1 = \beta \otimes A/K_1$.

We have three surjective maps:

$$\alpha_1 : P \twoheadrightarrow K \cap K_1, \quad \alpha_{r+1} : P \twoheadrightarrow K \cap K_2 \quad \beta : P \twoheadrightarrow K_1 \cap K_3$$

By Proposition 7.5, there is a surjective map

$$\beta_{r+1} : P \twoheadrightarrow K_3 \cap K_2 \quad \ni \quad \alpha_{r+1} \otimes \frac{A}{K_2} = \beta_{r+1} \otimes \frac{A}{K_2}, \quad \beta \otimes \frac{A}{K_3} = \beta_{r+1} \otimes \frac{A}{K_3}.$$

So, we have

$$(J,\omega)_{\mathbb{Z}} + \left(\widetilde{J_{r+1}}, \widetilde{\beta_{r+1}}\right)_{\mathbb{Z}} + \sum_{l=r+2}^{r+s} (J_l,\omega_l)_{\mathbb{Z}} = \left(\widetilde{J_1}, \widetilde{\beta_1}\right)_{\mathbb{Z}} + \sum_{l=2}^{r}(J_l,\omega_l)_{\mathbb{Z}}, \tag{13}$$

where $\widetilde{J_{r+1}} = K_3 \cap K_2$ and $\widetilde{J_1} = K_3 \cap K_1$. ($K$ *is removed from both sides and* K_3 *is inserted.*) It is clear $n(\text{Eqn}-13) < n(\text{Eqn}-12)$. Therefore, by induction, the Equation-13 would reduce to an Equation (*), so that $n(*) = 0$.

Now assume $K = \tilde{K}$. Let $\omega_K = \alpha_1 \otimes A/K$. Therefore, $K = \tilde{K}$ is a component of J_2. We denote $\tilde{\omega}_K = \alpha_2 \otimes A/K$. Using Equation-12, it follows that either (K, ω_K) or $(K, \tilde{\omega}_K)$ is a summand of $\sum_{t=r+1}^{r+s}(J_t,\omega_t)_{\mathbb{Z}}$. Without loss of generality, we assume that (K,ω_K) is a summand of $(J_{r+1},\omega_{r+1})_{\mathbb{Z}}$ and complete the induction exactly in the same manner, as above. This completes the proof. □

7.2. Comparison with Chow groups

In this section, we exploit the work of N. Mohan Kumar and M. P. Murthy [25,24] to compare the Euler class group $E(P)$ with the Chow group of zero cycles $CH^d(A)$, when A is a smooth affine algebra over an algebraically closed field and $n = rank(P) = d = \dim A$.

Definition 7.7. Let A be a Cohen Macaulay ring, with $\dim A = d$. Let $K_0(A)$ denote the Grothendieck group of projective A-modules. Let $F_0K_0(A) =$

$$\left\{ \left[\frac{A}{I}\right] \in K_0(A) : I \text{ is a locally complete intersection ideal, with } height(I) = d \right\}.$$

It was established in [14, Theorem 1.1] that $F_0K_0(A)$ is a subgroup of $K_0(A)$.

Let Q be a projective A-module with $rank(Q) = d - 1$, and $P = Q \oplus A$. Then, there is a surjective homomorphism

$$\varphi : E(P) \longrightarrow F_0K_0(A) \quad \text{sending} \quad [(I,\omega)] \mapsto \left[\frac{A}{I}\right]$$

Proof. Since A is Cohen Macaulay, for $(I,\omega) \in \mathcal{LO}^d(P)$, I is a locally complete intersection ideal. Now, consider the map $\mathcal{LO}^d_c(P) \longrightarrow F_0K_0(A)$, sending $(I,\omega) \mapsto \left[\frac{A}{I}\right]$. This map extends to a homomorphism $\varphi_0 : \mathbb{Z}\left(\mathcal{LO}^d_c(P)\right) \longrightarrow F_0K_0(A)$. Now, if (I,ω) is a global orientation, then ω lifts to a surjective map $f : P \twoheadrightarrow I$. Since $P = Q \oplus A$, it follows

$$\varphi_0(I,\omega) = \left[\frac{A}{I}\right] = \sum_{r=0}^{d}(-1)^r \left[\wedge^r P\right] = 0.$$

Therefore, φ_0 factors through a map $\varphi : E(P) \longrightarrow F_0K_0(A)$. For, $\left[\frac{A}{I}\right] \in F_0K_0(A)$, there is an isomorphism $\frac{P}{IP} \xrightarrow{\sim} \frac{I}{I^2}$, which gives rise to a surjective map $\omega : P \twoheadrightarrow \frac{I}{I^2}$. Therefore, $\varphi([(I,\omega)]) = \left[\frac{A}{I}\right]$. So, φ is surjective.

Now, we assume that the base field k is algebraically closed, and use [19].

Corollary 7.8. *Suppose A is a reduced affine algebra over an algebraically closed field k, with $\dim A = d \geq 2$. Assume A is Cohen Macaulay and that $F_0K_0(A)$ has no $(d-1)!$ torsion. Let Q be a projective A-module with rank$(Q) = d-1$, and $P = Q \oplus A$. Then, the map $\varphi : E(P) \longrightarrow F_0K_0(A)$ in (7.7) is an isomorphism.*

Proof. By Swan's Bertini theorem [25, pp. 586], it follows that $F_0K_0(A)$ coincides with the usual subgroup $F^dK_0(A)$ (see [8,24,19]), which is generated by the cycles of $A/\mathfrak{m}$, where $\mathfrak{m}$ runs through the smooth maximal ideals of height d. We only need to prove that φ is injective. Suppose $\varphi(x) = 0$ for some $x \in E(P)$. By Moving Lemma, we can write $x = [(I,\omega)]$, for some $(I,\omega) \in \mathcal{LO}^d(A)$. Therefore $\varphi(x) = \left[\frac{A}{I}\right] = 0$. Since $P = Q \oplus A$, the top Chern class $C^d(P^*) = \sum_{r=0}^{d}(-1)^r[\wedge^r P] = 0 \in F^dK_0(A)$ (see [24, Definition 3.5]). Therefore, $C^d(P^*) = \left[\frac{A}{I}\right] \in F^dK_0(A)$. By [19, Theorem 2.1], it follows ω lifts to surjective map $P \twoheadrightarrow I$. Therefore, $x = [(I,\omega)] = 0$. This establishes that φ is an isomorphism. The proof is complete. □

The condition in (7.8) that $F_0K_0(A)$ has no $(d-1)!$ torsion is a minor condition, due to the results of Levine [10] and Srinivas [29] (see [24, Lemma 2.10, Theorem 2.14]). Summarizing all the above, with smoothness hypotheses, we have the following.

Corollary 7.9. *Suppose A is smooth affine algebra over an algebraically closed field k, with $1/2 \in k$ and $\dim A = d \geq 3$. Let Q be a projective A-module with rank$(Q) = d-1$, and $P = Q \oplus A$. Then, the maps*

$$\pi_0\left(\mathcal{LO}(P)\right) \xleftarrow{\sim} E(P) \xrightarrow{\sim} F_0K_0(A) \xleftarrow{\sim} CH^d(A)$$

are isomorphisms, where $CH^d(A)$ denotes the Chow group of codimension d cycles.

Proof. The last isomorphism follows from Riemann-Roch theorem, because $F_0K_0(A)$ is divisible and does not have $(d-1)!$ torsion [24, Lemma 2.10, Theorem 2.14]. The second isomorphism follows from (7.8), while the first isomorphism follows from (7.3). The proof is complete. □

7.3. Some closing remarks

Before we close the main body of this article, we have the following remarks.

Remark 7.10. For the following comments, assume A is an essentially smooth affine rings, over an infinite perfect field k, with $1/2 \in k$ and $\dim A = d \geq 3$, and $X = \mathrm{Spec}\,(A)$.

1. The structure theorem [4, Theorem 4.21] illustrates that these monoids $\pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$ can assume a wide range of values.
2. Assume P does not have a unimodular element (see [26]). Then, there is no ideal preserving and homotopy preserving map $\mathcal{LO}(A^n) \longrightarrow \mathcal{LO}(P)$.
   (a) However, in a subsequent article [16], we prove that when $rank(P) = d$, then $\pi_0(\mathcal{LO}(P)) \cong \pi_0\left(\mathcal{LO}(\Lambda^d P \oplus A^{d-1})\right)$. Since the latter one is a group, $\pi_0(\mathcal{LO}(P))$ is a group. In particular, for $\pi_0(\mathcal{LO}(P))$ to be a group, it is not necessary that P splits as $P \cong Q \oplus A$.
   (b) **Open Problem:** It remains open, whether $\pi_0(\mathcal{LO}(P))$ is always a group or not, whenever $2n \geq d+3$.
   (c) In [16], we establish a natural additive map $\pi_0(\mathcal{LO}(P)) \longrightarrow CH^n(A)$, where $CH^n(A)$ denotes the Chow group of codimension n cycles.
3. If $\pi_0(\mathcal{LO}(P))$ is a group and $\mathbf{e}_0 \neq \mathbf{e}_1$, then the natural map $\mathcal{LO}(P) \longrightarrow \pi_0(\mathcal{LO}(P))$ (see (7.2)), does not factor through a group homomorphism, from $E(P)$ to $\pi_0(\mathcal{LO}(P))$. This is because the global orientations, map to $\mathbf{e}_0$.
4. Note (see [8]), that the total Chern class $C(P) = 1+C^1(P)+\cdots+C^n(P)$ takes value in the total Chow group $CH(X) = \oplus_{i=1}^d CH^i(X)$, which is an invariant of X. However, the homotopy obstruction group $\pi_0(\mathcal{LO}(P))$, which houses the Nori class $\varepsilon(P)$, is an invariant of P. This does not come as a surprise, because when $rank(P) = n = d$, the Euler class of P (as in [7]) takes value in $E^n(A, \wedge^d P) = E(\wedge^d P \oplus A^{d-1})$, which is dependent on P.

Appendix A. The motivic interpretation

In this section, we attempt to give a motivic interpretation to the homotopy obstruction sets, in analogy to the case when $P = A^n$ [2,23]. Four descriptions for the same was given in section 2, assuming $1/2 \in A$. For our purpose, in this section, it would be best to work with $\widetilde{\mathcal{Q}}'(P)$ and $\pi_0\left(\widetilde{\mathcal{Q}}'(P)\right)$. We assume $1/2 \in A$ in this section. Recall, with

$$\mathscr{B}_{2n+1} = \frac{k[X_1,\ldots,X_n;Y_1,\ldots,Y_n,Z]}{(\sum_{i=1}^n X_iY_i + Z^2 - 1)}, \quad \text{and} \quad Q'_{2n} = \mathrm{Spec}\,(\mathscr{B}_{2n+1}) \tag{A.1}$$

$[\mathrm{Spec}\,(A)\,, Q'_{2n}]_{\underline{Sch}} \cong$

$$Q'_{2n}(A) \cong \left\{(f_1,\ldots,f_n;g_1,\ldots,g_n,z) \in A^{2n+1} : \sum_{i=1}^n f_ig_i + z^2 = 1\right\}.$$

Also recall, $\pi_0(Q'_{2n})(A) \cong [Q'_{2n}, \mathrm{Spec}\,(A)]_{\mathbb{A}^1}$ where the right hand side denotes the set of all morphisms in the $\mathbb{A}^1$-homotopy category [23, Chapter 8] (also see [1, Theorem 1.1.1]). A similar interpretation for $\widetilde{\mathcal{Q}}'(P)$ and $\pi_0\left(\widetilde{\mathcal{Q}}'(P)\right)$ would be desirable.

We follow Swan [30, §1, 2]. Suppose Q is a projective A-module. Let $S(Q^*) = \bigoplus_{i\geq 0} S_i(Q^*)$ denote the symmetric algebra of Q^*. Let $Quad(Q) = \{\varphi \in Hom(Q, Q^*) :$

$\varphi^* = \varphi\}$ denote the A-module of all the quadratic forms on Q. Given $\varphi \in Quad(Q)$, let $B(\varphi) \in Hom(Q \otimes Q, A) \cong Q^* \otimes Q^*$ be the corresponding bilinear map. In fact, this association $\varphi \mapsto B(\varphi)$ induces a bijection $Quad(Q) \xrightarrow{\sim} S_2(Q^*)$ (see [30, § 2]).

Since A is commutative, all maps $f : Q^* \longrightarrow A$ extends to a map $S(Q^*) \longrightarrow A$. So, we have the commutative diagram of bijections:

$$\begin{tikzcd} Q \arrow[r, "ev", "\sim"'] \arrow[dr, "\lambda"'] & Hom(Q^*, A) \arrow[d, "\wr"] \\ & Hom(S(Q^*), A) \end{tikzcd}$$

$$\text{For } x \in Q, \quad f, g \in Q^* \quad \langle \lambda(x), f\rangle = f(x) \quad \langle \lambda(x), fg\rangle = f(x)g(x)$$

For a bilinear map $\beta \in Hom(Q \otimes Q, A) = Q^* \otimes Q^*$, we can write $\beta = \sum f_i \otimes g_i$ for some $f_i, g_i \in Q^*$ and

$$\langle \lambda(x), \beta\rangle = \sum f_i(x) g_i(x) = \beta(x, x).$$

Fix a quadratic form $\varphi : Q \longrightarrow Q^*$ and $B(\varphi) : Q \otimes Q \longrightarrow A$ be the corresponding bilinear map. More precisely, $B(\varphi)(x, y) = \varphi(x)(y)$. As usual, define $q : Q \longrightarrow A$ by $q(x) = B(x, x)$. Then,

$$\text{for } x \in Q \quad \langle \lambda(x), B(\varphi)\rangle = B(\varphi)(x, x) = q(x).$$

We introduce some notations.

Notations A.1. Suppose A is a commutative noetherian ring, with $1/2 \in A$, and $X = \mathrm{Spec}\,(A)$. For a quadratic space (Q, φ) over A, denote

$$\mathbb{S}(\varphi) = \{x \in Q : q(x) = 1\}, \ \mathscr{B}(\varphi) = \frac{S(Q^*)}{(B(\varphi) - 1)} \text{ and } \mathcal{X}(\varphi) = \mathrm{Spec}\,(\mathscr{B}(\varphi))\,.$$

Proposition A.2. *With notations as in (A.1), the following maps*

$$[X, \mathcal{X}(\varphi)]_{\underline{Sch}_A} \xrightarrow{\sim} Hom\,(\mathscr{B}(\varphi), A) \xleftarrow{\sim} \mathbb{S}(\varphi) \qquad \text{are bijections,}$$

where $[-,-]_{\underline{Sch}_A}$ *denotes the set of morphisms in* $\underline{Sch}_A$.

Proof. Follows from above discussions. $\square$

Remark A.3. Use the same notations, as in (A.1). Consider the presheaf

$$[-, \mathcal{X}(\varphi)]_{\underline{Sch}} : \underline{Sch}_A \longrightarrow \underline{Sets} \quad \text{sending} \quad Y \mapsto [Y, \mathcal{X}(\varphi)]_{\underline{Sch}_A}$$

In fact, for affine schemes $Y = \mathrm{Spec}\,(B) \in \underline{Sch}_A$, the following maps

$$[Y, \mathcal{X}(\varphi)]_{\underline{Sch}_A} \xrightarrow{\sim} Hom\,(\mathscr{B}(\varphi), B) \xleftarrow{\sim} \mathbb{S}\,(\varphi \otimes B) \quad \text{are bijections.}$$

One can make a similar statement for any scheme $Y \in \underline{Sch}_A$. Let $f : Y \longrightarrow X$ be the structure map, and f^* would denote the pullback. Redefine

$$\begin{cases} \mathbb{S}(f^*q) = \{x \in \Gamma(Y, f^*Q) : f^*q(x) = B(f^*\varphi)(x,x) = 1\} \\ \mathscr{B}(f^*\varphi) = \frac{S(f^*Q^*)}{(B(f^*\varphi)-1)\mathcal{O}_Y} \\ \mathcal{X}(f^*\varphi) = \mathrm{Spec}\,(\mathscr{B}(f^*\varphi))\,. \end{cases}$$

Then, the following maps

$$[Y, \mathcal{X}(\varphi)]_{\underline{Sch}_A} \xrightarrow{\sim} Hom\,(\mathscr{B}(\varphi), \Gamma(Y, \mathcal{O}_Y)) \xleftarrow{\sim} \mathbb{S}\,(f^*\varphi)$$

are bijections. (see [9, II, Ex. 2.4]).

Corresponding to the notations (A.1), we introduce the following notations.

Notations A.4. Let $(\mathbb{Q}(P), q) = \mathbb{H}(P) \perp A$ be as in (2.1). Denote the underlying projective module of $(\mathbb{Q}(P), q)$ by the same notation $\mathbb{Q}(P) := P^* \oplus P \oplus A$. Let $B : \mathbb{Q}(P) \times \mathbb{Q}(P) \longrightarrow A$ be the corresponding bilinear form. Define

$$\mathscr{B}(P) = \frac{S(P \oplus P^* \oplus A)}{(B-1)}, \quad \text{and denote} \quad Q'_P := \mathrm{Spec}\,(\mathscr{B}(P))\,.$$

Corollary A.5. *Use the notations, as in (A.4). Then, for $Y \in \underline{Sch}_A$ and the structure map $f : Y \longrightarrow X$, the following maps*

$$\begin{cases} \widetilde{\mathcal{Q}}'(P) = \mathbb{S}(q) \xrightarrow{\sim} [X, Q'_P]_{\underline{Sch}_A} \\ \widetilde{\mathcal{Q}}'(f^*P) = \mathbb{S}(f^*q) \xrightarrow{\sim} [Y, Q'_P]_{\underline{Sch}_A} \end{cases} \quad \textit{are bijections.}$$

Consequently, the association

$$Y \mapsto \widetilde{\mathcal{Q}}'(f^*P) \quad \text{defines a presheaf} \quad \underline{Sch}_A \longrightarrow \underline{Sets}.$$

Therefore, one can define

$$\pi_0(\widetilde{\mathcal{Q}}(P)) : \underline{Sch}_A \longrightarrow \underline{Sets} \quad \text{as a presheaf.}$$

Proof. Follows from (A.2). This completes the proof. $\square$

In analogy to the free case $P = A^n$, we raise the following question.

Question A.6. Suppose k is an infinite perfect field, with $1/2 \in k$ and A is an essentially smooth ring over k, with $\dim A = d$. Write $X = \mathrm{Spec}\,(A)$. Suppose P is a projective A-module with $rank(P) = n$. The question remains, whether a motivic interpretation can be given to the pre sheaf $\pi_0(\widetilde{\mathcal{Q}}(P))$. In particular, whether

$$\pi_0(\widetilde{\mathcal{Q}}(P))(A) \cong [X, Q_P]_{\mathcal{H}(A)}?$$

Here $\mathcal{H}(A)$ denotes the $\mathbb{A}^1$-homotopy category of smooth schemes over $X = \mathrm{Spec}\,(A)$, and $[X, Q_P]_{\mathcal{H}(A)}$ denotes the set of all maps $X \longrightarrow Q_P$ in $\mathcal{H}(A)$.

If P is free, then the question has an affirmative answer, when A is an infinite perfect field (see [23, Remark 8.10]).

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