Projective Modules and Complete Intersections

Satya Mandal
University of Kansas
Lawrence, Kansas 66045

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Preface

In these notes we give an account of the developments in research in *Projective Modules and Complete Intersections* since the proof of Serre's Conjecture due to Quillen and Suslin and the subsequent publication of T-Y Lam's book, *Serre's Conjecture*.

I expect these notes to be accessible to a wide range of readers, with or without a serious background in commutative algebra. These notes evolved out of class notes for a course on this topic that I taught several years ago at the University of Kansas to a group of students who had no prior serious exposure to commutative algebra. My students enjoyed the course. I would hope that the readers will find these notes enjoyable as well.

I need to thank a long list of people who helped me, directly or indirectly, to accomplish this goal. I thank Professor Amit Roy of the Tata Institute of Fundamental Research, Bombay, for the excellent training he gave me in my early career. I would also like to thank my friends Daniel Katz and Jeffrey Lang for their encouragement and for many helpful discussions. Thanks are also due to D. S. Nagaraj of the Institute of Mathematical Sciences, Madras, and to Ravi Rao and Raja Sridharan of the Tata Institute of Fundamental Research, Bombay, for many helpful discussions. I would like to thank my brothers, sister and our parents for their support. Thanks to my wife, Elsit, for her encouragement and patience and also to our little daughter Nila for being here with us on time for the occasion.

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Contents

Chapter 0

Introduction

In these notes on *Projective Modules and Complete Intersections* we present an account of the developments in research on this subject since the proof of the Conjecture of Serre due to Quillen and Suslin.

After two preliminary chapters, we start with the proof of Serre's Conjecture and some associated results of Quillen and Suslin in Chapter 3.

Chapter 4 includes the Basic Element Theory of Eisenbud and Evans and the proofs of the Eisenbud-Evans Conjectures. Our treatment of the Basic element theory incorporates the idea of *generalized dimension functions* due to Plumstead.

In Chapter 5, we discuss the theory of matrices that we need in the later chapters. We tried to avoid the theory of elementary matrices in these notes. Instead, we talk about the *Isotopy Subgroup* of the General Linear Group in this chapter.

The theory of Complete Intersections is discussed in Chapter 6. Among the theorems in this chapter are

1. the theorem of Eisenbud and Evans on the number of set theoretic generators of ideals in polynomial rings,

2. the theorem of Ferrand-Szpiro,

3. the theorem of Boratynski on the number of set theoretic generators of ideals in polynomial rings over fields,

4. the theorem of Ferrand-Szpiro-Mohan Kumar on the local complete intersection curves in affine spaces,

5. the theorem of Cowsik-Nori on curves in affine spaces.

To prove the theorem of Cowsik-Nori, we also give a complete proof of the fact that a curve in an affine space over a perfect field is integral and birational to its projection to an affine two subspace, after a change of variables.

In Chapter 7, we discuss the theory of Projective modules over polynomial rings in several variables over noetherian commutative rings. The techniques used in this chapter are almost entirely due to Lindel. Among the theorems in this chapter are

1. Lindel's Theorem on Bass-Quillen Conjecture,
2. the theorem of Bhatwadekar-Roy on the existence of Unimodular elements in projective modules,
3. Lindel's Theorem on the transitivity of the action of the group of transvections on the set of unimodular elements of a projective module.

A large portion of these notes evolved out of class notes for a course on this topic that I taught some years ago at the University of Kansas. The students in this class did not have any previous serious exposure to commutative algebra. My approach while conducting this course was to

a) state and explain results and proofs that have a potential to excite the students;

b) skip those proofs that may become technical;

c) state, explain and use results from commutative algebra as and when needed.

With this approach, I was able to cover the materials in Chapter 1-6. Among the theorems that I stated in this course *without proof* were the theorems of Eisenbud-Evans (Theorem 4.1.1), Plumstead (Theorem 4.3.1 and 4.3.2) and Sathaye-Mohan Kumar(Theorem 4.3.3). On the other hand, I proved the theorem of Ferrand-Szpiro (Theorem 6.1.3). I finished the course with the proof of the theorem of Ferrand-Szpiro-Mohan Kumar(Theorem 6.2.5) that

a locally complete intersection ideal of height $n-1$ in a polynomial ring $k[X_1, \ldots, X_n]$ over a field k is set theoretically generated by $n-1$ polynomials,
and with the statement of the Cowsik-Nori theorem(Theorem 6.3.1) that

any ideal of pure height $n-1$ in a polynomial ring $k[X_1, \ldots, X_n]$ over a field k of positive characteristic is set theoretically generated by $n-1$ elements.

Chapter 1

Preliminaries

In this chapter we shall put together some notations, some terminologies and preliminaries from commutative algebra that we will be using throughout these notes.

1.1 Localization

Suppose R is a commutative ring. For a subset S of R, we say that S is a *multiplicative* subset of R if 1 is in S and for s and t in S, st is also in S.

For a multiplicative subset S of R and an R-module M, we have

$$M_S = \{m/t \ : \ m \in M \text{ and } t \in S\}.$$

For m, n in M and t, s in S, we have $m/t = n/s$ if $u(sm - tn) = 0$ for some u in S. M_S is called the *localization* of M at the multiplicative set S. The following are some facts about localization.

Fact 1.1.1 Suppose R is a commutative ring and S is a multiplicative subset of R. Let M be an R-module. Then the following are easy to see.

(a) R_S is a ring and the map $R \to R_S$ that sends r to $r/1$ is a ring homomorphism.

(b) M_S becomes an R_S-module under the natural operations

$$m/s + n/t = (tm + sn)/st \quad \text{and} \quad (a/u)(m/t) = am/ut$$

for m, n in M , a in R and s, t, u in S.

(c) The natural map $i : M \longrightarrow M_S$ that sends m to $m/1$ is an R-linear map.

(d) The natural map $i : M \longrightarrow M_S$ has the following universal property :
Given an R_S-module N and an R-linear map $f : M \longrightarrow N$ there is a

unique R_S-linear map $F : M_S \longrightarrow N$ such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{i} & M_S \\ & f \searrow & \downarrow F \\ & & N \end{array}$$

commutes.

(e) For an element f in R, we write R_f for R_S and write M_f for M_S where $S = \{1, f, f^2, \ldots\}$. For a prime ideal $\wp$ of R, $R_\wp$ and $M_\wp$, respectively, denote R_S and M_S where $S = R - \wp$.

(f) We have $\quad M_S \approx M \otimes_R R_S \quad$ as R_S - modules.

(g) Let M, N be two R-modules and let $f : M \to N$ be an R-linear map. It follows from the universal property that there is an R_S-linear map $F : M_S \to N_S$ such that the following diagram

$$\begin{array}{ccc} M & \longrightarrow & N \\ \downarrow & & \downarrow \\ M_S & \longrightarrow & N_S \end{array}$$

commutes.

Definition 1.1.1 A homomorphism $i : R \to A$ of commutative rings is called *flat* if for all short exact sequences

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

of R-modules and R-linear maps, the induced sequence

$$0 \longrightarrow M' \otimes_R A \longrightarrow M \otimes_R A \longrightarrow M'' \otimes_R A \longrightarrow 0$$

is exact.

Proposition 1.1.1 For a commutative ring R and a multiplicative subset S of R, the natural map $i : R \longrightarrow R_S$ is flat.

Definition 1.1.2 Suppose R is a commutative ring and M is an R-module. We say that M is *finitely presented* if there is an exact sequence

$$0 \longrightarrow K \longrightarrow R^n \longrightarrow M \longrightarrow 0$$

of R-modules, for some nonnegative integer n and a finitely generated R-module K. Equivalently, M is finitely presented if there is an exact sequence

$$R^m \longrightarrow R^n \longrightarrow M \longrightarrow 0$$

of R-modules, where m and n are nonnegative integers.

Remark 1.1.1 We leave it as an exercise that *for a noetherian commutative ring R, an R-module M is finitely generated if and only if it is finitely presented.*

Notations 1.1.1 Let R be a commutative ring.

1. We denote the set of all prime ideals of R by $Spec(R)$.
2. The set of all maximal ideals of R will be denoted by $\max(R)$.
3. For an ideal I of R, $V(I)$ will denote the set of all prime ideals of R that contain I.
4. For an element f of R, $D(f)$ will denote the set of all the prime ideals of R that do not contain f.

Exercise 1.1.1 Suppose R is a ring and $S \subseteq T$ be two multiplicative subsets of R. Then the following diagram

$$\begin{array}{ccc} R & \longrightarrow & R_S \\ & \searrow & \downarrow \\ & & R_T \end{array}$$

of the natural maps commutes. Further if $\tilde{T}$ is the image of T in R_S then $(R_S)_{\tilde{T}}$ is naturally isomorphic to R_T. We say that *R_T is a further localization of R_S* to explain this phenomenon.

1.2 The Local-Global Principle

Lemma 1.2.1 Suppose R is a commutative ring and M is an R-module. Then the following are equivalent :

1. $M = 0$,
2. $M_\wp = 0$ for all $\wp$ in $Spec(R)$,
3. $M_m = 0$ for all maximal ideals m of R.

Proof. See the book of Kunz ([K1]), Chapter III.

Proposition 1.2.1 Suppose R is a commutative ring and M is an R-module. For two submodules M' and M'' of M, we have $M' = M''$ if and only if $M'_m = M''_m$ for all maximal ideals m of R.

Proof. See the book of Kunz ([K1]), Chapter IV.

Example 1.2.1 Suppose R is a Dedekind domain which is not a Principal ideal domain (PID). If I is an ideal of R that is not principal, then I is not isomorphic to R but $I_m \approx R_m$ for all maximal ideals m of R.

Corollary 1.2.1 Suppose I and J are two ideals of R. Then $I = J$ if and only if $I_m = J_m$ for all maximal ideals m of R containing $I \cap J$.

Proof. The corollary follows from Proposition 1.2.1 .

Corollary 1.2.2 Suppose that R is a commutative ring and M is an R-module. Let $\{m_i\}_{i\in I} = S$ be a subset of M. Then the set $\{m_i\}_{i\in I}$ generates M if and only if the image $\{m_i/1\}_{i\in I}$ of S in M_m generates M_m for all maximal ideals m of R.

Proof. Let N be the submodule of M generated by S. Now the proof is an immediate consequence of Proposition 1.2.1.

Lemma 1.2.2 Suppose R is a commutative ring and let $f_1, f_2, \ldots, f_r$ be elements of R. Then $D(f_1) \cup D(f_2) \cup \ldots \cup D(f_r) = \mathrm{S}pec(R)$ if and only if the ideal $Rf_1 + Rf_2 + \cdots + Rf_r = R$.

We leave the proof of this Lemma as an exercise.

Corollary 1.2.3 Suppose R is a commutative ring and assume that $Spec(R) = D(f) \cup D(g)$ for some f and g in R. Let M be an R-module.

(a) Suppose that M_f and M_g are finitely generated. Then M is finitely generated.

(b) Let $m_1, \ldots, m_r$ be elements in M such that their respective images generate both M_f and M_g. Then $m_1, \ldots, m_r$ generate M.

Proof. The corollary follows from Corollary 1.2.2.

Corollary 1.2.4 Suppose R is a commutative ring. Then a sequence

$$M' \xrightarrow{f} M \xrightarrow{g} M'$$

of R-modules and R-linear maps is exact if and only if the induced sequence

$$M'_m \longrightarrow M_m \longrightarrow M'_m$$

is exact for all maximal ideals m of R.

Proof. See the book of Kunz ([K1]), Chapter IV.

Corollary 1.2.5 Let $f : M \longrightarrow N$ be an R-linear map.

1. Then f is injective if and only if f_m is injective for all m in $\max R$.
2. Similarly, f is surjective if and only if f_m is surjective for all m in $\max R$.

Proof. It is immediate from Corollary 1.2.4.

Example 1.2.2 Suppose D is a Dedekind domain that is not a PID. Let I be an ideal that is not principal. Then I_m is one generated for all m in $\max R$. This is probably the simplest example to illustrate that the local number of generators and the global number of generators are not always the same. *Deriving the global number of generators from the local number of generators is one of our main interests in these notes.*

Definition 1.2.1 (Zariski Topology) For a noetherian commutative ring R, the Zariski Topology on $Spec(R)$ is defined by declaring $D(f)$ as the basic open sets, for f in R. Equivalently, the closed sets in $Spec(R)$ are $V(I)$, where I is an ideal in R.

Exercise 1.2.1 Let R be a commutative noetherian ring. Then $Spec(R)$ is connected if and only if R has no idempotent element other than 0 and 1.

1.3 Homomorphisms of Modules and Flatness

The main theorem in this section is about the commutativity of the tensor product for a flat extension with the module of homomorphisms of modules. Of particular interest are the cases of polynomial extensions and localizations.

Notations 1.3.1 Suppose R is a commutative ring and M and N are two R-modules. We shall denote the set of all R-linear maps from M to N by $Hom_R(M, N)$ or simply by $Hom(M, N)$. Note that $Hom(M, N)$ is also an R-module in a natural way.

Definition 1.3.1 Suppose R, M, N are as in Notations 1.3.1 and let S be a multiplicative subset of R. We define a natural map

$$\varphi \; : \; S^{-1}Hom_R(M,N) \longrightarrow Hom_{R_S}(M_S, N_S)$$

by defining $\varphi(f/t) : M_S \longrightarrow N_S$ as

$$\varphi(f/t)(m/s) \; = \; f(m)/st$$

for f in $Hom(M,N)$, m in M and s, t in S.

Exercise 1.3.1 Let R, M, N, S be as in Definition 1.3.1 and let M be finitely generated. Prove that the natural map in Definition 1.3.1 is injective. Further, if M is finitely presented then the natural map is an isomorphism.

In fact, Definition 1.3.1 and Exercise 1.3.1 are, respectively, the particular cases of Definition 1.3.2 and Theorem 1.3.1 that follow.

Definition 1.3.2 Let $i : R \longrightarrow A$ be a flat homomorphism of rings. Let M and N be two R-modules. Define the natural map

$$\varphi \; : \; Hom_R(M,N) \otimes A \longrightarrow Hom_A(M \otimes A, N \otimes A)$$

by setting $\varphi(f \otimes t)(m \otimes s) = f(m) \otimes st$ for f in $Hom(M,N)$, m in M, and s, t in A.

Theorem 1.3.1 *Let $i : R \longrightarrow A$ be a flat homomorphism of commutative rings and let M, N be two R-modules with M being finitely presented. Then the natural map*

$$\varphi \; : \; Hom_R(M,N) \otimes A \longrightarrow Hom_A(M \otimes A, N \otimes A)$$

is an isomorphism.

Proof. First assume that $M \approx R^n$ is free. In that case we have the commutative diagram :

$$\begin{array}{lcl} Hom(M,N) \otimes A & \xrightarrow{\varphi} & Hom(M \otimes A, N \otimes A) \\ \downarrow \wr & & \downarrow \wr \\ N^n \otimes A & \xrightarrow{\varphi'} & (N \otimes A)^n \, . \end{array}$$

Here N^n denotes the direct sum of n copies of N and φ' is the natural identification. Since the vertical maps are isomorphisms, φ is also an isomorphism.

In the general case, since M is finitely presented, there is an exact sequence

$$R^m \longrightarrow R^n \longrightarrow M \longrightarrow 0$$

of R-linear maps. This sequence will induce the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \to & Hom(M,N)\otimes A \to & Hom(R^n,N)\otimes A \to & Hom(R^m,N)\otimes A \\ & & \downarrow & \downarrow & \downarrow \\ 0 & \to & Hom(M',N') \to & Hom(A^n,N') \to & Hom(A^m,N') \end{array}$$

where $M' = M\otimes A$ and $N' = N\otimes A$.

Since the operation of taking $Hom(\ ,N)$ is left exact and $R \longrightarrow A$ is flat the first row of this diagram is exact. For similar reasons, the last row is also exact. By the case when M is free, the last two vertical maps are isomorphisms. Hence the first vertical map is also an isomorphism. This completes the proof of Theorem 1.3.1 .

Remark 1.3.1 All the rings we consider now onwards will be assumed to be noetherian and commutative. That is why any finitely generated module will also be a finitely presented module.

Remark 1.3.2 Let $A = R[X]$ be the polynomial ring over a noetherian commutative ring R. Let M and N be two finitely generated R-modules. It follows that

$$Hom(M\otimes R[X], N\otimes R[X]) \approx Hom(M,N)\otimes R[X].$$

Remark 1.3.3 Let R be a noetherian commutative ring and S be a multiplicative subset of R. For finitely generated R-modules M and N, we have

$$Hom_{R_S}(M_S,N_S) \approx Hom_R(M,N)\otimes_R R_S \approx \ (Hom_R(M,N))_S\,.$$

1.4 Definition of Projective Modules

Before we define projective modules, we want to discuss the splitting properties of exact sequences.

Definition 1.4.1 Suppose R is a commutative ring and let

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

be an exact sequence of R-modules and R-linear maps. We say that the sequence *splits* if there is an R-linear map $\zeta : M'' \longrightarrow M$ such that $go\zeta = Id_{M''}$.

Lemma 1.4.1 Suppose that R is a commutative ring and let

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

be an exact sequence of R-modules and R-linear maps. Then the following conditions are equivalent :

1. The sequence splits.
2. $M = M' \oplus N$ for some submodule N of M such that the restriction $g|_N : N \longrightarrow M''$ is an isomorphism.
3. There is a map $t : M \longrightarrow M'$ such that $tof = Id_{M'}$.
4. The natural map

$$\varphi \ : \ Hom(M'', M) \longrightarrow Hom(M'', M''),$$

that sends a map $h : M'' \longrightarrow M$ to goh, is surjective.

Proof. It is easy to see that (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3).

To see that (1) implies (4) let $\zeta : M'' \longrightarrow M$ be a split i.e. $go\zeta = Id_{M''}$. We have the following

$$\begin{array}{ccc} M & \xrightarrow{g} & M'' \\ \zeta \uparrow & \nearrow Id & \\ M'' & & \end{array}$$

commutative diagram. Given a map $h : M'' \longrightarrow M''$, we have $\varphi(\zeta oh) = go\zeta oh = h$. Hence φ is surjective. This establishes (4).

To see (4) implies (1), let $\varphi(\zeta) = Id$. Then ζ is a split of g. This completes the proof of the Lemma.

Corollary 1.4.1 Suppose R is a commutative noetherian ring and let

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be an exact sequence of finitely generated R-modules and R-linear maps. Then the sequence splits if and only if the induced exact sequences

$$0 \longrightarrow M'_m \longrightarrow M_m \longrightarrow M''_m \longrightarrow 0$$

split for all m in $\max(R)$.

Proof. It is immediate from Lemma 1.4.1 and Corollary 1.2.5.

Now we are ready to define projective modules.

Definition 1.4.2 Suppose R is a commutative ring and let P be an R-module. We say that P is a *projective* R-module if one of the following equivalent conditions hold :

1. Given a surjective R-linear map $f : M \longrightarrow N$ and an R-linear map $g : P \longrightarrow N$ there is an R-linear map $h : P \longrightarrow M$ such that the diagram

$$\begin{array}{ccccc} & & P & & \\ & h \swarrow & \downarrow g & & \\ M & \underset{f}{\longrightarrow} & N & \longrightarrow & 0 \end{array}$$

commutes.

2. Every exact sequence $0 \longrightarrow N \longrightarrow M \longrightarrow P \longrightarrow 0$ of R-modules and R-linear maps splits.

3. There is an R-module Q such that $P \oplus Q$ is free.

4. The functor $M \longrightarrow Hom_R(P, M)$ from the category of R-modules to itself is exact.

Proof.

(1) $\Rightarrow$ (2) follows by looking at the diagram

$$\begin{array}{cccc} & & P & \\ & h \swarrow & \downarrow Id & \\ M & \longrightarrow & P \longrightarrow & 0\,. \end{array}$$

(2) $\Rightarrow$ (3) We can find a surjective map $f : F \longrightarrow P$, where F is free. Take Q = kernel (f). Then $P \oplus Q \approx F$ is free.

(3) $\Rightarrow$ (4) Let $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ be an exact sequence of R-modules and R-linear maps. It is a general fact that

$$0 \longrightarrow Hom(P, M') \longrightarrow Hom(P, M) \longrightarrow Hom(P, M'')$$

is exact. So, we need only to show that the map

$$Hom(P, M) \longrightarrow Hom(P, M'')$$

is surjective. Let $F = P \oplus Q$ be free and let $h : P \longrightarrow M''$ be any R-linear map. If $h_0 : F \longrightarrow M''$ is the map such that $h_0|_P = h$ and $h_0|_Q = 0$ then there is an R-linear map $h_0' : F \longrightarrow M$ such that $goh_0' = h_0$. Let $h' = h_0|_P$, then $goh' = h$.

(4) $\Rightarrow$ (1) is obvious.

Theorem 1.4.1 *Let R be a commutative noetherian ring and let P be a finitely generated R-module. Then P is projective if and only if $P_\wp$ is a free $R_\wp$-module for all $\wp$ in $Spec(R)$ if and only if P_m is free for all m in* $\max(R)$.

Proof. Immediate from Corollary 1.4.1 and condition (2) of Definition 1.4.2.

Chapter 2

Patching Modules and Other Preliminaries

Projective Modules are one of our main interests in these notes. Fiber product is one of the basic tools that will be used later in these notes to construct Projective modules. In section 2.1, we shall discuss the techniques of Fiber Product. We shall use Fiber Product to develop the technique of Patching homomorphisms of Modules and also to construct Projective modules by patching techniques. In section 2.5, we discuss the correspondence between vector bundles and Projective modules. This correspondence is also used to construct a classical example of a nonfree projective module.

2.1 Fiber Product of Modules

One of the main references for this section is the book ([Mi]) of Milnor.

Definition 2.1.1 Let R be a commutative ring and let $f_1 : M_1 \longrightarrow N$ and $f_2 : M_2 \longrightarrow N$ be homomorphisms of R-modules. The *fiber product* of M_1 and M_2 over N is a triple (M, g_1, g_2) where M is an R-module, $g_1 : M \longrightarrow M_1$ and $g_2 : M \longrightarrow M_2$ are R-linear maps such that $f_1 o g_1 = f_2 o g_2$ and the triple is universal in the sense that given any other triple (M', g_1', g_2') of this kind with $f_1 o g_1' = f_2 o g_2'$ there is a unique homomorphism $h : M' \longrightarrow M$ such that $g_1 o h = g_1'$ and $g_2 o h = g_2'$.

Remark 2.1.1 We say that

$$\begin{array}{ccc} M & \xrightarrow{g_1} & M_1 \\ \downarrow g_2 & & \downarrow f_1 \\ M_2 & \xrightarrow{f_2} & N \end{array}$$

is a *fiber product diagram* to mean M is the fiber product of M_1 and M_2 over N as defined in Definition 2.1.1. Later in these notes, by a *fiber product diagram we shall mean a combination of more than one such fiber product diagrams connected by maps between them.*

Remark 2.1.2 The definition of fiber product makes sense in any category, although existence is not guaranteed. We will mainly be interested in

1. the category of commutative rings and ring homomorphisms;
2. the category of R-modules and R-linear maps, where R is a commutative ring.

Proposition 2.1.1 Let R be a commutative ring and let $f_1 : M_1 \longrightarrow N$ and $f_2 : M_2 \longrightarrow N$ be two homomorphisms of R-modules. Then the fiber product of M_1 and M_2 over N exists and is unique up to an isomorphism.

Proof. Let $M = \{(m_1, m_2) \in M_1 \oplus M_2 : f_1(m_1) = f_2(m_2)\}$ and let

$$g_1 : M \to M_1 \quad \text{and} \quad g_2 : M \to M_2$$

be the natural projections. It is easy to check that

$$\begin{array}{ccc} M & \xrightarrow{g_1} & M_1 \\ \downarrow g_2 & & \downarrow f_1 \\ M_2 & \xrightarrow{f_2} & N \end{array}$$

is a fiber product of M_1 and M_2 over N. For the uniqueness part one can see the book of Kunz ([K1]), Chapter III.

Definition 2.1.2 As mentioned in Remark 2.1.2, the fiber product of commutative rings

$$\begin{array}{ccc} R & \longrightarrow & R_1 \\ \downarrow & & \downarrow \\ R_2 & \longrightarrow & R_0 \end{array}$$

is defined as in Definition 2.1.1. So, in this diagram R is the fiber product of R_1 and R_2 over R_0 in the category of commutative rings.

Remark 2.1.3 In these notes we shall be concerned with *fiber product of modules, only in the context of a given fiber product of rings as above.* We shall be considering fiber product of an R_1-module M_1, an R_2-module M_2 over an R_0-module N. Here follow some examples of fiber products.

Example 2.1.1 Let R be a commutative ring and let I, J be ideals of R. Then

$$\begin{array}{ccc} R/I \cap J & \longrightarrow & R/I \\ \downarrow & & \downarrow \\ R/J & \longrightarrow & R/I+J \end{array}$$

is a fiber product diagram.

Further, if M is an R-module then

$$\begin{array}{ccc} M/(IM \cap JM) & \longrightarrow & M/IM \\ \downarrow & & \downarrow \\ M/JM & \longrightarrow & M/(I+J)M \end{array}$$

is a fiber product diagram of R-modules.

Example 2.1.2 Let R be a commutative ring and s, t be elements in R such that $Rs + Rt = R$. Then

$$\begin{array}{ccc} R & \longrightarrow & R_s \\ \downarrow & & \downarrow \\ R_t & \longrightarrow & R_{st} \end{array}$$

is a fiber product diagram of commutative rings.

Further, if M is an R-module, then

$$\begin{array}{ccc} M & \longrightarrow & M_s \\ \downarrow & & \downarrow \\ M_t & \longrightarrow & M_{st} \end{array}$$

is a fiber product diagram of R-modules.

It is important to understand the proofs of Examples (2.1.1) and (2.1.2), which we leave as exercise.

Proposition 2.1.2 Let R be a commutative ring and

$$\begin{array}{ccc} M & \xrightarrow{g_1} & M_1 \\ \downarrow g_2 & & \downarrow f_1 \\ M_2 & \xrightarrow{f_2} & N \end{array}$$

be a commutative diagram of R-modules. This diagram is a fiber product diagram if and only if for each pair of elements m_1 in M_1 and m_2 in M_2 with $f_1(m_1) = f_2(m_2)$ there is a unique element m in M with $g_1(m) = m_1$ and $g_2(m) = m_2$.

The proof of this Proposition 2.1.2 is obvious from the construction in Proposition 2.1.1.

Exercise 2.1.1 Suppose R is a commutative ring and

$$\begin{array}{ccc} M & \longrightarrow & M_1 \\ \downarrow & & \downarrow \\ M_2 & \longrightarrow & M_0 \end{array}$$

is a fiber product diagram of R-modules. Let t be a nonzero element in R. Then

$$\begin{array}{ccc} M_t & \longrightarrow & M_{1t} \\ \downarrow & & \downarrow \\ M_{2t} & \longrightarrow & M_{0t} \end{array}$$

is also a fiber product diagram of R_t-modules.

2.2 Patching Homorphisms of Modules

In this section, we shall be patching homomorphisms of modules in the context of fiber product diagrams of rings as follows.

Proposition 2.2.1 Let R be a commutative ring and let s and t be elements in R such that $Rs + Rt = R$. Suppose M and M' are two R-modules. Let $f_1 : M_s \longrightarrow M'_s$ be a R_s-linear map and $f_2 : M_t \longrightarrow M'_t$ be a R_t-linear map such that $(f_1)_t = (f_2)_s$.

1. Then there is an R-linear map $f : M \longrightarrow M'$ such that $(f)_s = f_1$ and $(f)_t = f_2$.
2. Further, if f_1 and f_2 are injective (*respectively*, surjective, isomorphism) then so is f.

Proof. The proof follows from the following fiber product diagram :

$$\begin{tikzcd}
M \arrow[rr] \arrow[dd] \arrow[dr, "f"] & & M_s \arrow[dd] \arrow[dr, "f_1"] & \\
& M' \arrow[rr] \arrow[dd] & & M'_s \arrow[dd] \\
M_t \arrow[rr] \arrow[dr, "f_2"] & & M_{st} \arrow[dr, "f_0"] & \\
& M'_t \arrow[rr] & & M'_{st}
\end{tikzcd} .$$

Here $f_0 = (f_1)_t = (f_2)_s$ and $f : M \longrightarrow M'$ is obtained by the properties of fiber product diagrams. Now the theorem follows from the properties of fiber product diagrams.

Remark 2.2.1 The Proposition 2.2.1 could as well be proved directly. We gave this proof because we shall be using such fiber product diagrams in our later discussions.

Further, the Proposition 2.2.1 also holds in the following situation: whenever we are given two fiber product diagrams of R-modules

$$\begin{array}{ccc} M & \longrightarrow & M_1 \\ \downarrow & & \downarrow \\ M_2 & \longrightarrow & M_0 \end{array} \qquad \text{and} \qquad \begin{array}{ccc} M' & \longrightarrow & M'_1 \\ \downarrow & & \downarrow \\ M'_2 & \longrightarrow & M'_0 \end{array}$$

and R-linear maps maps $f_i : M_i \longrightarrow M'_i$ for $i = 0, 1, 2$ so that the diagrams

$$\begin{array}{ccc} M_i & \stackrel{f_i}{\to} & M'_i \\ \downarrow & & \downarrow \\ M_0 & \stackrel{f_0}{\to} & M'_0 \end{array}$$

commute for $i = 1, 2$ then there is an R-linear map $f : M \longrightarrow M'$ with respective properties as in Proposition 2.2.1.

We conclude this Chapter with the following exercise.

Exercise 2.2.1 Let R be a commutative ring and let S be a multiplicative subset of R. Let M and N be finitely generated R-modules and $f : M_S \longrightarrow N_S$ be an R_S-linear map. Then there is an element t in S and an R_t-linear map $g : M_t \longrightarrow N_t$ such that $(g)_S = f$.

Further, if f is injective (*respectively*, surjective, isomorphism) then we can pick t in S, so that g is also injective (*respectively*, surjective, isomorphism).

2.3 Elementary facts about Projective Modules

We start with some easy remarks.

Remark 2.3.1 If P is a finitely generated projective R-module, then $P \oplus Q$ is a free R-module of finite rank for some R-module Q.

Proposition 2.3.1 Let R be a commutative ring.

1. If P is a projective R-module and S is a multiplicative subset of R then P_S is a projective R_S-module.

2. If P is a projective R-module and I is an ideal of R then P/IP is a projective R/I-module.

3. More generally, if $R \longrightarrow A$ is a ring homomorphism and if P is a projective R-module then $P \otimes_R A$ is a projective A-module.

4. Direct summand of a projective module is projective.

Proposition 2.3.2 Let R be a noetherian commutative local ring and let P be a finitely generated R-module. Then P is projective if and only if P is free.

Proof of Proposition 2.3.2 follows form Nakayama's Lemma, which we state here for the sake of completeness.

Lemma 2.3.1 (Nakayama) Let R-be a commutative ring and M be a finitely generated R-module. Suppose I is an ideal and $IM = M$. Then there is an x in I such that $(1 + x)M = 0$. In particular, if R is a local ring and I is a proper ideal then $M = 0$.

Proof. See the book of Matsumura ([Mt]), Chapter 1.

Proof of Proposition 2.3.2. The proof follows from Lemma 2.3.1. One can see the book of Kunz ([K1]), Chapter IV.

Proposition 2.3.3 Let R be a commutative noetherian ring and let P be a finitely generated R-module. Then P is projective if and only if P_m is free for all maximal ideals m of R.

Proof. Suppose P is projective. Then P_m is projective and hence free by Proposition 2.3.2. The converse follows from Corollary 1.4.1. One can see the book of Kunz ([K1]), Chapter IV.

Before we conclude this section we define the rank of a projective module.

Definition 2.3.1 Let R be a commutative noetherian ring and let P be a finitely generated projective R-module. We define the rank function as

$$rank(P, \wp) = rank(P_{\wp})$$

for prime ideals $\wp$ in $Spec(R)$. If $rank(P, \wp) = r$ is constant for all $\wp$ in $Spec(R)$, then we say that P has rank r.

Exercise 2.3.1 For a finitely generated projective R-module P, $rank(P, \wp)$ is a continuous function from $Spec(R)$ to $\{0, 1, 2, \ldots\} = \mathcal{N}$. Hence if R has no nontrivial idempotent, then $rank(P, \wp) = r$ is constant (see Exercise 1.2.1).

2.4 Modules over Principal Ideal Domains

Before we state our main theorem in this section, we define torsion free modules.

Definition 2.4.1 Let R be a commutative ring and let M be an R-module. We say that M is torsion free if $rm \neq 0$ for all nonzero divisors r in R and $m \neq 0$ in M.

The following is the main theorem in this section.

Theorem 2.4.1 *Let R be a principal ideal domain (PID) and M be a finitely generated torsion free R-module. Then M is free. In particular, finitely generated projective modules over R are free.*

Proof. Let S be the set of all nonzero elements in R. Then $S^{-1}M$ is a finite dimensional vector space over R_S. Therefore, there is an isomorphism, $f : S^{-1}M \longrightarrow S^{-1}R^n$ for some n. Let $m_1, \ldots, m_k$ be generators of M and let $f(m_i/1) = x_i/s$ for some x_i in R^n and some s in S. Define an R-linear map $g : M \longrightarrow R^n$ by $g(m) = x$ if $f(m) = x/s$. This map is well defined and since M is torsion free the map is also injective. So, we can assume that M is a submodule of R^n.

Now we shall prove that any submodule M of R^n is free, by induction on n.

If $n = 1$, then M is an ideal of R and hence free of rank one.

Assume $n > 1$ and let $p_n : R^n \longrightarrow R$ be the projection to the last coordinate. Let $I = p_n(M)$. If $I = 0$, then M is a submodule of R^{n-1} and M is free by induction. If $I \neq 0$, then I is free of rank one and

$$0 \longrightarrow M' \longrightarrow M \longrightarrow I \longrightarrow 0$$

is exact where $M' = \text{kernel } p_n \cap M$. Again by induction M' is free. Since this sequence is split exact, $M \approx M' \oplus I$ is free.

The following is a more explicit version of Theorem 2.4.1 that we need for our future use.

Proposition 2.4.1 Let R be a principal ideal domain and $M \neq 0$ be a submodule of a free module R^n and $e_1, \ldots, e_n$ be a basis of R^n. Then, after relabeling the basis, there are nonzero elements $f_1, f_2, \ldots, f_r$ in R such that $M = \oplus_{i=1}^r R f_i e_i$, where $r = \text{rank } M$.

Proof. We use induction on n. If $n = 1$ then M is an ideal of $R = Re_1$. So, $M = Rf_1e_1$ for some nonzero f_1 in R.

Now assume that $n > 1$ and let $p_n : R^n \longrightarrow R$ be the projection to $R \approx Re_n$. Let $I = p_n(M)$. If $I = 0$ then M is a submodule of $\oplus_{i=1}^{n-1} Re_i$ and hence by induction $M = \oplus_{i=1}^r R f_i e_i$. If $I \neq 0$ then $I = Rf_r$ for some nonzero f_r in R. If $M' = \text{kernel } p_n \cap M$, then

$$0 \longrightarrow M' \longrightarrow M \longrightarrow I = Rf_r \longrightarrow 0$$

is a split exact sequence. Since M' is a submodule of $\oplus_{i=1}^{n-1} Re_i$, the proposition follows by induction.

2.5 Vector Bundles and Projective Modules

The purpose of this section is to give some of the obvious and the classical examples of projective modules that are not free. This is also an opportunity to talk about the correspondence between the topological vector bundles and the projective modules over the ring of continuous functions.

Example 2.5.1 Let A be a commutative ring and let $R = A \times A$. Then $P = 0 \times A$ is a projective R-module that is not free.

The following is the classical example of a nonfree projective module that is not so obvious.

Example 2.5.2 (Sw4) Let

$$R = \mathbb{R}[X, Y, Z]/(X^2 + Y^2 + Z^2 - 1) = \mathbb{R}[x, y, z]$$

be the algebraic coordinate ring of the sphere S^2. Define a map $f : R^3 \longrightarrow R$ by sending $e_1 = (1,0,0)$ to x, $e_2 = (0,1,0)$ to y and $e_3 = (0,0,1)$ to z. Since $x^2 + y^2 + z^2 = 1$, we have f is surjective. Let $P = \text{kernel } (f)$. Then $P \oplus R \approx R^3$ and hence P is a projective R-module. It is well known that P is not free. This follows from the fact that the Tangent Bundle of the sphere S^2 is not trivial.

The rest of this section is devoted to justify the validity of Example 2.5.2. This will also be an opportunity to mention the correspondence between vector bundles and projective modules. We shall only be able to sketch this theory. One can see the papers of Swan ([Sw1], [Sw4]) for more details on this theory.

Definition 2.5.1 Suppose S is a connected compact real manifold. A (real) *vector bundle* V over S of rank n is a topological space V with a continuous map $\pi : V \longrightarrow S$ such that

1. For x in S, the fiber $\pi^{-1}x$ is a vector space over $\mathbb{R}$ of dimension n;
2. For x in S, there is an open neighborhood U of x and a homeomorphism $\varphi : U \times \mathbb{R}^n \longrightarrow \pi^{-1}U$ such that $\pi o \varphi = p$, where $p : U \times \mathbb{R}^n \longrightarrow U$ is the projection map. Also, for all y in U, the restriction of φ to $y \times \mathbb{R}^n \longrightarrow \pi^{-1}y$ is an $\mathbb{R}$-linear homomorphism.

The maps between vector bundles over S are defined in the obvious way. A vector bundle V over S is said to be *trivial* if it is isomorphic to the vector bundle $S \times \mathbb{R}^n$ for some nonnegative integer n, in the category of vector bundles over S.

Definition 2.5.2 Let S be as above and let $\pi : V \to S$ be a vector bundle over S. A *section* of V is a continuous map $s : S \to V$ such that $\pi o s = id$.

Given a connected compact manifold S, the ring of all continuous functions from S to $\mathbb{R}$ will be denoted by $C(S)$.

Definition 2.5.3 Let S be as above and let V be a vector bundle over S. Let $P(V)$ be the set of all sections of V. Then $P(V)$ has a $C(S)$-module structure under the operations :

1. for s and t in $P(V)$ define $(s + t)(x) = s(x) + t(x)$,
2. for s in $P(V)$ and f in $C(S)$ define $(fs)(x) = f(x)s(x)$.

Proposition 2.5.1 (Sw4) Let S be as above and V be a vector bundle of rank n over S. Then $P(V)$ is a projective $C(S)$-module of rank n.

Proof. See the paper of Swan ([Sw4]).

Theorem 2.5.1 (Swan) *Let S be a connected compact Hausdorff space. Then $V \to P(V)$ gives an equivalence of categories between the category of vector bundles on S and the category of projective $C(S)$-modules of finite rank. In particular, V is trivial if and only if $P(V)$ is free.*

Proof. See the paper of Swan ([Sw4]).

Example 2.5.3 Let $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ be the sphere. Then $T = \{(x, y, z, u, v, w) \in S^2 \times \mathbb{R}^3 : xu + yv + zw = 0\}$ is the tangent bundle on S^2. It is also known that T is not trivial. Then $P(T)$ = set of all continuous functions $s = (u, v, w) : S^2 \to \mathbb{R}^3$ such that $xu(x) + yv(y) + zw(z) = 0$ for all x in S^2. So, it follows from Theorem 2.5.1 that $P(T)$ is not a free $C(S^2)$-module.

Now we are ready to give a proof of Example 2.5.2.

Proof of Example 2.5.2. Our notations here are as in Example 2.5.2 and Example 2.5.3. Clearly, P=kernel $f = \{(u, v, w)$ in $R^3 : ux + vy + wz = 0\}$. Note that R is a subring of $C(S^2)$ and $P(T) \approx P \otimes_R C(S^2)$. Since $P(T)$ is not free, P cannot be free.

2.6 Some Constructions of Projective Modules

In this section we shall give some of the standard constructions of projective modules. We start with some definitions.

Definition 2.6.1 Let R be a commutative ring and let $(x_1, x_2, \ldots, x_n)$ be in R^n. We say that $(x_1, x_2, \ldots, x_n)$ is a *unimodular row* if

$$x_1y_1 + x_2y_2 + \cdots + x_ny_n = 1$$

for some $y_1, y_2, \ldots, y_n$ in R. The set of all unimodular rows in R^n will be denoted by $U_n(R)$.

Definition 2.6.2 Let R be a commutative ring and let M be a finitely generated R-module. We say that M is *stably free* if $M \oplus R^k \approx R^{n+k}$ for some nonnegative integers n and k. Obviously, a stably free module is projective. Therefore, Example 2.5.2 is also stably free.

Construction 2.6.1 Let R be a commutative ring and let $(x_1, x_2, \ldots, x_n)$ be a unimodular row in R^n. Define a map $f : R^n \to R$ by sending the standard basis $e_1, e_2, \ldots, e_n$ to $x_1, x_2, \ldots, x_n$, respectively. Let $P =$ kernel of f. Then $P \oplus R \approx R^n$. So, P is stably free.

Exercise 2.6.1 Let R be a commutative ring.

1) Let P be as in Construction 2.6.1. Then P is free if and only if the unimodular row $(x_1, \ldots, x_n)$ is the first row of an invertible $n \times n$ - matrix.

2) More generally, let F and F' be two finitely generated free modules over a commutative ring A. Let

$$\alpha : F \longrightarrow F'$$

be a surjective A-linear map and P be the kernel of α . Then P is free if and only if the matrix of α can be completed into an invertible matrix.

The following is the construction of projective modules by the method of patching that will be of great use for us in the later sections.

Construction 2.6.2 Let R be a commutative ring and let s_1, s_2 be in R be such that $Rs_1 + Rs_2 = R$. Let P_i be a finitely generated projective R_{s_i} - module for $i = 1, 2$. Let $f : (P_1)_{s_2} \longrightarrow (P_2)_{s_1}$ be an isomorphism and let

$$\begin{array}{ccccc} P & & \longrightarrow & & P_2 \\ \downarrow & & & & \downarrow \\ P_1 & \to & (P_1)_{s_2} & \xrightarrow{f} & (P_2)_{s_1} \end{array}$$

be a fiber product diagram. Then since $P_{s_1} \approx P_1$ and $P_{s_2} \approx P_2$ are projective, P is also projective by Proposition 2.3.3.

For similar constructions of projective modules see the book of Milnor ([Mi]).

Chapter 3

Extended Modules over Polynomial Rings

In this chapter we prove various theorems of Quillen and Suslin that lead to the proof of Serre's Conjecture that finitely generated projective modules over polynomial rings over fields are free. Serre's conjecture was proved, independently, by Quillen and Suslin using two different methods. We will, essentially, be following Quillen's methods.

3.1 Quillens Theorem

In this section we shall prove Theorem 3.1.1 of Quillen that a module over a polynomial ring is extended if it is locally extended. We start with the definition of extended modules.

Definition 3.1.1 Let $A \to B$ be an extension of commutative noetherian rings. A B-module M is said to be *extended* from A (or simply an extended module) if $M \approx N \otimes_A B$ for some A-module N. We will mainly be concerned with the situations where B will be a polynomial ring over A.

Proposition 3.1.1 Let $R = A[X]$ be a polynomial ring over a commutative noetherian ring A and let M be a finitely generated R-module that is extended from A. Then $M \approx \bar{M} \otimes_A R[X]$ where $\bar{M} = M/XM$.

Proof. Since M is extended, $M \approx N \otimes_A A[X]$ for some A-module N. Then $\bar{M} =$

$M/XM \approx (N \otimes_A A[X]) \otimes_R R/XR \approx N \otimes_A (A[X] \otimes_R R/XR) \approx N \otimes_A A \approx N.$

So, $M \approx \bar{M} \otimes_A R$.

The main theorem in this section is the theorem of Quillen ([Q]) as follows.

Theorem 3.1.1 (Quillen) *Let $R = A[X]$ be a polynomial ring over a noetherian commutative ring A and let M be a finitely generated R-module. If $M_m = (A\backslash m)^{-1}M$ is extended from A_m, for all m in $\max(A)$, then M is extended from A.*

We shall use the following two lemmas to prove Theorem 3.1.1.

Lemma 3.1.1 (Quillen) Let A be a commutative ring and R be an A-algebra (that is not necessarily commutative). Let f be in A and X be a variable. Let θ be a unit in $1 + XR_f[X]$. Then there is an integer $k \geq 0$ such that for any g_1, g_2 in A with $g_1 - g_2$ in f^kA, there exists a unit ψ in $1 + XR[X]$ such that $\psi_f(X) = \theta(g_1X)\theta(g_2X)^{-1}$.

Proof. Let $\theta(X) = \sum_{i=0}^{p} a_iX^i$ and $\theta(X)^{-1} = \sum_{i=0}^{p} b_iX^i$ with a_i, b_i in R_f and $a_0 = b_0 = 1$. Let r be a nonnegative integer and let Y, Z be indeterminates. Then

$$\theta((Y + f^rZ)X)\theta(YX)^{-1} = 1 + [\theta((Y + f^rZ)X) - \theta(YX)]\theta(YX)^{-1}$$

$$= 1 + ZX \sum_{i=1}^{p} \sum_{j=0}^{p} \sum_{n=0}^{i-1} f^r a_i b_j (Y + f^rZ)^{i-1-n} Y^{n+j} X^{i-1+j} .$$

If r is large enough then there are C_{ij} in R such that $(C_{ij})_f = f^r a_i b_j$. So, there is ϕ in $1+ZXR[Y, Z, X]$ such that $\phi_f(Y, Z, X) = \theta((Y+f^rZ)X)\theta(YX)^{-1}$. Replacing Y, Z by $Y + f^rZ, -Z$ we see that there is ϕ' in $1+ZXR[Y, Z, X]$ such that $\phi'_f(Y, Z, X) = \theta(YX)\theta((Y+f^rZ)X)^{-1}$. Then we have $(\phi\phi')_f = (\phi'\phi)_f = 1$. So, if we write $\phi\phi' = 1+ZXh_1$ and $\phi'\phi = 1+ZXh_2$, then there is a nonnegative integer s such that $f^sh_1 = f^sh_2 = 0$. It follows that $\phi(Y, f^sZ, T)$ is a unit. The lemma now follows with $k = r+s$ and $\psi(X) = \phi(g_2, f^sz, X)$ where $g_1 = g_2+f^kz$.

Lemma 3.1.2 (Quillen) Let $R = A[X]$ be a polynomial ring over a commutative noetherian ring A and let s, t in A be such that $As + At = A$. Let M_0 be an A-module and $M = M_0 \otimes_A A[X] = M_0[X]$ and N be an R-module. Suppose $f_1 : M_s \to N_s$ and $f_2 : M_t \to N_t$ are two isomorphisms such that $(f_1)_t \equiv (f_2)_s$ (modulo X). Then there is an isomorphism $f : M \to N$ such that $(f)_s \equiv f_1$ (modulo X) and $(f)_t \equiv f_2$ (modulo X).

Proof. Let $\theta = (f_1)_t^{-1} o (f_2)_s : M_{st} \to M_{st}$. It follows that $\theta \equiv Id$ (modulo X). We look at θ as an element in $\mathrm{End}(M_{st}) = End(M_{0st})[X]$ and apply Quillen's Lemma 3.1.1. So,

$$\theta = \varphi_0 + \varphi_1 X + \varphi_2 X^2 + \cdots + \varphi_r X^r$$

for some φ_i in $\mathrm{End}(M_{0st})$ and $\varphi_0 = Id_{M_{0st}}$. Apply Lemma 3.1.1 for the extension $A_s \to End(M_{0s})$. We see that there is an integer $k_1 \geq 0$ such that whenever $g_1 - g_2$ is in $t^{k_1} A_s$ for some g_1, g_2 in A_s, there is a unit ψ_1 in 1+X $\mathrm{End}(M_{0s})[X]$ such that $(\psi_1)_t = \theta(g_1 X)\theta(g_2 X)^{-1}$. Similarly, there is an integer $k_2 \geq 0$, such that whenever $g_1 - g_2$ is in $s^{k_2} A_t$, there is a unit ψ_2 in $1 + X$ $\mathrm{End}(M_{0t})[X]$ such that $\psi_{2s} = \theta(g_1 X)\theta(g_2 X)^{-1}$. Fix $k \geq max\{k_1, k_2\}$. Since $at^k + bs^k = 1$ for some a, b in A, we have

$$\theta(X)\theta(bs^k X)^{-1} = (\psi_1)_t$$

for some unit ψ_1 in $1 + X$ $\mathrm{End}(M_{0s})[X]$ and

$$\theta(bs^k X) = \theta(bs^k X)\theta(0)^{-1} = (\psi_2)_s$$

for some unit ψ_2 in $1 + X$ End $(M_{0t})[X]$.

So,

$$(f_1)_t^{-1} o (f_2)_s = \theta(X) = [\theta(X)\theta(bs^k X)^{-1}][\theta(bs^k X)\theta(0)^{-1}] = (\psi_1)_t(\psi_2)_s.$$

Hence $(f_1\psi_1)_t = (f_2\psi_2^{-1})_s$.

So, there is an isomorphism $f : M \to N$ such that $f_s = f_1\psi_1$ and $f_t = f_2\psi_2^{-1}$. Since $\psi_1 \equiv Id$ (modulo X), we have $(f)_s \equiv f_1$ (modulo X). Similarly, $(f)_t \equiv f_2$ (modulo X). This completes the proof of Lemma 3.1.2.

Now we are ready to give a proof of Theorem 3.1.1 of Quillen.

Proof of Theorem 3.1.1 of Quillen. Let $N = M/XM$. Because of Proposition 3.1.1, we need to prove that $M \approx N \otimes A[X]$. Let

$$J = \{t \text{ in } A : M_t \approx N_t \otimes A[X]\} = \{t \text{ in } A : M_t \text{ is extended from } A_t\}.$$

We shall first prove that J is an ideal of A. Clearly, 0 is in J and also for a in A and t in J we have at is in J.

Now let s and t be in J. We want to show that $s + t$ is in J. By replacing A by A_{s+t}, we can assume that $As + At = A$. We have two isomorphisms

$$f_1 : (N \otimes A[X])_s \longrightarrow M_s \qquad \text{and} \qquad f_2 : (N \otimes A[X])_t \longrightarrow M_t\,.$$

We can assume that $f_1 \equiv Id$ (modulo X) and $f_2 \equiv Id$ (modulo X). Hence by Lemma 3.1.2 $N \otimes A[X] \approx M$.

So, we have proved that the set J defined above is an ideal of A.

Now we prove that $J = A$. Otherwise, J is contained in a maximal ideal m of A. Since $M_m \approx N_m \otimes A[X]$, it follows that $M_t \approx N_t \otimes A[X]$ for some t in $A \backslash m$. This means t is in J. This contradicts that J is contained in m.

So, $J = A$. Hence 1 is in J and $M \approx N \otimes A[X]$. Therefore the proof is complete.

Remark 3.1.1 It follows from the proof of Theorem 3.1.1 that for a finitely generated $A[X]$-module M, we have $J = \{t \text{ in } A : M_t \text{ is extended from } A_t\}$ is an ideal. It is also clear that $\sqrt{J} = J$.

3.2 Projective Modules over Polynomial Rings

In this section we prove various important theorems of Quillen and Suslin, including the Conjecture of Serre. Recall that a polynomial f in a polynomial ring $R = A[X]$ is called a *monic polynomial* if the coefficient of the leading term of f is a unit in A. The following theorem of Horrocks plays an important role in this theory.

Theorem 3.2.1 (Horrocks) *Let (A, m) be a commutative noetherian local ring and let $R = A[X]$ be the polynomial ring. Suppose P is a finitely generated projective R-module such that P_f is free for some monic polynomial f in R. Then P is free.*

Before we prove Theorem 3.2.1 of Horrocks, we derive the following Theorems 3.2.2 and 3.2.3, both proved, independently, by Quillen and Suslin from Theorem 3.2.1.

Theorem 3.2.2 (Quillen, Suslin) *Let A be a noetherian commutative ring and let $R = A[X]$ be the polynomial ring. Suppose that P is a finitely generated projective R-module such that P_f is free for some monic polynomial f in R. Then P is free.*

Proof. Here we give the proof of Quillen. For any maximal ideal m of A, $P_m = (A - m)^{-1}P$ is free by Horrocks' Theorem 3.2.1. So, P_m is extended for all maximal ideals m of A. By Theorem 3.1.1 of Quillen, we have P is extended from A. So, $P \approx N \otimes A[X]$ for some A-module N. It remains to show that N is free.

Now look at $R_X = A[X, X^{-1}]$ and let $Y = X^{-1}$ and $g(Y) = X^{-d}f(X)$ where $d =$ degree of f. Since f is monic, the constant term of g is a unit. Hence $g(Y)A(Y) + YA[Y] = A[Y]$.

The diagram

$$\begin{array}{ccccc} A[Y] & \longrightarrow & A[Y,Y^{-1}] & = & A[X,X^{-1}] \\ \downarrow & & \downarrow & & \\ A[Y]_g & \longrightarrow & A[Y]_{Yg} & = & A[X,X^{-1}]_f \end{array}$$

is a fiber product diagram.

Let $r = \text{rank } P_f$ and let F be a free $A[Y]_g$ - module of rank r. Since P_{Xf} is free there is an $A[Y]_g$ -linear map $\varphi : F \to P_{Xf}$ so that the induced map $F_g \to P_{Xf}$ is an isomorphism. Let P' be the projective $A[Y]$ -module given by the fiber product diagram

$$\begin{array}{ccc} P' & \longrightarrow & P_X \\ \downarrow & & \downarrow \\ F & \xrightarrow{\varphi} & P_{Xg}\,. \end{array}$$

Since $P'_Y \approx P_X$ is extended from A, by the argument of the first paragraph, P' is also extended from A. Therefore $P' \approx N' \otimes A[Y]$ for some finitely generated A-module N'.

The natural map $P' \to P'_g \approx F$ will induce an isomorphism

$$P'/YP' \approx P'_g/YP'_g \approx F/YF$$

(this is possible because $g(0) = 1$). Since F is free, P'/YP' is also free. Hence $N' \approx P'/YP'$ is free.

Now it follows that

$$N \approx P/(X-1)P \approx P_X/(X-1)P_X \approx P'_Y/(Y-1)P'_Y \approx P'/(Y-1)P' \approx N'$$

is free. So, $P \approx N \otimes A[X]$ is free. So, the proof of Theorem 3.2.2 is complete.

Exercise 3.2.1 Let P be a finitely generated $R = A[X]$-module and let N be a finitely generated projective A-module. Suppose that $P_f \approx N[X]_f$ for some monic polynomial f. Then $P \approx N[X]$.

The following theorem, proved independently by Quillen and Suslin, settles the conjecture of Serre about projective modules over polynomial rings over fields.

Theorem 3.2.3 (Quliien,Suslin) *Let $R = A[X_1, \ldots, X_n]$ be a polynomial ring over a principal ideal domain A. Then any finitely generated projective R-module is free.*

Proof. We prove this theorem by induction on n. If $n = 0$, then any finitely generated projective R-module is free by Theorem 2.4.1.

Now assume that $n > 0$. Let S be the set of all monic polynomials in $A[X_1]$ and let $A' = S^{-1}A[X_1]$. Since $\dim(A') = 1$ and A' is a unique factorization domain, A' is a principal ideal domain. By induction it follows that $S^{-1}P$ is a free $S^{-1}R = A'[X_2, \ldots, X_n]$-module. So, there is a monic polynomial f in $A[X_1]$, such that P_f is free. Hence P is free by Theorem 3.2.2 and the proof is complete.

The rest of this section is devoted to proving Theorem 3.2.1 of Horrocks.

3.2.1 The Proof of the Theorem of Horrocks

The proof of Theorem 3.2.1 that we give here evolves out of some techniques that was developed by Amit Roy ([R]) and is due to Budh Nashier and Warren Nichols([NN]). We follow the notations as in Theorem 3.2.1.

First note that since A is local, $R = A[X]$ has no idempotent element other than 0 and 1. So, P has a constant rank. First we shall establish Theorem 3.2.1 for the rank one case. We shall split the proof of this part into several lemmas.

Lemma 3.2.1 Let R be any noetherian commutative ring and let P be a projective R-module of rank one. Then P is isomorphic to a projective ideal (also called an invertible ideal) of R.

Proof. Let S be the set of all nonzero divisors of R. Then since $S^{-1}R$ is a semilocal ring and since P has constant rank one, $S^{-1}P$ is isomorphic to $S^{-1}R$. Hence $P_s \approx R_s$ for some s in S. By removing the denominators (as in Theorem 2.4.1) we can assume that there is an injective map $f : P \to R$. So $P \approx f(P)$, which is an ideal.

Lemma 3.2.2 Let $R = A[X]$ be as in Theorem 3.2.1 and I be an ideal of R. Assume that I contains a monic polynomial. Then each nonzero element f in $I + mR/mR$ is the image of a monic polynomial in I.

Proof. The proof is done by careful use of the division algorithm. Let $k = A/m$ and let f be a nonzero element in $I + mR/mR$. Then there is an element g_1 in I such that $\bar{g}_1 = f$. (" $-$ " bar denotes the image in $R/mR \approx k[X]$). Let degree$(f) = r$ and hence degree $(g_1) = n \geq r$. Let $g_1(X) = a_nX^n + a_{n-1}X^{n-1} + \cdots + a_0$ with a_i in A. Now, if $n = r$, then g_1 is monic and the proof is complete in this case. Now assume that $n > r$. So, a_n is in m.

We claim that I contains a monic polynomial of degree $r+1$. To see this, let $F_1(X)$ be a monic polynomial in I of degree $d > r+1$. We can assume that the leading coefficient of F_1 is one. Now degree$(X^{d-1-r}f(X)) = d-1$. We also have $\overline{X^{d-1-r}g_1(X)} = X^{d-1-r}f(X)$.

Write $k = d+n-r-1$ and let $g_2(X) = X^{d-1-r}g_1(X) - a_nX^{k-d}F_1(X)$. Then degree$(g_2) \leq k-1$ and $\bar{g}_2 = X^{d-1-r}f(X)$ has degree $d-1$. Repeating this process, we can find a polynomial $g'(X)$ in I of degree $d-1$, so that $\bar{g'} = X^{d-1-r}f(X)$. Since $X^{d-1-r}f(X)$ also has degree $d-1$, we have g' is monic.

So, we found a monic polynomial g' in I of degree $d-1$. Again repeating this process, we see that there is a monic $F(X)$ in I of degree $r+1$. We can also assume that the leading coefficient of F is one.

Now $G(X) = g_1(X) - a_nX^{n-r-1}F(X)$ has degree $n-1$ and $\bar{G} = \bar{g}_1 = f(X)$. Repeating this process we get a polynomial G_1 in I of degree r so that $\bar{G}_1 = f$. Hence G_1 is also monic. This completes the proof of Lemma 3.2.2.

Lemma 3.2.3 Suppose R is a noetherian commutative ring . Let I be an invertible ideal of R and let J be an ideal of R contained in I. Then there is an ideal L of R such that $LI = J$.

Proof. Let $L = \{f(X) \text{ in } R : f(X)I \subseteq J\}$. We check that for every maximal ideal m of R, we have $(LI)_m = J_m$. Note that LI is contained in J. If LI is not contained in m then $(LI)_m = R_m = J_m$.

Assume that LI is contained in m. So, either L or I is contained in m. Since J is contained in L, it follows that J is also contained in m.

Now $L_m = \{f \text{ in } R_m : fI_m \subseteq J_m\}$.

If I is not contained in m, then $(LI)_m = L_m = J_m$. If I is contained in m, then $I_m = gR_m$ for some nonzero divisor g and $L_m = \{f \text{ in } R_m : fg \text{ is in } J_m\}$. Hence $(LI)_m = L_m(gR_m) = J_m$. So, the proof of Lemma 3.2.3 is complete.

Lemma 3.2.4 Let $R = A[X]$ be a polynomial ring over a noetherian commutative ring A and let I be an invertible ideal that contains a monic polynomial. Then I is a principal ideal.

Proof. Let $J = I + mR$ and let $J/mR = fk[X]$ for some $f(X)$ in $k[X] = R/mR$. By Lemma 3.2.2, there is a monic polynomial $F(X)$ in I, so that $\bar{F} = f$ (here " – " bar means the images in $k[X]$). So, it follows that $J = RF + mR$. By Lemma 3.2.3, let $I \cap mR = IL$ for some ideal L of R. Now $mI \subset I \cap mR$. Since $IL \subseteq mR$ and since I is not contained in mR, we have $L \subseteq mR$. Hence $I \cap mR = IL \subseteq mI$. So, it follows that $I \cap mR = mI$.

We have $I + mR = J = RF + mR$. So, $I = RF + I \cap mR = RF + mI$, where F is a monic polynomial. Now $I/FR = M$ is a finitely generated R/FR -module and $M = mM$. Since m is contained in the radical of R/FR, it follows from Nakayama's Lemma that $M = 0$. Hence $I = RF$. So, the proof of Lemma 3.2.4 is complete.

As Lemma 3.2.1 and Lemma 3.2.4 settle the proof of the rank one case of Theorem 3.2.1 of Horrocks, we proceed to prove the higher rank case of the theorem. We need the following lemma.

Lemma 3.2.5 Let R be a principal ideal domain and let M be a finitely generated free R-module. Let p be a nonzero element of M. Then M has a basis $p_1, p_2, \ldots, p_r$ such that $p = ap_1$ for some a in R.

Proof. Let $q_1, \ldots, q_r$ be a basis of M and $p = a_1q_1 + \cdots + a_rq_r$ for $a_1, \ldots a_r$ in R. We can assume that $a_1 \neq 0$. Let $aR = a_1R + \cdots + a_rR$. Then $a_i = ab_i$ for $b_1, \ldots, b_r$ in R. So, if $a = c_1a_1 + \cdots + c_ra_r$ then $c_1b_1 + \cdots + c_rb_r = 1$. Let $p_1 = b_1q_1 + b_2q_2 + \cdots + b_rq_r$. Define an R-linear map $\varphi : M \longrightarrow Rp_1$ by sending q_i to c_ip_1 for $i = 1$ to r. Then $\varphi(p_1) = p_1$. So φ splits and $M = ker\varphi \oplus Rp_1$.

Now let $p_2, \ldots, p_r$ be a basis of ker φ. Hence $p_1, p_2, \ldots, p_r$ is a basis of M and $ap_1 = p$. So the proof of Lemma 3.2.5 is done.

Now we are ready to prove Theorem 3.2.1 of Horrocks.

Proof of Theorem 3.2.1. Let $n = \text{rank}P$. We give a proof by induction on n.

If $n = 1$, then by Lemma 3.2.1 P is isomorphic to an ideal I of R. Since $P_f \approx R_f$, we can also assume that $I_f = R_f$. Hence I contains a monic polynomial. Hence I is a principal ideal by Lemma 3.2.4. Therefore P is free.

Now assume that $n \geq 2$. Let $p_1, \ldots, p_n$ be a basis of P_f. Let " $-$ " bar denote "(modulo mR)" and let $k = A/m$. Since p_1 is not in mP, we have $\bar{p}_1$ is nonzero. By Lemma 3.2.5, we can find $q_1, q_2, \ldots, q_n$ in P, so that $\bar{q}_1, \ldots, \bar{q}_n$ is a basis of $\bar{P}$ and $\bar{p}_1 = a\bar{q}_2$ for some a in $k[X]$.

Now $f^kq_1 = a_1p_1 + \cdots + a_np_n$ for some $k \geq 1$ and $a_1, \ldots, a_n$ in R. Let $p = q_1 + X^tp_1$ for some $t \geq 0$, so that $a_1 + f^kX^t = f_1$ is a monic polynomial. Hence $f^kp = f^kq_1 + f^kX^tp_1 = (a_1 + f^kX^t)p_1 + a_2p_2 + \cdots + a_np_n$.

Let $T = 1 + mR$. Since $\bar{p} = \bar{q}_1 + X^t\bar{p}_1 = \bar{q}_1 + X^ta\bar{q}_2$, we see that $\bar{p}, \bar{q}_2, \ldots, \bar{q}_n$ generates $\bar{P}$. Hence by Nakayama's Lemma we have $p, q_2, \ldots, q_n$ generate P_T. Since P_T is a projective R_T-module of rank n, it follows that $p, q_2, \ldots, q_n$ is a basis of P_T.

Let $P' = P/Rp$. Then P'_T is free of rank $n-1$, with basis $q_2, \ldots, q_n$. Also P_{ff_1} has a basis $p, p_2, \ldots, p_n$. Hence P'_{ff_1} is also free of rank $n-1$.

Since ff_1 is monic, for any given maximal ideal M, either M does not contain ff_1 or it does not intersect with T. Hence P' is a projective R-module. Since P'_{ff_1} is free, by induction P' is also free. Hence P is free. This completes the proof of Theorem 3.2.1 of Horrocks.

The following exercise is the Laurent polynomial version of Theorem 3.2.2 of Quillen and Suslin.

Exercise 3.2.2 Let $R = A[X, X^{-1}]$ be the Laurent polynomial ring over a noetherian commutative ring A and let P be a finitely generated projective R-module. If P_f is free for some Laurent polynomial f so that the coefficients of the highest and the lowest degree terms of f are units (we say f is doubly monic), then P is free. (see [Ma5]).

Chapter 4

Modules over Commutative Rings

In this Chapter we shall talk about the theory of basic elements over noetherian commutative rings. Unless otherwise specified all our rings are commutative noetherian and modules are finitely generated.

4.1 The Basic Element Theory

In some sense, the concept of basic element is the best possible generalization of the concept of basis of vector spaces. This concept was thoroughly studied by Eisenbud and Evans ([EE1]). This theory was also successful in giving an unified treatment to prove almost all the important results in this area at that time.

Before we give the definitions, we introduce the following notations.

Notations 4.1.1 For a finitely generated A-module M, the minimal number of generators of M will be denoted by $\mu(M)$. For a prime ideal $\wp, \mu(M_\wp)$ will denote the minimal number of generators of $M_\wp$ as an $A_\wp$-module.

Definition 4.1.1 Let A be a noetherian commutative ring and let M be a finitely generated A-module.

1. An element m of M is said to be a *basic element* of M at a prime ideal $\wp$ if m is not in $\wp M_\wp$.

2. An element m of M is said to be a *basic element* of M if m is basic in M at all the prime ideals $\wp$ of A. We also say that m is basic in M on a subset X of *Spec*(A) if m is basic in M at all prime ideals in X.

3. For a nonnegative integer w, a submodule M' of M is said to be $w-fold\ basic$ in M at a prime ideal $\wp$, if $\mu((M/M')_\wp) \leq \mu(M_\wp) - w$.

4. For a set of elements $\{m_1, \ldots, m_t\}$ of M, we say that $\{m_1, \ldots, m_t\}$ is $w-fold\ basic$ in M at $\wp$ if $M' = \sum_{i=1}^t Am_i$ is w-fold basic in M at $\wp$. We also say that $m_1, \ldots, m_t$ are $w-fold$ basic in M at $\wp$ to mean that the set $\{m_1, \ldots, m_t\}$ is $w-fold$ basic in M at $\wp$.

Remark 4.1.1 Suppose A is a commutative noetherian ring and M is a finitely generated A-module.

1. Clearly, an element m of M is basic in M at a prime ideal $\wp$ if and only if m is 1-fold basic in M at $\wp$. It also follows from Nakayama's Lemma that m is a basic element of M at a prime ideal $\wp$ if and only if m extends to a minimal set of generators of $M_\wp$.

2. Suppose that a subset $\{m_1, \ldots, m_t\}$ of M is w-fold basic in M at a prime ideal $\wp$ and let $x_1, \ldots, x_t$ be elements in $\wp M$. Then $\{m_1 + x_1, \ldots, m_t + x_t\}$ is also w-fold basic in M at $\wp$.

The idea of generalized dimension function was formally introduced by Plumstead ([P]). It will be best to introduce this notion at this point.

Definition 4.1.2 Suppose A is a commutative noetherian ring. Let X be a subset of $Spec(A)$ and let $\mathcal{N} = \{0, 1, 2, \ldots\}$ be the set of all the nonnegative integers. Let $d : X \to \mathcal{N}$ be a function.

1. We define a partial ordering on X by defining $\wp_1 \ll \wp_2$ if either $\wp_1 = \wp_2$ or if $\wp_1 \subseteq \wp_2$ and $d(\wp_1) > d(\wp_2)$ for $\wp_1, \wp_2$ in X.

2. A function $d : X \to \mathcal{N}$ is said to be a *generalized dimension function* if for any ideal I of A, $V(I) \cap X$ has only a finitely many minimal elements with respect to the partial ordering $\ll$.

The idea of generalized dimension is useful mainly in the natural situations. Some examples are as follows.

Example 4.1.1 Let A be a commutative ring.

1. Let $d_1 : Spec(A) \to \mathcal{N}$ be defined as $d_1(\wp) = \dim(A/\wp)$.

2. For an ideal I of A let $d_2 : V(I) \to \mathcal{N}$ be the restriction of d_1.

3. For an integer t let $X_t = \{\wp \text{ in } Spec(A) \ : \ height(\wp) \leq t\}$. Let $d_3 : X_t \to \mathcal{N}$ be defined as

   $$d_3(\wp) = \max\{n : \text{ there is a chain } \wp = \wp_1 \subset \wp_2 \subset \ldots \subset \wp_n \text{ with } \wp_i \in X_t\}.$$

Then d_1, d_2, d_3 are generalized dimension functions. See ([P]) for more examples.

The following is the theorem of Eisenbud and Evans ([EE1]). We give the *generalized dimension version* of the theorem which is due to Plumstead ([P]).

Theorem 4.1.1 (Eisenbud-Evans) *Let A be a noetherian commutative ring and let $d : X \to \mathcal{N}$ be a generalized dimension function on a subset X of $Spec(A)$. Let M be a finitely generated A-module.*

(i) *Suppose $\mu(M_\wp) > d(\wp)$ for all $\wp$ in X, then M has a basic element on X.*

(iia) *Let M' be a submodule of M, such that M' is $(d(\wp)+1)$-fold basic in M at $\wp$, for all $\wp$ in X. Then M' contains an element that is basic in M on X.*

(iib) *Let $m_1, m_2, \ldots, m_r$ be elements in M that are $(d(\wp)+1)$-fold basic in M at $\wp$, for all $\wp$ in X. If (a, m_1) is basic in $A \oplus M$ on X, then there is an element*

$$m' = a_2 m_2 + a_3 m_3 + \cdots + a_r m_r$$

for some $a_2, \ldots, a_r$ in A such that $m_1 + am'$ is basic in M on X.

Remark 4.1.2 We want to emphasize that Theorem 4.1.1 applies mainly in the natural situations as in Example 4.1.1. With $d = d_1$, as in Example 4.1.1, it follows from (1) that if $\mu(M_\wp) > \dim A$ for all prime ideals $\wp$, then M has a basic element.

Proof of Eisenbud-Evans Theorem 4.1.1. Since (i) follows from (iia) and (iia) follows from (iib), we need to prove (iib) only. We shall need a few lemmas that follow.

Lemma 4.1.1 Let X be a subset of $Spec(A)$ and let (X, d) be a generalized dimension function. Let M be a finitely generated A-module and N be a submodule. Suppose X' is a subset of X and ω is a nonnegative integer. Assume that for all $\wp$ in X for which there is a prime $\wp'$ in X' with $\wp' \neq \wp$ and $\wp \ll \wp'$, N is ω-fold basic in M at $\wp$. Then N is ω-fold basic in M at all but finitely many primes in X'.

We need the following notations that was introduced by Eisenbud and Evans.

Notations 4.1.2 For a finitely generated module M over a noetherian commutative ring A and a positive integer t, define

$$I_t(M, A) = \sum\{\text{ann}(M/N) : N \textit{ is a submodule of } M \textit{ generated by } t \textit{ elements}\}.$$

We also define $I_0(M, A) = ann(M)$ and $I_{-1}(M, A) = \{0\}$.

The following are obvious:

1. $I_t(M, A)$ is contained in $I_{t+1}(M, A)$ for $t = -1, 0, 1, \ldots$.
2. For a prime ideal $\wp$ of A,we have $I_t(M, A)$ is contained in $\wp$ if and only if $\mu(M_\wp) > t$.
3. If $\mu(M) = t_0$ then $I_t(M, A) = R$ for all integers $t \geq t_0$.
4. Since A is assumed to be noetherian, there are only finitely many ideals $I_t(M, A)$.

Proof of Lemma 4.1.1. We prove that for $\wp'$ in X', if N is not ω-fold basic in M at $\wp'$, then $\wp'$ is minimal over $I_t(M/N, A) \cap X$ with respect to $\ll$, for some t and that will complete the proof by (4) of the Notations 4.1.2.

Now suppose that $\wp'$ is in X' and is not minimal over $I_t(M/N, A) \cap X$ for all t. Now let $\mu((M/N)_{\wp'}) = r + 1$ for some $r = -1, 0, 1, \ldots$. So, $I_r(M/N, A) \subseteq \wp'$ and $I_t(M/N, A)$ is not contained in $\wp'$ for all $t > r$, by Notations 4.1.2. By assumption there is a prime ideal $\wp$ in X such that $I_r(M/N, A) \subseteq \wp \subseteq \wp'$ and $d(\wp) > d(\wp')$. So, N is ω-fold basic at $\wp$. Now $\mu((M/N)\wp) \geq r + 1$. Since

$$r + 1 = \mu(M/N)_{\wp'}) \geq \mu((M/N)_\wp),$$

we have $\mu((M/N)_\wp) = \mu((M/N)_{\wp'}) = r + 1$. Hence

$$\mu((M/N)_{\wp'}) = \mu((M/N)_\wp) \leq \mu(M_\wp) - \omega \leq \mu(M_{\wp'}) - \omega.$$

Therefore N is ω-fold basic in M at $\wp'$. This completes the proof.

Lemma 4.1.2 Suppose M is a finitely generated module over a commutative noetherian ring A. Let $\{m_1, \ldots, m_r\}$ be a set of elements in M. For $i = 1$ to k let ω_i be integers with $\omega_i < r$ and let $\wp_1, \ldots, \wp_k$ be prime ideals in A. Suppose $\{m_1, \ldots, m_r\}$ is ω_i-fold basic at $\wp_i$ for $i = 1$ to k and suppose that (a, m_1) is basic in $A \oplus M$ at $\wp_i$ for $i = 1$ to k. Then there are elements $a_1, a_2, \ldots, a_{r-1}$ in A such that

1. $(a, m_1 + aa_1m_r)$ is basic in $A \oplus M$ at $\wp_i$ for $i = 1$ to k and

2. $\{m_1 + aa_1m_r, m_2 + a_2m_r, \ldots, m_{r-1} + a_{r-1}m_r\}$ is ω_i-fold basic at $\wp_i$, for $i = 1$ to k.

Proof. Since for an element m' in M, the map $A \oplus M \to A \oplus M$ that sends (x, m) to $(x, m + xm')$ is an isomorphism, (1) of the lemma is valid for any choice of $a_1, \ldots, a_{r-1}$. So, we have to prove (2) only. Now we shall proceed by induction on k.

If $k = 0$, then there is nothing to prove. So, we assume that $k > 0$. We can also assume they $\wp_k$ is minimal among $\{\wp_1, \ldots, \wp_k\}$ and hence $\bigcap_{i=1}^{k-1} \wp_i \not\subseteq \wp_k$.

Now suppose that

$$m'_1 = m_1 + aa'_1m_r, m'_2 = m_2 + a'_2m_r, \ldots, m'_{r-1} = m_{r-1} + a'_{r-1}m_r$$

are ω_i-fold basic in M at $\wp_i$ for some $a'_1, a'_2, \ldots, a'_{r-1}$ in A. We will show that we can choose $a''_1, \ldots, a''_{r-1}$ in A such that

$$(*) \begin{cases} \text{for any } c \ \text{ in } \ A\backslash\wp_k, \\ m''_1 = m'_1 + aa''_1cm_r, \ m''_2 = m'_2 + a''_2cm_r, \ldots, \\ m''_{r-1} = m'_{r-1} + a''_{r-1}cm_r \\ \text{are } \ \omega_k - \text{fold basic at } \ \wp_k. \end{cases}$$

If we choose c in $\bigcap_{i=1}^{k-1} \wp_i\backslash\wp_k$ then we will be done by Remark 4.1.1.

By changing notations, to prove $(*)$, we need to prove the following : Suppose $\{m_1, m_2, \ldots, m_r\}$is ω-fold basic in M at a prime ideal $\wp$ of A, with $\omega < r$. And suppose (a, m_1) is basic in $A \oplus M$ at $\wp$. Then we can find $a_1, a_2, \ldots, a_{r-1}$ such that for any c in $A\backslash\wp$, we have $\{m_1 + aa_1cm_r, m_2 + a_2cm_r, \ldots, m_{r-1} + a_{r-1}cm_r\}$ is ω-fold basic in M at $\wp$.

If $\{m_1, \ldots, m_{r-1}\}$ is already ω-fold basic in M at $\wp$, then we choose $a_i = 0$ for $i = 1$ to $r - 1$. So, we assume that $\{m_1, \ldots, m_{r-1}\}$ is not ω-fold basic at $\wp$.

If $\bar{m}_1, \ldots, \bar{m}_{r-1}$ are the images of $m_1, \ldots, m_{r-1}$, respectively, in $M_\wp/\wp M_\wp$, then we claim that $\bar{m}_1, \ldots, \bar{m}_{r-1}$ are linearly dependent in $M_\wp/\wp M_\wp$. Otherwise let $n = \mu((M/(m_1, \ldots, m_{r-1}))_\wp)$. Let $x_1, \ldots, x_n$ be elements in M, so that their images in $(M/\sum_{i=1}^{r-1} Am_i + \wp M)_\wp$ form a basis. Then it follows that the images $\bar{x}_1, \ldots, \bar{x}_n, \bar{m}_1, \ldots, \bar{m}_{r-1}$ are linearly independent in $M_\wp/\wp M_\wp$. Hence $n + r - 1 = \mu(M_\wp)$. We also have $\mu((M/(m_1, \ldots, m_{r-1}))_\wp) > \mu(M_\wp) - \omega$. So $n > n + r - 1 - \omega$, i.e. $\omega \geq r$, which is a contradiction that $\omega < r$. Hence the claim above is established.

So, we assume that $\bar{m}_1, \ldots, \bar{m}_{r-1}$ are linearly dependent in $M_\wp/\wp M_\wp$. Let

$$t = \max\{\ell : \bar{m}_\ell \textit{ is a linear combination of } \bar{m}_1, \ldots, \bar{m}_{\ell-1} \textit{ in } M_\wp/\wp M_\wp\}.$$

Let $a_t = 1$ and $a_i = 0$ for $i = 1$ to $r - 1$ with $i \neq t$. We let

$$m''_1 = m_1 + aa_1cm_r, m''_2 = m_2 + a_2cm_r, \ldots, m''_{r-1} = m_{r-1} + a_{r-1}cm_r.$$

We check that $\{m''_1, \ldots, m''_{r-1}\}$ is ω-fold basic in M at $\wp$ for all c in $A\backslash\wp$. To see this let W be the subspace of $M_\wp/\wp M_\wp$ generated by the images of $m_1, \ldots, m_r$.

Then it follows that W is also generated by the images of $m''_1, \ldots, m''_{r-1}$. Hence we have,

$$\mu((M/\sum_{i=1}^{r} Am_i)_\wp) = \mu(M_\wp) - dim(W) = \mu((M/\sum_{i=1}^{r-1} Am''_i)_\wp.$$

The proof of Lemma 4.1.2 is complete.

The following is an obvious consequence of Lemma 4.1.2 that will be convenient to use for the proof of Theorem 4.1.1.

Lemma 4.1.3 Suppose M is a finitely generated module over a commutative noetherian ring A. Let $\{m_1, \ldots, m_r\}$ be a set of elements in M. For $i = 1$ to k, let ω_i be integers and let $\wp_1, \ldots, \wp_k$ be prime ideals in A. Suppose $\{m_1, \ldots, m_r\}$ is ω_i-fold basic at $\wp_i$ for $i = 1$ to k and suppose that (a, m_1) is basic in $A \oplus M$ at $\wp_i$ for $i = 1$ to k. Then there are elements $a_1, a_2, \ldots, a_{r-1}$ in A such that

1. $(a, m_1 + aa_1m_r)$ is basic in $A \oplus M$ at $\wp_i$ for $i = 1$ to k and
2. $\{m_1 + aa_1m_r, m_2 + a_2m_r, \ldots, m_{r-1} + a_{r-1}m_r\}$ is $\min\{r-1, \omega_i\}$-fold basic at $\wp_i$, for $i = 1$ to k.

Proof. Let us assume that $\omega_i < r$ for $i = 1$ to k' and that $\omega_i \geq r$ for $i = k' + 1$ to k. By Lemma 4.1.2 we can find

$$m''_1 = m_1 + aa_1m_r, m''_2 = m_2 + a_2m_r, \ldots, m''_{r-1} = m_{r-1} + a_{r-1}m_r$$

such that $\{m''_1, \ldots, m''_{r-1}\}$ is ω_i-fold basic at $\wp_i$ for $i = 1$ to k'.

Since $\min\{r-1, \omega_i\} = \omega_i$, for $i = 1$ to k', we see that $\{m''_1, \ldots, m''_{r-1}\}$ is $\min\{r-1, \omega_i\}$-fold basic at $\wp_i$.

For $i = k'+1$ to k we have $\min\{r-1, \omega_i\} = r-1$ and that $\{m_1, \ldots, \ldots, m_r\}$ is ω_i-fold basic at $\wp_i$. It follows that, for $i = k' + 1$ to k,

$$\mu((M/\sum_{1}^{r-1} Am''_i)_{\wp_i}) \leq \mu((M/\sum_{1}^{r} Am_i)_{\wp_i}) + 1 \leq \mu(M_{\wp_i}) - \omega_i + 1 \leq \mu(M_{\wp_i}) - r + 1.$$

Hence $\{m''_1, \ldots, m''_{r-1}\}$ is $\min\{r-1, \wp_i\}$-fold basic at these prime ideals as well. So, the proof of Lemma 4.1.3 is complete.

Now we are ready to prove Theorem 4.1.1 of Eisenbud and Evans. As we mentioned before, we prove only (iib) of Theorem 4.1.1.

Proof of (iib) of Theorem 4.1.1. We use the notations as in Theorem 4.1.1. We define that a set of elements $\{n_1, n_2, \ldots, n_u\}$ in M is d-basic if $\{n_1, \ldots, n_u\}$ is $\min\{u, d(\wp) + 1\}$-fold basic in M at all $\wp$ in X.

It follows that an element m in M is basic in M on X if and only if $\{m\}$ is d-basic. It also follows from the hypothesis that $\{m_1, m_2, \dots, m_r\}$ is d-basic.

If $r > 1$, we will show that there are elements $a_1, \dots, a_{r-1}$ in A such that $\{m_1 + aa_1 m_r, m_2 + a_2 m_r, \dots, m_{r-1} + a_{r-1} m_r\}$ is d-basic.

Let $N = \sum_{i=1}^{r} A m_i$. We claim that there are only finitely many primes $\wp$ in X such that N is not $\min\{r, d(\wp)+2\}$-fold basic at $\wp$. To see this let $X_s = \{\wp \textit{ in } X : d(\wp) = s\}$ for a nonnegative integer s. If $\wp'$ is in X and $\wp' \ll \wp$ for some $\wp$ in X_s with $\wp \neq \wp'$, then $d(\wp') > d(\wp) = s$. So, $\min\{r, d(\wp')+1\} \geq \min\{r, s+2\}$ and hence N is $\min\{r, s+2\}$-fold basic at $\wp'$. By Lemma 4.1.1, there are only finitely many primes $\wp$ in X_s such that N is not $\min\{r, s+2\}$-fold basic at $\wp$. Hence the claim is established.

Let $E = \{\wp \text{ in } X : N \text{ is not } \min\{r, d(\wp)+2\}\text{-fold basic in } M \text{ at } \wp\}$. Since E is finite, by Lemma 4.1.3, there are elements $a_1, a_2, \dots, a_{r-1}$ in A such that

$$N' = A(m_1 + aa_1 m_r) + A(m_2 + a_2 m_2) + \cdots + A(m_{r-1} + a_{r-1} m_r)$$

is $\min(r-1, d(\wp)+1)$-fold basic at $\wp$, for all $\wp$ in E.

Now, if $\wp$ is in $X \backslash E$, then $\mu((M/N)_\wp) \leq \mu(M_\wp)$ - $\min(r, d(\wp)+2)$ and hence

$$\mu((M/N')_\wp) \leq \mu((M/N)_\wp) + 1 \leq \mu(M_\wp) - \min\{r-1, d(\wp)+1\}.$$

So, N' is $\min\{r-1, d(\wp)+1\}$-fold basic at $\wp$.

Hence $\{m_1 + aa_1 m_r, m_2 + a_2 m_r, \dots, m_{r-1} + a_{r-1} m_r\}$ is d-basic. By induction, the proof of (iib) of Theorem 4.1.1 of Eisenbud and Evans is complete.

4.2 Applications of Eisenbud-Evans Theorem

In this section, we derive most of the main theorems about modules over noetherian commutative rings, in this theory, as applications of the Eisenbud-Evans Theorem.

First we derive the theorem of Serre about splitting projective modules.

Theorem 4.2.1 (Serre) *Let A be a noetherian commutative ring of dimension d and let P be a finitely generated projective A-module such that rank $(P_\wp) > d$ for all $\wp$ in Spec(A). Then $P \approx Q \oplus A$ for some projective A-module Q.*

To prove this theorem, we need the following definition and the lemma.

Definition 4.2.1 Let M be a module over a noetherian commutative ring A and let m be an element of M. We define the *order ideal* of m as

$$O(m, M) = O(m) = \{f(m) : f : M \to A \textit{ is an } A-\textit{linear map}\}.$$

We say that m is *unimodular in* M if $O(m, M) = A$, i.e. there is an A-linear map $f : M \to A$ such that $f(m) = 1$. Equivalently we have, m is a unimodular element in M if and only if $M = N \oplus Am \approx N \oplus A$ for a submodule N of M. It is also easy to see that a unimodular elment of M is also a basic element in M.

The set of all unimodular elements in M will be denoted by $Um(M)$.

Exercise 4.2.1 Suppose A is a noetherian commutative ring.

1. Then an element $(x_1, x_2, \ldots, x_n)$ in the free module A^n is a unimodular element if and only if $(x_1, x_2, \ldots, x_n)$ is a unimodular row.

2. For a finitely generated A-module M and m in M, we have
$$O(m/1, S^{-1}M) = S^{-1}O(m, M)$$
for any multiplicative subset S of A.

Lemma 4.2.1 Let P be a finitely generated projective module over a noetherian commutative ring A and let p be in P. Then p is unimodular in P if and only if p is basic in P.

Proof. Since $P_\wp$ is free $A_\wp$-module for all $\wp$ in $Spec(A)$ the lemma follows immediately from (1) and (2) of Exercise 4.2.1.

Proof of Serre's Theorem 4.2.1. Let $\delta : Spec(A) \to \{0, 1, 2, \ldots\}$ be the usual dimension function (see Example 4.1.1). Since rank $(P_\wp) > \dim A \geq \delta(\wp)$ for all $\wp$ in $Spec(A)$, by Eisenbud-Evans Theorem 4.1.1, P has a basic element p. By Lemma 4.2.1, there is an A-linear map $f : P \to A$ so that $f(p) = 1$. Let Q be the kernel of f. Then the exact sequence
$$0 \longrightarrow Q \longrightarrow P \xrightarrow{f} A \longrightarrow 0$$
splits. Hence $P \approx Q \oplus A$. In fact, $P = Q \oplus Ap$. So, the proof is complete.

Next we state and prove the cancellation theorem of Bass.

Theorem 4.2.2 (Bass) *Let P be a finitely generated projective module over a noetherian commutative ring A, with rank $(P_\wp) > \dim A$, for all $\wp$ in $Spec(A)$. Suppose that $P \oplus Q \approx P' \oplus Q$ for some finitely generated projective A-modules P' and Q. Then $P \approx P'$. (We say that P has the cancellation property).*

Proof. Since $Q \oplus Q'$ is free, for some finitely generated A-module Q', we can assume, by downward induction, that $Q = A$. Hence we have an isomorphism $f : P' \oplus A \longrightarrow P \oplus A$. Let $f(0,1) = (p,a)$. Since (p,a) is unimodular, it is also basic in $P \oplus A$. By (iib) of Theorem 4.1.1, we can find an element p' in P such that $p + ap'$ is basic and hence unimodular in P. Let $p_0 = p + ap'$, then there is an A-linear map $g : P \longrightarrow A$ such that $g(p_0) = 1$. Now, we define isomorphisms $f_i : P \oplus A \longrightarrow P \oplus A$ for $i = 1, 2, 3$ as follows :

1. $f_1(m,x) = (m + xp', x)$,
2. $f_2(m,x) = (m, x + (1-a)g(m))$,
3. $f_3(m,x) = (m - xp_0, x)$

for m in P and x in A.
It is easy to check that if $F = f_3of_2of_1of$, then $F(0,1) = (0,1)$. Hence $F(0 \oplus A) = 0 \oplus A$. So, F will induce an isomorphism from $P' \approx (P' \oplus A)/(0 \oplus A)$ to $P \approx (P \oplus A)/(0 \oplus A)$. Hence the proof is complete.

Remark 4.2.1 Because of Example 2.5.2, Theorem 4.2.2 of Bass is the best possible in this generality.

The following is the theorem of Forster and Swan about the number of generators of modules.

Theorem 4.2.3 (Forster, Swan) *Let M be a finitely generated module over a noetherian commutative ring A. Let*

$$n = \max\{\mu(M_\wp) + \dim(A/\wp) : \wp \textit{ is a prime ideal with } M_\wp \neq 0\}.$$

Then M is generated by n-elements.

Proof. By replacing A by $A/ann(M)$, we can assume that $M_\wp \neq 0$ for all $\wp$ in $Spec(A)$.

Let $\mu(M) = u > n$. Then there is an exact sequence

$$0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$$

where $F = A^u$ is a free A-module of rank u. Now $\mu((F/K)_\wp) = \mu(M_\wp)$ for all $\wp$ in $Spec(A)$. Since $u > n$, we have $u > \mu(M_\wp) + \dim(A/\wp)$ for all $\wp$ in $Spec(A)$. Hence by (iia) of Theorem 4.1.1, K contains a basic element m of F. By Lemma 4.2.1, Am is a free direct summand of F. Hence $F = F' \oplus Am \approx F' \oplus A$ for some projective A-module F' of rank $u-1$. Since $u - 1 > \dim A$, by Theorem 4.2.2 of Bass, F' is free of rank $u-1$. Since m is in K ,we see that F' maps onto

M and hence $\mu(M) \leq u - 1$. This contradicts the fact that $\mu(M) = u$. Hence the theorem is proved.

The following theorem on the *set theoretic generation* of ideals was proved, independently, by Eisenbud-Evans and Storch. For polynomial rings over fields the theorem is due to Kronecker.

Theorem 4.2.4 (Kronecker, Eisenbud-Evans, Storch) *Let A be a noetherian commutative ring of dimension d and let I be an ideal of A. Then there are elements $f_1, \ldots, f_{d+1}$ such that $\sqrt{(f_1, f_2, \ldots, f_{d+1})} = \sqrt{I}$. (We say that I is set theoretically generated by $f_1, \ldots, f_{d+1}$).*

Proof. Let $\bar{A} = A/ann(I)$ and let $M = I^{d+1}$. Then M is an $\bar{A}$-module and $\mu(M_\wp) \geq d + 1 > \dim \bar{A}$ for all $\wp$ in $Spec(\bar{A})$. Hence there is $(f_1, \ldots, f_{d+1})$ in $I^{d+1} = M$ that is basic in M (as an $\bar{A}$-module). It is easy to see that for $\wp$ in $Spec(A)$, the ideal $(f_1, \ldots, f_{d+1}) \subseteq \wp$ if and only if $I \subset \wp$. Hence $\sqrt{(f_1, \ldots, f_{d+1})} = \sqrt{I}$ and the proof is complete.

Remark. There is an alternative proof of Theorem 4.2.4 by usual "prime avoidance argument". We shall be using the "prime avoidance argument" quite extensively in our later chapters.

The following is the stable range theorem of Bass.

Theorem 4.2.5 (Bass) *Let A be a commutative noetherian ring of dimension d. Let $(f_1, \ldots, f_n)$ be a unimodular row in A^n. If $n - 1 > \dim A = d$, then there exist $g_1, g_2, \ldots, g_{n-1}$ in A such that $(f_1 + g_1 f_n, f_2 + g_2 f_n, \ldots, f_{n-1} + g_{n-1} f_n)$ is a unimodular row.*

Proof. Let $F = A^{n-1}$ and let $m = (f_1, \ldots, f_{n-1})$ be in F. Since (m, f_n) is unimodular in $F \oplus A$ and since $\mu(F_\wp) = n - 1 > d$, by Eisenbud-Evans Theorem 4.1.1, there is $m' = (g_1, \ldots, g_{n-1})$ in F such that $m + f_n m'$ is basic in F. That means that $(f_1 + f_n g_1, \ldots, f_{n-1} + f_n g_{n-1})$ is a unimodular row. So, the proof of Theorem 4.2.5 is complete.

The following is the Plumstead's ([P]) version of Theorem 4.2.7 of Eisenbud and Evans ([EE1]) on generators of modules.

Theorem 4.2.6 (Eisenbud-Evans) *Let A be a noetherian commutative ring and let $d : X \longrightarrow \mathcal{N}$ be a generalized dimension function on a subset X of*

Spec(A). Let M be a finitely generated A-module and let N be a submodule of M. Suppose $m_1, m_2, \ldots, m_k$ be elements of M such that

$$(Am_1 + Am_2 + \cdots + Am_k + N)_\wp = M_\wp$$

for all $\wp$ in X. Assume that

$$k \geq \max\{\mu(M_\wp) + d(\wp) : \wp \;\; in \;\; X \;\; with \;\; N_\wp \not\subseteq \wp M_\wp\}.$$

Then there exist elements $m_i' = m_i + n_i$ where n_i is in N,for $i = 1$ to k, such that

$$(Am_1' + Am_2' + \cdots + Am_k')_\wp = M_\wp$$

for all $\wp$ in X. (We say that $m_1', m_2', \ldots, m_k'$ generate M on X).

Proof of Theorem 4.2.6. Let

$$N = Am_{k+1} + Am_{k+2} + \cdots + Am_t$$

for some $m_{k+1}, m_{k+2}, \ldots, m_t$ in N. We will show that if $t > \mu(M_{\wp'}) + d(\wp')$ for some $\wp'$ in X with m_t not in $\wp' M_{\wp'}$, then there are $a_1, a_2, \ldots, a_{t-1}$ in A such that

$$(A(m_1 + a_1 m_t) + A(m_2 + a_2 m_t) + \cdots + A(m_{t-1} + a_{t-1} m_t))_\wp = M_\wp$$

for all $\wp$ in X.

Let $X' = \{\wp$ in X : m_t is not in $\wp M_\wp\}$ and let

$$t_0 = \max\{\mu(M_\wp) + d(\wp) : \wp \;\; in \;\; X'\}.$$

If $t_0 = 0$, then X' is empty, i.e. m_t is in $\wp M_\wp$ for all $\wp$ in X. If we take $a_1 = a_2 = \cdots = a_{t-1} = 0$, then the assertion holds by Nakayama's Lemma. So, we assume that $t > t_0 > 0$ and use induction on t_0.

First we show that if $t_0 = \mu(M_\wp) + d(\wp)$ for some $\wp$ in X', then $\wp$ is minimal in $V(I_r(M, A)) \bigcap X$ with respect to $\ll$ for some r. Let $\mu(M_\wp) = u + 1$. Then it follows that $I_u(M, A) \subseteq \wp$ and $I_{u+1}(M, A) \not\subseteq \wp$. If $\wp$ is not minimal over $I_u(M, A) \bigcap X$ for $\ll$, then there is $\wp$ in X such that $I_u(M, A) \subseteq \wp' \subseteq \wp$ and $d(\wp') > d(\wp)$. Hence $\mu(M_{\wp'}) = u + 1 = \mu(M_\wp)$. So,

$$\mu(M_{\wp'}) + d(\wp') = \mu(M_\wp) + d(\wp') > \mu(M_\wp) + d(\wp) = t_0.$$

Since $\mu(M_\wp) = u + 1$, and m_t is not in $\wp M_\wp$, we see that $M_\wp$ is generated by $m_t, x_1, \ldots, x_u$ for some $x_1, \ldots, x_u$ in M. Hence $M_{\wp'}$ is also generated by $m_t, x_1, \ldots x_u$. Since $\mu(M_{\wp'}) = u + 1$, we have $m_t, x_1, \ldots, x_u$ is , in fact, a minimal set of generators of $M_{\wp'}$. So, m_t is not in $\wp' M_{\wp'}$ and hence $\wp'$ is in X'. This is a contradiction because $\mu(M_{\wp'}) + d(\wp') > t_0$. Hence we have established that if $t_0 = \mu(M_\wp) + d(\wp)$ for some $\wp$ in X', then $\wp$ is minimal over

$I_r(M, A) \bigcap X$ for some r. Hence there are only finitely many primes $\wp$ in X' such that $t_0 = \mu(M_\wp) + d(\wp)$.

Let $E = \{\wp \text{ in } X' : \mu(M_\wp) + d(\wp) = t_0\}$. Now E is a finite set, and $m_1, m_2, \ldots, m_t$ is 1-fold basic in M on E. Since $(m_1, 1)$ is basic in $M \oplus A$, by repeated application of Lemma 4.1.2, there are $b_2, b_3, \ldots, b_t$ in A such that $m' = m_1 + b_2 m_2 + \cdots + b_t m_t$ is basic in M on E.

Now let $M' = M/Am'$. Then for $\wp$ in X', with $\mu(M_\wp) + d(\wp) = t_0$, we have $\mu(M'_\wp) < \mu(M_\wp)$. Let " $-$ " bar denote the images in M'. Then

$$(A\bar{m}_2 + A\bar{m}_3 + \cdots + A\bar{m}_t)_\wp = M'_\wp$$

for all $\wp$ in X. Also $\bar{m}_t$ is not in $\wp M'_\wp$ implies that $\wp$ is in X'. Hence

$$t'_0 = \max\{\mu(M'_\wp) + d(\wp) \ : \ \wp \textit{ in } X \textit{ and } \bar{m}_t \textit{ is not in } \wp M'_\wp\} < t_0.$$

Hence by induction, there are $a_2, \ldots, a_{t-1}$ in A such that

$$(A\overline{m_2 + a_2 m_t} + \cdots + A\overline{m_{t-1} + a_{t-1} m_t})_\wp = M'_\wp,$$

for all $\wp$ in X. Therefore

$$(Am' + A(m_2 + a_2 m_t) + \cdots + A(m_{t-1} + a_{t-1} m_t))_\wp = M_\wp$$

for all $\wp$ in X. Write $a_1 = b_t - \sum_{i=2}^{t-1} b_i a_i$. Then $m' = m_1 + b_2 m_2 + \cdots + b_t m_t =$

$$(m_1 + a_1 m_t) + b_2(m_2 + a_2 m_t) + b_2(m_3 + a_3 m_t) + \cdots + b_{t-1}(m_{t-1} + a_{t-1} m_t).$$

Hence

$$((A(m_1 + a_1 m_t) + A(m_2 + a_2 m_t) + \cdots + A(m_{t-1} + a_{t-1} m_{t-1})_\wp = M_\wp$$

for all $\wp$ in X. So, the proof of Theorem 4.2.6 is complete.

The following is the original version of the theorem of Eisenbud and Evans on generators of modules.

Theorem 4.2.7 (Eisenbud-Evans, [EE1]) *Let A be a noetherian commutative ring and let M be a finitely generated A-module . Suppose N is a submodule of M and $m_1, m_2, \ldots, m_k$ are elements of M such that*

$$Am_1 + Am_2 + \cdots + Am_k + N = M.$$

Assume that

$$k \geq \max\{\mu(M_\wp) + \dim(A/\wp) : \wp \ \textit{ in } \ Spec(A) \ \textit{ with } \ N_\wp \not\subseteq \wp M_\wp\}.$$

Then there exist elements $m'_i = m_i + n_i$ where n_i is in N, for $i = 1$ to k, such that $\{m'_1, m'_2, \ldots, m'_k\}$ generates M.

Proof. Let $d : Spec(A) \to \mathcal{N}$ be the usual dimension function (see Example 4.1.1). Now the theorem follows by a direct application of Theorem 4.2.6.

Although it is not an application of Eisenbud-Evans Theorem, this may be the best place to give a proof of the theorem on cancellation of rank one projective modules as follows.

Theorem 4.2.8 *Let A be a noetherian commutative ring and let L be a projective A-module of constant rank one. Then L has the cancellative property, i.e. $L \oplus Q \approx L' \oplus Q$ for some finitely generated projective A-modules L' and Q implies that $L \approx L'$.*

Proof. Let $L \oplus Q \approx L' \oplus Q$. By tensoring with $L'^{-1} = \text{Hom }(L', A)$, we get $LL'^{-1} \oplus Q' \approx A \oplus Q'$ where $Q' = Q \otimes L'^{-1}$. Since it is enough to prove that $LL'^{-1} \approx A$, we can assume that $L' = A$.

So, we have $L \oplus Q \approx A \oplus Q$. Since $Q \oplus Q_1 \approx A^n$ is free for some Q_1, we have $L \oplus A^n \approx A^{n+1}$ for some integer $n \geq 0$.

Let $f : L \oplus A^n \longrightarrow A^{n+1}$ be an isomorphism. Let

$$e_2 = (0, 1, 0, \ldots, 0), e_3 = (0, 0, 1, 0, \ldots, 0), \ldots, e_{n+1} = (0, 0, \ldots, 1)$$

be the standard basis of A^n in $L \oplus A^n$ and let $e'_1, e'_2, \ldots, e'_{n+1}$ be the standard basis of A^{n+1}. For x in L, let

$$\begin{pmatrix} f(x) \\ f(e_2) \\ f(e_3) \\ \vdots \\ f(e_{n+1}) \end{pmatrix} = u(x) \begin{pmatrix} e'_1 \\ e'_2 \\ \vdots \\ e'_{n+1} \end{pmatrix}$$

where $u(x)$ is an $(n+1) \times (n+1)$- matrix in $\mathcal{M}_{n+1}(A)$. Define a map $F : L \longrightarrow A$ as $F(x) = \det u(x)$. Note that F is an A-linear map. Since $L_\wp \approx A_\wp$ for all $\wp$ in $Spec(A)$, and $f_\wp : A_\wp^{n+1} \longrightarrow A_\wp^{n+1}$ is an isomorphism, $F_\wp(e)$ is a unit in $A_\wp$ if e is a generator of $L_\wp$. Hence $F_\wp$ is an isomorphism for all $\wp$ in $Spec(A)$. So, $F : L \longrightarrow A$ is an isomorphism. So, the proof of Theorem 4.2.8 is complete.

This proof of Theorem 4.2.8 is an improvization of the usual proof given by taking exterior power.

4.3 The Modules over Polynomial Rings

In this section we shall discuss some of the main results, in this theory, about modules over polynomial rings. We shall split this section under several subheadings.

4.3.1 Eisenbud-Evans Conjectures

In [EE2], Eisenbud and Evans proposed three conjectures, about modules over polynomial rings as follows.

Conjecture 4.3.1 Let $R = A[X]$ be a polynomial ring in a single variable X over a noetherian commutative ring A of dimension d.

> **Conjecture I.** Let M be a finitely generated R-module with $\mu(M_\wp) \geq d+1$ for all $\wp$ in $Spec(R)$. Then M has a basic element. In particular, if P is a finitely generated projective R-module with rank $(P_\wp) \geq d+1$ for all $\wp$ in $Spec(R)$, then P has a free direct summand.
>
> **Conjecture II.** Let P be a finitely generated projective R-module with rank$(P_\wp) \geq d+1$ for all $\wp$ in $Spec(R)$. Then P has the cancellative property i.e. $P \oplus Q \approx P' \oplus Q$ for some finitely generated projective R-modules Q and P' implies that $P \approx P'$.
>
> **Conjecture III.** Let M be a finitely generated projective R-module and let
>
> $$e(M) = \max\{\mu(M_\wp) + \dim R/\wp : \wp \textit{ is in } Spec(R) \textit{ with } \dim R/\wp < d+1\}$$
>
> then M is generated by $e(M)$ elements.

The Conjecture III was proved first, by Sathaye ([Sa]) for affine domains A over infinite fields and then was proved completely by Mohan Kumar ([MK1]). Later Plumstead ([P]) proved the conjectures I and II. Plumstead also gave a proof of the conjecture III. In these notes we shall give the proofs of Plumstead ([P]) for all the three conjectures above.

4.3.2 Some Preliminaries from Plumstead's Work

The following is a version of the Quillen's Lemma 3.1.1.

Lemma 4.3.1 (Quillen,[P],[Ma2]) Let A be a commutative ring and R be an A-algebra (that is not necessarily commutative). Let f be an element in A and X be a variable. Let θ be a unit in $1 + XR_f[X]$. Then there is an integer $k \geq 0$, such that for any g_1, g_2 in A with $g_1 - g_2$ in $f^k A$, there is a unit ψ in $1 + XR[X]$ such that $\psi_f(X) = \theta_1(g_1 X)\theta(g_2 X)^{-1}$.

Further, if $h : R \longrightarrow R'$ is a ring homomorphism and the image of θ in $R'_f[X]$ is one, then ψ can be chosen with the property that the image of ψ in $R'[X]$ is also one.

Proof. The first part of Lemma 4.3.1 is in fact the statement of Lemma 3.1.1. The proof of the last part is exactly similar to the proof of Lemma 3.1.1, where we also have $h(a_i) = 0 = h(b_i)$ for all $i > 0$. (Here a_i, b_i are as in the proof of Lemma 3.1.1.

For an A-module M we write $M[X]$ for $M \otimes_A A[X]$. We define *isotopy of module isomorphisms* as follows. This definition of isotopy of isomorphisms is an important idea for us.

Definition 4.3.1 Let A be a noetherian commutative ring and let M, M' be A-modules. Suppose $f, g : M \longrightarrow M'$ be two isomorphisms. We say that f *is isotopic to* g if there is an isomorphism $\varphi : M[X] \longrightarrow M'[X]$, where X is a variable, such that $\varphi(0) = f$ and $\varphi(1) = g$.

Further, if $h : A \longrightarrow A'$ is a homomorphism of commutative rings then we say that f *is isotopic to* g *relative to* h, if we can also chose φ such that $\varphi \otimes_A A' = f \otimes_A A'[X]$ is a constant map.

Remark 4.3.1 Most often we will consider isotopies of isomorphisms relative to a map $A \longrightarrow A/I$, for an ideal I of A.

Remark 4.3.2 It is also easy to see that the isotopy of automorphisms of a module is an equivalence relation.

The following is the Plumstead's version of the Quillen's Lemma 3.1.2 about patching via isotopic isomorphisms.

Lemma 4.3.2 (Plumstead) Let A be a commutative noetherian ring and let s_1, s_2 in A be such that $As_1 + As_2 = A$. Let M and M' be finitely generated A-modules and let $f_i : M_{s_i} \longrightarrow M'_{s_i}$ be isomorphisms for $i = 1, 2$. If $(f_1)_{s_2}$ is isotopic to $(f_2)_{s_1}$, then there is an isomorphism $f : M \longrightarrow M'$. Further, if $h : A \longrightarrow A'$ is a homomorphism of commutative rings and $(f_1)_{s_2}$ is isotopic to $(f_2)_{s_1}$ relative to h, then the isomorphism $f : M \longrightarrow M'$ can be chosen such that $f_{s_i} \otimes A' = (f_i) \otimes A'$ for $i = 1, 2$.

Proof. Let $g = (f_2)_{s_1}^{-1} o (f_1)_{s_2} : M_{s_1 s_2} \longrightarrow M_{s_1 s_2}$. If $g = (g_2)_{s_1} o (g_1)_{s_2}$ for some isomorphisms $g_i : M_{s_i} \longrightarrow M_{s_i}$ for $i = 1, 2$ then $(f_1 o g_1^{-1})_{s_2} = (f_2 o g_2)_{s_1}$. In that case, by Proposition 2.2.1, there will be an isomorphism $f : M \longrightarrow M'$ with $(f)_{s_1} = f_1 o g_1^{-1}$ and $(f)_{s_2} = f_2 o g_2$.

Now since g is isotopic to $Id_{M_{s_1s_2}}$, there is an isomorphism

$$F(X) : M_{s_1s_2}[X] \longrightarrow M_{s_1s_2}[X]$$

such that $F(0) = Id$ and $F(1) = g$. Hence $g = F(1)F(a)^{-1}F(a)$ for any a in A.

We consider F as an element of $End(M)_{s_1s_2}[X]$ and apply Quillen's Lemma 4.3.1, for $A_{s_2} \longrightarrow End(M_{s_1s_2})$ with $''f = s_1''$ and then for $A_{s_1} \longrightarrow End(M_{s_1s_2})$ with $''f = s_2''$. So, there is an integer $k \geq 0$ such that $F(X)F(aX)^{-1}$ is in the image of $Aut(M_{s_2}[X])$ if $1-a$ is in s_1^kA and $F(aX) = F(aX)F(0X)^{-1}$ is in the image of $Aut(M_{s_1}[X])$ if a is in s_2^kA. Since $As_1 + As_2 = A$, we have $1 = cs_1^k + ds_2^k$ for some c, d in A. Taking $a = ds_1^k$ and $X = 1$, we have $F(1)F(a)^{-1} = (g_2)_{s_1}$ and $F(a) = (g_1)_{s_1}$, for some isomorphisms $g_i : M_{s_i} \longrightarrow M_{s_i}$ for $i = 1, 2$. So, we have $g = (F(1)F(a)^{-1})F(a) = (g_2)_{s_1}o(g_1)_{s_2}$ as desired.

For the relative case, we apply Lemma 4.3.1 relative to the maps

$$End(M)_{s_i} \longrightarrow End(M \otimes A')_{s_i}$$

for $i = 1, 2$. In that case we can assume that $F(X) \otimes A'$ is $Id_{M\otimes A'[X]}$. Hence we can assume $g_i \otimes A' = Id$. Therefore

$$(f)_{s_1} \otimes A' = (f_1 o g_1^{-1}) \otimes A' = f_1 \otimes A' \quad \text{and} \quad (f)_{s_2} \otimes A' = (f_2 \otimes g_2) \otimes A' = f_2 \otimes A'$$

as desired. So, the proof of Lemma 4.3.2 is complete.

The following patching lemma is a consequence of Lemma 4.3.2.

Lemma 4.3.3 (Plumstead) Let A be a commutative noetherian ring and let $R = A[X]$ be the polynomial ring. Let s_1, s_2 in A be such that $As_1 + As_2 = A$. For two R-modules M and M', let

$$f_1 : M_{s_1} \longrightarrow M'_{s_1} \quad \text{and} \quad f_2 : M_{s_2} \longrightarrow M'_{s_2}$$

be two isomorphisms such that $(f_1)_{s_2} \equiv (f_2)_{s_1}$ (modulo X). Also assume that $M_{s_1s_2}$ is extended from $A_{s_1s_2}$. Then there is an isomorphism $f : M \longrightarrow M'$ such that $(f)_{s_i} \equiv f_i$ (modulo X).

Proof. Since $M_{s_1s_2}$ is extended, $M_{s_1s_2} \approx \bar{M}_{s_1s_2}[X]$, where $\bar{M} = M/XM$. Consider $\omega = (f_2^{-1})_{s_1}o(f_1)_{s_2}$ as an element of $End(\bar{M}_{s_1s_2})[X]$. Clearly, $\omega(0) = Id_{\bar{M}_{s_1s_2}}$. Therefore $F(T) = \omega(XT)$ will define an isotopy from $Id_{M_{s_1s_2}}$ to ω, relative to the map $R_{s_1s_2} \longrightarrow R_{s_1s_2}/XR_{s_1s_2}$. Hence the lemma follows from Lemma 4.3.2.

The following is a lemma of Plumstead ([P]) on *patching basic elements.*

Lemma 4.3.4 (Plumstead) Let $R = A[X]$ be a polynomial ring over a reduced noetherian commutative ring A and let M be a finitely generated R-module. Let s_1 and s_2 in A be such that $As_1 + As_2 = A$. (Barring " –" will denote "(modulo X)" in this lemma). Let m_i be a basic element in M_{s_i} on $Spec(R_{s_i})$ for $i = 1, 2$ such that $\bar{m}_1 = (z)_{s_1}$ and $\bar{m}_2 = (z)_{s_2}$ for some z in $\bar{M}$. Let $N_1 = M_{s_1}/R_{s_1}m_1$ and $N_2 = M_{s_2}/R_{s_2}m_2$ and assume that $(N_1)_{s_2}$ and $(N_2)_{s_1}$ are extended projective $R_{s_1 s_2}$ - modulus. Then there is a basic element m in M such that $\bar{m} = z$.

Proof. Since $\bar{m}_1 = (z)_{s_1}$ and $\bar{m}_2 = (z)_{s_2}$, we have

$$(\bar{N}_1)_{s_2} \approx \bar{M}_{s_1 s_2}/A_{s_1 s_2}z \approx (\bar{N}_2)_{s_1}.$$

We identify both $(\bar{N}_1)_{s_2}$ and $(\bar{N}_2)_{s_1}$ with $N = (\bar{M}/Az)_{s_1 s_2}$ in the natural way. Since $(N_1)_{s_2}$ and $(N_2)_{s_1}$ are extended projective modules, there is an isomorphism $f_0 : (N_1)_{s_2} \longrightarrow (N_2)_{s_1}$ such that $\bar{f}_0 = Id_N$. The sequences

$$0 \longrightarrow R_{s_1 s_2} \xrightarrow{m_1} M_{s_1 s_2} \longrightarrow (N_1)_{s_2} \longrightarrow 0$$

and

$$0 \longrightarrow R_{s_1 s_2} \xrightarrow{m_2} M_{s_1 s_2} \longrightarrow (N_2)_{s_1} \longrightarrow 0$$

are split exact sequences.

We can find splittings

$$\lambda_1 : (N_1)_{s_2} \longrightarrow M_{s_1 s_2} \quad \text{of} \quad M_{s_1 s_2} \longrightarrow (N_1)_{s_2} \quad \text{and}$$

$$\lambda_2 : (N_2)_{s_1} \longrightarrow M_{s_1 s_2} \quad \text{of} \quad M_{s_1 s_2} \longrightarrow (N_2)_{s_1} \quad \text{such that} \quad \bar{\lambda}_1 = \bar{\lambda}_2.$$

Using λ_1 and λ_2, we can define an isomorphism $f : M_{s_1 s_2} \longrightarrow M_{s_1 s_2}$ such that the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R_{s_1 s_2} & \xrightarrow{m_1} & M_{s_1 s_2} & \longrightarrow & (N_1)_{s_2} & \longrightarrow & 0 \\
 & & || & & \downarrow f & & \downarrow f_0 & & \\
0 & \longrightarrow & R_{s_1 s_2} & \xrightarrow{m_2} & M_{s_1 s_2} & \longrightarrow & (N_2)_{s_1} & \longrightarrow & 0
\end{array}$$

commutes. After tensoring with $R_{s_1 s_2}/XR_{s_1 s_2}$, the above diagram reduces to

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bar{R}_{s_1 s_2} & \xrightarrow{z} & \bar{M}_{s_1 s_2} & \longrightarrow & N & \longrightarrow & 0 \\
 & & || & & \downarrow \bar{f} & & \downarrow Id & & \\
0 & \longrightarrow & \bar{R}_{s_1 s_2} & \xrightarrow{z} & \bar{M}_{s_1 s_2} & \longrightarrow & N & \longrightarrow & 0 \ .
\end{array}$$

Since $\bar{\lambda}_1 = \bar{\lambda}_2$ we have $\bar{f} = Id$.

Now let M' (*respectively* N') be the R-module found by patching M_{s_1} and M_{s_2} via f (*respectively* N_{s_1} and N_{s_2} via f_0). We get the following two fiber product diagrams.

$$\begin{tikzcd}
R \arrow[rr] \arrow[dr, "g'"] \arrow[dd] & & R_{s_2} \arrow[dr, "m_2"] \arrow[dd] & \\
& M' \arrow[rr] \arrow[dd] & & M_{s_2} \arrow[dd] \\
R_{s_1} \arrow[rr] \arrow[dr, "m_1"] & & R_{s_1s_2} \arrow[dr] & \\
& M_{s_1} \arrow[rr, "F"] & & M_{s_1s_2}
\end{tikzcd}$$

$$\begin{tikzcd}
M' \arrow[rr] \arrow[dr, "g"] \arrow[dd] & & Ms_2 \arrow[dr] \arrow[dd] & \\
& N' \arrow[rr] \arrow[dd] & & N_2 \arrow[dd] \\
M_{s_1} \arrow[rr, "F"] \arrow[dr] & & M_{s_1s_2} \arrow[dr] & \\
& N_1 \arrow[rr, "F_0"] & & N_{2s_1}
\end{tikzcd}$$

Here $F : M_{s_1} \to M_{s_1s_2}$ is the composition map

$$M_{s_1} \to M_{s_1s_2} \xrightarrow{f} M_{s_1s_2}$$

and $F_0 : N_1 \to N_{2s_1}$ is the composition map

$$F_0 : N_1 \to N_{1s_2} \xrightarrow{f_0} N_{2s_1}.$$

In these diagrams, g' and g are found by the properties of fiber product and all rectangles commute. Let $h_i : M'_{s_i} \longrightarrow M_{s_i}$ for $i = 1, 2$ be the natural isomorphisms. Since $(h_2)_{s_1} o (h_1^{-1})_{s_2} = f \equiv Id$ (modulo X), by Lemma 4.3.3 there is an isomorphism $h : M' \longrightarrow M$ such that $h_{s_i} \equiv h_i$ (modulo X). Let $g'(1) = m'$ and $h(m') = m$. Then, it follows that m' is basic in M' and hence m is basic in M. Since $(\bar{m})_{s_i} = \overline{h(m')}_{s_i} = \overline{h_i(m'_i)} = (z)_{s_i}$, it follows that $\bar{m} = z$. This completes the proof of Lemma 4.3.4.

The following Lemma of Plumstead is about patching generators of modules over polynomial rings.

Lemma 4.3.5 (Plumstead) Suppose $R = A[X]$ is a polynomial ring over a noetherian commutative ring A and let M be a finitely generated R-module.

Let s_1, s_2 in A be such that $As_1 + As_2 = A$ and let $z_1, z_2, \ldots, z_k$ generate $\bar{M} = M/XM$. (Barring " $-$ " will denote "(modulo X)" in this lemma).

Let $m'_1, \ldots, m'_k$ generate M_{s_1} and let $m''_1, m''_2, \ldots, m''_k$ generate M_{s_2}, such that $\bar{m}'_i = (z_i)_{s_1}$ and $\overline{m''_i} = (z_i)_{s_2}$ for $i = 1$ to k. For $i = 1, 2$ let $g_i : R^k_{s_i} \longrightarrow M_{s_i}$ be the maps defined by the corresponding generators and let $L_i =$ kernel (g_i). Assume that $M_{s_1 s_2}$ is projective and $(L_1)_{s_2}, (L_2)_{s_1}$ are extended. Then there exist $m_1, m_2, \ldots, m_k$ in M that generate M and $\bar{m}_i = z_i$ for $i = 1$ to k.

Proof. The sequences

$$0 \longrightarrow (L_1)_{s_2} \longrightarrow R^k_{s_1 s_2} \xrightarrow{g_1} M_{s_1 s_2} \longrightarrow 0$$

and

$$0 \longrightarrow (L_2)_{s_1} \longrightarrow R^k_{s_1 s_2} \xrightarrow{g_2} M_{s_1 s_2} \longrightarrow 0$$

are split exact sequences (here we continue to denote $(g_1)_{s_2}$ by g_1 and $(g_2)_{s_1}$ by g_2). Since $(L_1)_{s_2}$ and $(L_2)_{s_1}$ are extended and since

$$(\bar{L}_1)_{s_2} = \ker(\bar{g}_1) = \ker(\bar{g}_2) = (\bar{L}_2)_{s_1},$$

we have L_1 and L_2 are isomorphic. Also since $\bar{g}_1 = \bar{g}_2$, we can find splittings

$$\lambda_i : M_{s_1 s_2} \longrightarrow R^k_{s_1 s_2} \quad \text{of} \quad g_i \quad \text{such that} \quad \bar{\lambda}_1 = \bar{\lambda}_2.$$

Using the splittings λ_1 and λ_2, and an isomorphism $f_0 : (L_1)_{s_2} \longrightarrow (L_2)_{s_1}$, we can define an isomorphism $f : R^k_{s_1 s_2} \longrightarrow R^k_{s_1 s_2}$ such that the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (L_1)_{s_2} & \longrightarrow & R^k_{s_1 s_2} & \xrightarrow{g_1} & M_{s_1 s_2} & \longrightarrow & 0 \\
 & & \downarrow f_0 & & \downarrow f & & || & & \\
0 & \longrightarrow & (L_2)_{s_1} & \longrightarrow & R^k_{s_1 s_2} & \xrightarrow{g_2} & M_{s_1 s_2} & \longrightarrow & 0
\end{array}$$

commutes and $\bar{f} = Id$.

Let Q be the R-module found by patching $R^k_{s_1}$ and $R^k_{s_2}$ via f. We get the following fiber product diagram.

$$\begin{tikzcd}
Q \arrow[rr] \arrow[dr, "g"] \arrow[dd] & & R^k_{s_2} \arrow[dr, "g_2"] \arrow[dd] & \\
 & M \arrow[rr] \arrow[dd] & & M_{s_2} \arrow[dd] \\
R^k_{s_1} \arrow[rr, "F"] \arrow[dr, "g_1"] & & R^k_{s_1 s_2} \arrow[dr] & \\
 & M_{s_1} \arrow[rr] & & M_{s_1 s_2}
\end{tikzcd}$$

Here $F : R^k_{s_1} \to R^k_{s_1 s_2}$ is the composition map

$$R^k_{s_1} \to R^k_{s_1 s_2} \xrightarrow{f} R^k_{s_1 s_2}.$$

The map g is found by the properties of fiber product diagrams. Since g_1 and g_2 are surjective, so is g. Let $f_i : Q_{s_i} \longrightarrow R^k_{s_i}$ be the natural isomorphisms for $i = 1, 2$. Since $\overline{f_2 o(f_1^{-1})} = \bar{f} = Id$, by Lemma 4.3.3 there is an isomorphism $h : Q \longrightarrow R^k_{s_i}$ such that $(\bar{h})_{s_1} = \bar{f}_1$ and $(\bar{h})_{s_2} = \bar{f}_2$. If $e'_1, \ldots, e'_k$ are elements of Q that correspond to the natural basis of R^k and if $g(e'_i) = m_i$, then $m_1, m_2, \ldots, m_k$ generate M and $\bar{m}_i = z_i$ for $i = 1$ to k. This completes the proof of Lemma 4.3.5.

Before we give the proofs of the conjectures of Eisenbud and Evans, we give two examples of generalized dimension functions constructed by Plumstead ([P]).

Example 4.3.1 (Plumstead) Let R be a commutative ring and let

$$d_i : V_i \longrightarrow \mathcal{N},$$

for $i = 1$ to k, be generalized dimension functions on subsets V_i of $Spec(R)$. Define

$$d : \bigcup_{i=1}^{k} V_i \longrightarrow \mathcal{N} \quad \text{by} \quad d(\wp) = \max\{d_i(\wp) : \wp \in V_i \ for\ i = 1\ to\ k\}.$$

Then d is a generalized dimension function.

The proof of Example 4.3.1 is easy and is left to the reader.

Example 4.3.2 (Plumstead) Let A be a commutative noetherian ring and let the radical of A contain an element s with $\dim(A/sA) < \dim A$. Then there is a generalized dimension function $d : Spec(A[X]) \longrightarrow \mathcal{N}$ such that $d(\wp) \leq \dim A$ for all $\wp$ in $Spec(A[X])$.

Proof. Let $V_1 = V(s)$ be the set of all prime ideals $\wp$ of $Spec(A[X])$ that contain s and $d_1 : V_1 \longrightarrow \mathcal{N}$ be defined as

$$d_1(\wp) = \dim(A[X]/\wp)$$

for $\wp$ in V_1. Let V_2 be the set of all prime ideals in $Spec(A[X])$ of height less than or equal to $\dim A$. Define $d_2 : V_2 \longrightarrow \mathcal{N}$ as $d_2(\wp) =$

$$\max\{k :\ there\ is\ a\ chain\ \wp_0 = \wp \subset \wp_1 \subset \wp_2 \subset \ldots \subset \wp_k\ with\ \wp_i\ in\ V_2\}.$$

Since d_1 and d_2 are generalized dimension functions and $Spec(A[X]) = V_1 \bigcup V_2$, the example follows from Example 4.3.1.

4.3.3 The Proofs of Eisenbud-Evans Conjectures

Under this subheading we shall give the proofs of the conjectures stated in the above subsection 4.3.1. We shall give the proofs of Plumstead ([P]). First we prove conjecture II.

Theorem 4.3.1 (Plumstead) *Let $R = A[X]$ be a polynomial ring over a noetherian commutative ring A and let P be a finitely generated projective R-module with rank $(P_\wp) \geq \dim R/\wp$ for all minimal primes $\wp$ of R. Then P has the cancellative property, i.e. $P \oplus Q \approx P' \oplus Q$ for some finitely generated projective R-modules P' and Q implies that $P \approx P'$.*

Proof. First since $Q \oplus Q' = R^k$ is free for some projective R-module Q, we can assume that $Q = R$. Hence we have $R \oplus P \approx R \oplus P'$.

If A has any nontrivial idempotent then $A \approx A_1 \times A_2 \times \ldots \times A_k$ and $R = A[X] \approx A_1[X] \times A_2[X] \times \ldots \times A_k[X]$ for some rings $A_1, A_2, \ldots, A_k$ with no nontrivial idempotent element. Since it is enough to prove the theorem for each $A_i[X]$, we can assume that A has no nontrivial idempotent element. Hence we can assume that rank $(P_\wp) = r$ is constant for all $\wp$ in $Spec(R)$. So, we have rank $(P_\wp) = r \geq \dim R$ for all $\wp$ in $Spec(R)$.

Also note that we can assume that A is a reduced ring.

Let $\varphi : R \oplus P' \longrightarrow R \oplus P$ be an isomorphism and let $\varphi(1, 0) = (f, p)$.
Let " – " barring denote "(modulo X)". Since (f, p) is unimodular in $R \oplus P$, $(\bar{f}, \bar{p})$ is also unimodular in $A \oplus \bar{P}$. Since rank $(\bar{P}) > \dim A$, by Theorem 4.1.1, there is an element p_0 in P such that $\overline{p + fp_0}$ is unimodular in $\bar{P}$. We can use this to define an automorphism $\varphi' : R \oplus P \longrightarrow R \oplus P$ such that $\bar{\varphi}'(\bar{f}, \bar{p}) = (1, 0)$ (see the proof of Theorem 4.2.2).

By replacing φ by $\varphi'\varphi$, we can assume that $\varphi(1, 0) = (f, p)$ and $(\bar{f}, \bar{p}) = (1, 0)$. So, the map $g : \bar{P}' \longrightarrow \bar{P}$ defined by $g(\bar{x}) = \bar{y}$, where $\varphi(0, x) = (a, y)$ for x in P' and y in P and a in R, is an isomorphism.

Now let $S = \{s$ in $A :$ s is not in any minimal prime ideal of A $\}$ be the set of all nonzero divisors of A. Since

$$R_S \approx A_S[X] \approx (k_1 \times k_2 \times \ldots \times k_\ell)[X],$$

where $k_1, \ldots, k_\ell$ are fields, P'_S and P_S are extended. Hence P'_s and P_s are extended from A_s for some s in S. So, there is an isomorphism $h_1 : P'_s \longrightarrow P_s$ such that $\bar{h}_1 = g_s$.

Let $S' = 1 + sA$. Then s is in the radical of $A_{S'}$. So, by Example 4.3.2, there is a generalized dimension function $d : Spec(A_{S'}[X]) \longrightarrow \mathcal{N}$ such that

$$d(\wp) \leq \dim A_{S'} < \ rank(P_{S'})$$

for all $\wp$ in $Spec(A_{S'}[X])$. We can write $f = 1 + Xf_0$ and $p = Xp_0$ for some p_0 in P and f_0 in R. Now, (f, p_0) is basic in $(R \oplus P)_{S'}$ and rank $(P_{S'}) > d(\wp)$ for all $\wp$ in $Spec(A_{S'}[X])$. By Theorem 4.1.1, there is a p' in $P_{S'}$ such that $p_0 + fp'$ is unimodular in $P_{S'}$. Hence there is an $R_{S'}$-linear map $\lambda : P_{S'} \longrightarrow R_{S'}$ such that $\lambda(p_0 + fp') = 1$.

Now we construct $R_{S'}$-linear automorphisms

$$\varphi_i : R_{S'} \oplus P_{S'} \longrightarrow R_{S'} \oplus P_{S'}$$

for $i = 1$ to 3 as follows :

$$\varphi_1(a, q) = (a, q + aXp'),$$

$$\varphi_2(a, q) = (a - f_0\lambda(q), q),$$

$$\varphi_3(a, q) = (a, q - a(p + fXp'))$$

for (a, q) in $R_{S'} \oplus P_{S'}$. It follows that

$$\varphi_3\varphi_2\varphi_1(f, p) = (1, 0) \quad \text{and} \quad \overline{\varphi_3\varphi_2\varphi_1}(\bar{a}, \bar{q}) = (\overline{a - f_0\lambda(q)}, \bar{q}).$$

Since $\varphi_3\varphi_2\varphi_1\varphi(1, 0) = (1, 0)$, it will induce an isomorphism $h'_2 : P'_{S'} \longrightarrow P_{S'}$ such that $(\bar{h}'_2) = g_{S'}$. Hence there is an element t in $S' = 1 + sA$ and an isomorphism $h_2 : P'_t \longrightarrow P_t$ such that $\bar{h}_2 = (g)_t$.

Since $(h_1)_t \equiv (h_2)_s$ (modulo X), we have $P' \approx P$ by Lemma 4.3.3. This completes the proof of Theorem 4.3.1.

Now we shall prove the conjecture I.

Theorem 4.3.2 (Plumstead) *Let $R = A[X]$ be a polynomial ring over a noetherian commutative ring A and let M be a finitely generated R-module with $\mu(M_\wp) \geq \dim(R/\wp)$ for all minimal prime ideals $\wp$ in $Spec(R)$. Suppose z is a basic element in $\bar{M} = M/XM$. (Again, " – " barring will denote "(modulo X)"). Then M has a basic element m such that $\bar{m} = z$.*

Proof. Note that it is enough to establish the theorem when A is a reduced ring. Also note that $\bar{M}$ has a basic element by Theorem 4.1.1.

First we shall assume that M is torsion free. Let

$$S = \{s \text{ in } A : s \text{ is not in any minimal prime ideal of } A\}$$

be the set of all nonzero divisors of A. We have $R_S \approx k_1[X] \times \ldots \times k_r[X]$, for some fields $k_1, \ldots, k_r$. Since M_S is torsion free it follows that M_S is a projective $R_S = A_S[X]$-module and is extended from A_S. So, we can find a basic element m'_1 in M_S such that $\bar{m}'_1 = (z)_S$. Also, since m'_1 is unimodular, $M_S/R_S m'_1$ is an extended projective module (see Theorem 2.4.1). Hence, we can find a nonzero divisor s_1 in S and a basic element m_1 in M_{s_1} such that

1. $\bar{m}_1 = (z)_{s_1}$,

2. M_{s_1} is an extended projective R_{s_1} -module and

3. $N_1 = M_{s_1}/R_{s_1}m_1$ is also an extended projective R_{s_1}-module.

So, the sequence

$$0 \longrightarrow R_{S_1} \xrightarrow{m_1} M_{S_1} \longrightarrow N_1 \longrightarrow 0$$

is a split exact sequence .

Write $S' = 1 + s_1A$. Then s_1 is in the radical of $A_{S'}$. By Example 4.3.2, there is a generalized dimension function

$$d : Spec(A_{S'}[X]) \longrightarrow \mathcal{N} \quad \text{such that} \quad d(\wp) \leq \dim A_{S'}$$

for all $\wp$ in $Spec(A_{S'}[X])$. Let ω be an element in $M_{S'}$ such that $\bar{\omega} = (z)_{S'}$. Then (ω, X) is basic in $M_{S'} \oplus R_{S'}$. Since $\mu((M_{S'})_\wp) > d(\wp)$ for all $\wp$ in $Spec(A_{S'}[X])$ there is a ω_0 in $M_{S'}$ such that $m_2' = \omega + X\omega_0$ is basic in $M_{S'}$. Further, since $M_{S's_1}$ is projective, the sequence

$$0 \longrightarrow R_{S's_1} \xrightarrow{m_2'} M_{S's_1} \longrightarrow M_{S's_1}/R_{S's_1}m_2' \to 0$$

is a split exact sequence. Hence

$$R_{S's_1} \oplus N_{1_{S'}} \approx M_{S's_1} \approx R_{S's_1} \oplus (M_{S's_1}/R_{S's_1}m_2').$$

Since rank $((N_{1_{S'}})_\wp) \geq \dim(R_{S's_1}/\wp)$ for all minimal primes $\wp$ in $Spec(R_{S's_1})$ by the cancellation Theorem 4.3.1, we have $N_{1_{S'}} \approx (M_{S's_1}/R_{S's_1}m_2')$.

Hence we can find an element s_2 in S' and an element m_2 in M_{s_2} such that

1. $(m_2)_{S'} = m_2'$ and $\bar{m}_2 = (z)_{s_2}$;

2. if $N_2 = M_{s_2}/R_{s_2}m_2$ then

$$0 \longrightarrow R_{s_2} \xrightarrow{m_2} M_{s_2} \longrightarrow N_2 \longrightarrow 0$$

   is an exact sequence. Also, if we invert s_1 the sequence splits.

3. Further, $(N_1)_{s_2} \approx (N_2)_{s_1}$ are extended projective $A_{s_1s_2}[X]$-modules.

It follows from (1) and (2) that m_2 is basic in M_{s_2}. So, by Lemma 4.3.4, there is a basic element m in M such that $\bar{m} = z$. So, the proof of the theorem is complete in the case when M is torsion free.

The general case requires a little more adjustment (see ([Ma2])). Again, we shall assume that A is reduced and let

$$T = \{m \ in \ M : fm = 0 \ for \ some \ nonzero \ divisor \ f \ in \ R\}$$

be the torsion submodule of M. By replacing M by M/XT, we can assume that $XT = 0$. We write $N = M/T$. Then the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\ & & || & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & T & \longrightarrow & \bar{M} & \longrightarrow & \bar{N} & \longrightarrow & 0 \end{array}$$

commutes and the rows are exact.

Let ω be a lift of z in M. Since (ω, X) is basic in M, by Theorem 4.1.1, $\omega + X\omega_0$ is basic in M at all the minimal primes of R, for some w_0 in M. By replacing ω by $\omega + X\omega_0$, we assume that ω is basic in M at all the minimal primes of R.

If $\wp$ is a minimal prime of A, then it follows that the image of ω in $N_{(\wp,X)} \approx (M/T)_{(\wp,X)}$ is nonzero. Hence, by Artin-Rees lemma, ω does not belong to $T_{(\wp,X)} + X^k M_{(\wp,X)}$ for large enough k, for all minimal primes $\wp$ in $Spec(A)$.

Write $L = T + R\omega + X^k M$, for some large enough integer k. Let z_1 be the image of ω in $\bar{L}$. It follows that

1. z_1 is basic in $\bar{L}$ and $L_\wp \approx M_\wp$ for all minimal primes $\wp$ of R;
2. T is the torsion submodule of L and, since k is large enough, the image of z_1 in $L/T + XL$ is basic at all the minimal primes in $Spec(A)$;
3. if ω' is a basic element in L, with $\bar{\omega}' = z_1$, then ω' is also a lift of z. Hence ω' will also be a basic element of M with $\bar{\omega}' = z$.

Therefore, by replacing M by L, we can assume that the image of z in $\bar{T} \approx M/T + XM$ is basic at all the minimal primes of $Spec(A)$ and also $XT = 0$.

Let S be the set of all nonzero divisors of A and z_0 be the image of z in $\bar{N}$ where $N = M/T$. Since N_S is torsion free, N_S is an extended projective $A_S[X]$-module. Hence there is a basic element y_1' in N_S such that $\bar{y_1'} = (z_0)_S$. Also note that y' is unimodular in N_S. We can pick an element m_1' in M_S that is a lift of y_1' and $\bar{m_1'} = (z)_S$. Then m_1' is also unimodular in M_S and $M_S \approx N_S \oplus T_S$. We can assume that m_1' is in N_S. So, $M_S/R_S m_1' \approx N_S/R_S y_1' \oplus T_S$. So, $N_S/R_S' y_1'$ is an extended projective module.

So, we can find an s_1 in S, a unimodular element y_1 in N_{s_1} and a unimodular element m_1 in M_{s_1} such that

1. N_{s_1} is an extended projective $A_{s_1}[X]$-module and $M_{s_1} \approx N_{s_1} \oplus T_{s_1}$;
2. m_1 is in $N_{s_1}, \bar{m}_1 = (z)_{s_1}$, image of m_1 in N_{s_1} is y_1;
3. $N_{s_1}/R_{s_1} y_1 = K_1$ is an extended projective $A_{s_1}[X]$-module and
$$K = M_{s_1}/R_{s_1} m_1 \approx K_1 \oplus T_{s_1}.$$

Hence the sequence

$$0 \longrightarrow R_{s_1} \xrightarrow{m_1} M_{s_1} \longrightarrow K \approx K_1 \oplus T_{s_1} \longrightarrow 0$$

is a split exact sequence.

Therefore

$$M_{s_1} \approx R_{s_1} \oplus K \approx R_{s_1} \oplus K_1 \oplus T_{s_1} \quad \text{and} \quad \bar{M}_{s_1} \approx A_{s_1} \oplus \bar{K}_1 \oplus T_{s_1}.$$

Under this identification we have $z = (1, 0, 0)$.

Now write $S' = 1 + s_1 A$. As in the "torsion free case", we can find a basic element m'_2 in $M_{S'}$ with $\bar{m}'_2 = (z)_{S'}$. Again since

$$\bar{T} = T \quad \text{and} \quad \bar{m}'_2 = (z)_{S'} = (1, 0, 0)$$

in $\bar{M}_{S's} = (A_{s_1} \oplus K_1 \oplus T_{s_1})_{S'}$, we have m'_2 is in

$$R_{S's_1} m_1 \oplus K_{1S'} \approx N_{S's_1} \subseteq N_{S's_1} \oplus T_{S's_1} \approx M_{S's_1}.$$

Hence m'_2 is also unimodular in $R_{S's_1} m_1 \oplus (K_1)_{S'} \approx N_{S's_1}$. Therefore

$$K'_2 = N_{S's_1} / R_{S's_1} m'_2 \approx K_{1S'}$$

by the cancellation Theorem 4.3.1.

So, as before, we can pick s_2 in S' and a basic element m_2 in M_{s_2} such that

1. $\bar{m}_2 = (z)_{s_2}$ and m_2 is in $R_{s_1 s_2} m_1 \oplus K_1 s_2 \approx N_{s_1 s_2}$,
2. $K_2 = (R_{s_1 s_2} m_1 \oplus K_{1 s_2}) / R_{s_1 s_2 m_2} \approx (K_1)_{s_2}$ are extended projective modules.

Let $\alpha : (\bar{K}_1)_{s_2} \longrightarrow \bar{K}_2$ be a fixed isomorphism and let $f_0 : (K_1)_{s_2} \longrightarrow K_2$ be the extension of α. As in Lemma 4.3.4, we can find an isomorphism

$$f : R_{s_1 s_2} m_1 \oplus K_{1 s_2} \longrightarrow R_{s_1 s_2} m_1 \oplus K_{1 s_2}$$

such that $\bar{f} = Id$ and the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R_{s_1 s_2} & \xrightarrow{m_1} & R_{s_1 s_2} m_1 \oplus K_{1 s_2} & \longrightarrow & K_{1 s_2} & \longrightarrow & 0 \\ & & || & & \downarrow f & & \downarrow f_0 & & \\ 0 & \longrightarrow & R_{s_1 s_2} & \xrightarrow{m_2} & R_{s_1 s_2} m_1 \oplus K_2 & \longrightarrow & K_2 & \longrightarrow & 0 \end{array}$$

commutes.

Since $\bar{f} = Id$ and $R_{s_1 s_2} m_1 \oplus K_{1 s_2}$ is extended, f is isotopic to identify relative to the map $R_{s_1 s_2} \longrightarrow A_{s_1 s_2}$. As $M_{s_1 s_2} \approx R_{s_1 s_2} m_1 \oplus K_{1 s_2} \oplus T_{s_1 s_2}$, we can extend f to an isomorphism $F : M_{s_1 s_2} \longrightarrow M_{s_1 s_2}$ by defining $F(t) = t$ for $t \in T_{s_1 s_2}$.

Hence F is also isotopic to identity relative to the map $R_{s_1s_2} \longrightarrow A_{s_1s_2}$. Also the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & R_{s_1s_2} & \xrightarrow{m_1} & M_{s_1s_2} \\ & & || & & \downarrow F \\ 0 & \longrightarrow & R_{s_1s_2} & \xrightarrow{m_2} & M_{s_1s_2} \end{array}$$

commutes and $\bar{F} = Id$.

Rest of the proof is similar to Lemma 4.3.4 (in this case, we have to use Lemma 4.3.2 instead of Lemma 4.3.3). This completes the proof of Theorem 4.3.2.

Now we shall prove the conjecture III. This conjecture was first proved by Sathaye ([Sa]) for affine domains over infinite fields and then Mohan Kumar ([MK1]) proved the conjecture completely.

Theorem 4.3.3 (Sathaye-Mohan Kumar, [Sa,MK2]) *Suppose $R = A[X]$ is a polynomial ring over a noetherian commutative ring A and M is a finitely generated R-module. Let k be an integer with $k \geq e(M) =$*

$$\max\{\mu(M_\wp) + \dim(R/\wp) : \wp \textit{ is in } Spec(R) \textit{ and } \dim(R/\wp) < \dim R\}.$$

Then M is generated by k elements. Further (Plumstead ([P])), if $z_1, z_2, \ldots, z_k$ generate M/XM, then there exist $m_1, m_2, \ldots, m_k$ that generate M and such that $\bar{m}_i = z_i$ for $i = 1$ to k (barring " – " means "(modulo X)" in this theorem).

Proof. First note that $M/XM = \bar{M}$ is generated by k elements by Theorem 4.2.3. Here we shall give the proof of Plumstead ([P]).

If N is the nil radical of A, then replacing A by A/N and M by M/NM, we can assume that A is reduced.

Now we shall use induction on $\dim A$. Assume $\dim A = 0$. Since A is reduced, we have $A \approx k_1 \times k_2 \times \ldots \times k_r$ and $A[X] \approx k_1[X] \times k_2[X] \times \ldots \times k_r[X]$ where $k_1, k_2, \ldots, k_r$ are fields. Since it is enough to establish the theorem for each $k_i[X]$, we can assume that A is a field and hence $R = A[X]$ is a principal ideal domain. Let $T = \{m \text{ in } M : fm = 0 \text{ for some nonzero } f \text{ in } R\}$ be the torsion submodule of M and let $F = M/T$. Since F is torsion free, F is a free R-module (see Theorem 2.4.1). Hence $M \approx T \oplus F$.

If $T = 0$, then since $M \approx F$ is free, M is extended from A. Hence the theorem holds in this case. If $T \neq 0$, then there is a maximal ideal $\wp_0$ of R such that $T_{\wp_0} \neq 0$ and hence $\mu(M_{\wp_0}) >$ rank F. So,

$$\mu(M_{(0)}) + \dim(R) = rank\ (F) + 1 \leq \mu(M_{\wp_0}) \leq e(M) \leq k.$$

Therefore,

$$k \geq \max\{\mu(M_\wp) + \dim R/\wp : \wp \textit{ in } Spec(R)\}.$$

If $m'_1, m'_2, \ldots, m'_k$ in M are such that $\bar{m}'_1 = z_1, \bar{m}'_2 = z_2, \ldots, \bar{m}'_k = z_k$, then $M = Rm'_1 + Rm'_2 + \cdots + Rm'_k + XM$. By Theorem 4.2.7, there are elements

$$m_1 = m'_1 + n_1, \quad \ldots, \quad m_k = m'_k + n_k$$

for $n_1, n_2, \ldots, n_k$ in XM, such that

$$M = Rm_1 + Rm_2 + \cdots + Rm_k.$$

Since $\bar{m}_1 = z_1, \ldots, \bar{m}_k = z_k$, the theorem is established when $\dim A = 0$.

Now we shall assume that A is reduced and $\dim A > 0$. Again let

$$T = \{m \textit{ in } M : fm = 0 \textit{ for some nonzero divisor } f \textit{ in } A\}$$

be the torsion submodule of M and let $M' = M/T$. Suppose that we can establish the theorem for M', then there are elements $m'_1, \ldots, m'_k$ in M' such that

$$M' = Rm'_1 + Rm'_2 + \cdots + Rm'_k \qquad \text{and} \quad \bar{m}'_1 = z'_1, \bar{m}'_2 = z'_2, \ldots, \bar{m}'_k = z'_k$$

in $\bar{M}' \approx M/(T + XM)$, where $z'_1, \ldots, z'_k$ are, respectively, the images of the elements $z_1, z_2, \ldots, z_k$ in $\bar{M}'$. Let $u_1, u_2, \ldots, u_k$ in M be such that the images of $u_1, \ldots, u_k$ in M' are, respectively, $m'_1, m'_2, \ldots, m'_k$. In fact, we can also assume that $\bar{u}_1 = z_1, \bar{u}_2 = z_2, \ldots, \bar{u}_k = z_k$. Hence

$$M = \sum_{i=1}^{k} Ru_i + XM = \sum_{i=1}^{k} Ru_i + X(\sum_{i=1}^{k} Ru_i + T) = \sum_{i=1}^{k} Ru_i + XT.$$

Let $J = \text{ann }(T)$ be the annihilator of T and $V(J) = \{\wp \text{ in } Spec(R) : J \subseteq \wp\}$. Since $T_\wp = 0$ for all minimal primes $\wp$ of R, $V(J)$ will not contain any of the minimal prime ideals of R. So, for $\wp$ in $V(J)$, we have $k \geq \mu(M_\wp) + \dim R/\wp$. By Theorem 4.2.6, there will be elements $n_1, n_2, \ldots, n_k$ in XT such that

$$m_1 = u_1 + n_1, \quad m_2 = u_2 + n_2, \quad \ldots, \quad m_k = u_k + n_k$$

will generate M on $V(J)$. If $\wp$ is in $Spec(R) \backslash V(J)$, then $M_\wp \approx M'_\wp$ and hence $m_1, m_2, \ldots, m_k$ will generate $M_\wp$. Hence $M_\wp = (Rm_1 + \cdots + Rm_k)_\wp$ for all $\wp$ in $Spec(R)$. Since $\bar{m}_1 = z_1, \ldots, \bar{m}_k = z_k$, the theorem is also established for M, assuming that the theorem holds for M'. So, by replacing M by M', we assume that M is torsion free, i.e. $T = 0$.

Let $S = \{s \text{ in } A : s \text{ is not in any minimal prime ideal of } A\}$ be the set of all nonzero divisors of A. Since A_S is a finite product of fields, $R_S \approx A_S[X]$ is a finite product of principal ideal domains. Since M_S is torsion free, M_S is a projective R_S-module. Since $\dim A_S = 0$, it follows that there are elements $u_1, u_2, \ldots u_k$ in M_S that generate M_S and $\bar{u}_1 = (z_1)_S, \ldots, \bar{u}_k = (z_k)_S$. If L is the kernel of the natural map for $R_S^k \longrightarrow M_S$ that sends the standard basis $e_1, e_2, \ldots, e_k$, respectively, to $u_1, u_2, \ldots, u_k$, then L is also an extended projective A_S-module. It follows that we can find an s in S and $\omega_1, \ldots, \omega_k$ in M_s that generate M_s and $\bar{\omega}_1 = (z_1)_s, \ldots, \bar{\omega}_k = (z_k)_s$ and the kernel L_1 of the natural map $f_1 : R_s^k \longrightarrow M_s$ that sends the standard basis $e_1, \ldots, e_k$ of R_s^k, respectively, to $\omega_1, \ldots, w_k$, is an extended projective A_s-module.

Let $S' = 1 + sA$. Then $\dim(A/sA) < \dim A$. Hence, by induction, there are elements $m_1'', m_2'', \ldots, m_k''$ in M so that the images of these elements in M/sM generate M/sM and the images of m_i'' in $M/(s, X)M$ are same as that of z_i for $i = 1$ to k. By modifying m_i'', we can assume that $\bar{m}_i'' = z_i$ for $i = 1$ to k. So, it follows that

$$M = \sum_{i=1}^{k} Rm_i'' + sXM \quad \text{and hence} \quad M_{S'} = \sum_{i=1}^{k} R_{S'}m_i'' + sXM_{S'}.$$

Let $D(s) = \{\wp \text{ in } Spec(R_{S'}) : s \text{ is not in } \wp\}$. Then $d(\wp) = \dim(R_{S'}/\wp)_s$ will define a generalized dimension function $d : D(s) \longrightarrow \mathcal{N}$. For all $\wp$ in $D(s)$, we have $d(\wp) < \dim R$. So, $k \geq \mu(M_\wp) + d(\wp)$ for all $\wp$ in $D(s)$. By Theorem 4.2.6, there are elements $m_i' = m_i'' + n_i$ for n_i in $sXM_{S'}$, for $i = 1$ to k such that

$$(R_{S'}m_1' + \cdots + R_{S'}m_k')_\wp = M_\wp$$

for all $\wp$ in $D(s)$. Also, since

$$M_{S'} = \sum_{i=1}^{k} R_{S'}m_i + sXM_{S'} \quad \text{we have} \quad \left(\sum_{i=1}^{k} R_{S'}m_i'\right)_\wp = M_\wp$$

for all $\wp$ in $Spec(R_{S'})$ that contain s. Hence $m_1', m_2', \ldots, m_k'$ generate $M_{S'}$ and $\bar{m}_1' = (z_1)_{S'}, \ldots, \bar{m}_k' = (z_k)_{S'}$.

Now, let L' be the kernel of the map $R_{S'}^k \longrightarrow M_{S'}$ that sends the standard basis $e_1, e_2, \ldots, e_k$ of $R_{S'}^k$, respectively, to $m_1', m_2', \ldots, m_k'$. Since $M_{S's}$ is projective,

$$M_{S's} \oplus L_s' \approx R_{S's}^k \approx M_{S's} \oplus L_{1S'}.$$

Also for any minimal prime $\wp$ of $R_{S's}$,we have $k \geq \mu(M_\wp) + \dim(R_{S's}/\wp)$. So, rank $(L_{1S'\wp})$ = k - rank $(M_{S's\wp}) \geq \dim(R_{S's/\wp})$. Hence by Theorem 4.3.1, $L_{1S'} \approx L_s'$. So, L_s' is an extended projective $A_{S's}$-module.

It follows that we can find an element t in S' and elements $m_1'', m_2'', \ldots, m_k''$ in M_t such that

1. $m_1'', m_2'', \ldots, m_k''$ generate M_t,
2. $\bar{m}_1'' = (z_1)_t, \ldots, \bar{m}_k'' = (z_k)_t$ and
3. if L_2 is the kernel of the natural map $f_2 : R_t^k \longrightarrow M_t$ that sends the standard basis $e_1, e_2, \ldots, e_k$ of R_t^k, respectively, to $m_1'', m_2'', \ldots, m_k''$, then L_{2s}, L_{1t} are extended A_{st}-projective modules and $L_{2s} \approx L_{1t}$.

Now the proof of Theorem 4.3.3 is complete by Lemma 4.3.5.

Chapter 5

The Theory of Matrices

In this Chapter we shall discuss some of the basic notations and facts about matrices and also prove two theorems of Suslin that we shall need in our later Chapters. Although part of this theory is available in other sources (for example, Murthy's Notes([GM])), our approach is widely different, especially in section 5.2 and 5.3. In section 5.2, we avoid the theory of elementary matrices and instead we talk about the subgroup of automorphisms that are isotopic to the identity. In section 5.3, our proof of Suslin's theorem is not what one would find in other sources. In both these sections, the reader will once again find our biases toward the techniques of patching isotopic isomorphisms due to Quillen over the techniques of elementary matrices due to Suslin.

5.1 Preliminaries about Matrices

In this section A will always denote a noetherian commutative ring.

Notations 5.1.1 Let A be a commutative ring.

1. $\mathbf{M}_n(A)$ will denote the set of all $n \times n$-matrices with entries in A,
2. $GL_n(A)$ will denote the group of all invertible matrices in $\mathbf{M}_n(A)$,
3. $SL_n(A)$ will denote the subgroup of all $n \times n$-matrices in $GL_n(A)$ with determinant one,
4. I_n or I will denote the $n \times n$ identity matrix.

Definition 5.1.1 For an element λ in A, and for $i, j = 1, 2, \ldots, n$ with $i \neq j$, $e_{ij}(\lambda)$ will denote the matrix whose diagonal entries are one, the (i, j)th entry is λ and rest of the entries are zero. $E_n(A)$ will denote the subgroup of $SL_n(A)$

generated by $\{e_{ij}(\lambda) : \lambda \text{ in } A,\ 1 \leq i, j \leq n, i \neq j\}$. The elements of $E_n(A)$ are called *elementary matrices.*

Remark 5.1.1 The following is a list of standard facts :

1. $E_n(A) = \{\prod_{k=1}^{m} e_{i_k j_k}(\lambda_k) : \lambda_k \in A, 1 \leq i_k, j_k \leq n, i_k \neq j_k\}$.
2. Any triangular matrix whose diagonal entries are one is an elementary matrix.
3. If $\alpha = \prod_{k=1}^{m} e_{i_k j_k}(\lambda_k)$ is an elementary matrix, then α is isotopic to the identity map if we consider α as an isomorphism $\alpha : A^n \longrightarrow A^n$. The isotopy is given by $\alpha(X) = \prod_{k=1}^{m} e_{i_k j_k}(X\lambda_k)$.
4. If $\phi : A \longrightarrow B$ is a surjective ring homomorphism then the induced homomorphism $E_n A \longrightarrow E_n B$ is surjective.
5. Given a $n \times m$-matrix α and λ in A, then if β is got from α by adding λ-times the ith row of α to the jth row of α, then $\beta = \epsilon\alpha$ for some ϵ in $E_n(A)$. A similar statement holds for columns.
6. If A is a local ring or a Euclidean ring, then $SL_n(A) = E_n(A)$ for all integers $n \geq 1$. More generally, if α is in $\mathbf{M}_n(A)$, there are ϵ_1, ϵ_2 in $E_n(A)$ such that $\epsilon_1 \alpha \epsilon_2$ is a diagonal matrix.

The following are some standard results.

Lemma 5.1.1 (Whitehead) Let α and β be in $GL_n(A)$. Then there are elementary matrices ϵ_1, ϵ_2 in $E_{2n}(A)$ such that

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \epsilon_1 = \begin{pmatrix} \alpha\beta & 0 \\ 0 & I_n \end{pmatrix}$$

and

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \epsilon_2 = \begin{pmatrix} \beta & 0 \\ 0 & \alpha \end{pmatrix}.$$

Proof. Take

$$\epsilon_1 \;=\; \begin{pmatrix} 1 & \beta - 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & \beta^{-1} - 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\beta & 1 \end{pmatrix}.$$

It also follows that

$$\epsilon_2 \;=\; \begin{pmatrix} \alpha^{-1}\beta & 0 \\ 0 & (\alpha^{-1}\beta)^{-1} \end{pmatrix} \quad \text{is in} \quad E_{2n}(A)$$

and

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \epsilon_2 = \begin{pmatrix} \beta & 0 \\ 0 & \alpha \end{pmatrix}.$$

This completes the proof of Lemma 5.1.1.

Theorem 5.1.1 (Suslin) *$E_n(A)$ is a normal subgroup of $GL_n(A)$ for any integer $n \geq 3$.*

We need the following lemmas to prove Theorem 5.1.1.

Lemma 5.1.2 (Vaserstein) Let α be an $r \times s$ -matrix and β be an $s \times r$-matrix. Suppose $I + \alpha\beta$ is in $GL_r(A)$. Then $I + \beta\alpha$ is in $GL_s(A)$ and

$$\begin{pmatrix} I + \alpha\beta & 0 \\ 0 & (I + \beta\alpha)^{-1} \end{pmatrix}$$

is in $E_{r+s}(A)$.

Proof. Note that $(I + \beta\alpha)^{-1} = I - \beta(1 + \alpha\beta)^{-1}\alpha$ and

$$\begin{pmatrix} I + \alpha\beta & 0 \\ 0 & (I + \beta\alpha)^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} I & 0 \\ (I + \beta\alpha)^{-1}\beta & I \end{pmatrix} \begin{pmatrix} I & -\alpha \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -\beta & I \end{pmatrix} \begin{pmatrix} I & (1 + \alpha\beta)^{-1}\alpha \\ 0 & 1 \end{pmatrix}$$

is in $E_{r+s}(A)$.

Corollary 5.1.1 Let $v = (v_1, v_2, \ldots, v_r)$ and $w = (w_1, w_2, \ldots, w_r)$, with v_i, w_i in A, be two row matrices such that $v_1w_1 + \cdots + v_rw_r = 0$. Then

$$\begin{pmatrix} 1 + v^t w & 0 \\ 0 & 1 \end{pmatrix}$$

is in $E_{r+1}(A)$.

Proof. Immediate from Lemma 5.1.2.

Corollary 5.1.2 Let $v = (v_1, v_2, \ldots, v_r)$ and $w = (w_1, \ldots w_r)$, with v_i, w_i in A, be two row matrices such that $\sum_{i=1}^{r} w_i v_i = 0$. If $w_i = 0$ for some $i = 1$ to r, then $I + v^t w$ is in $E_r(A)$.

Proof. If $w_r = 0$, then $\sum_{i=1}^{r-1} v_i w_i = 0$. Taking $v' = (v_1, \ldots, v_{r-1})$ and $w' = (w_1, \ldots, w_{r-1})$, it follows from Corollary 5.1.1 that

$$\begin{pmatrix} 1 + v'^t w' & O^t \\ O & 1 \end{pmatrix} \quad \text{is in} \quad E_r(A), \text{ where } O = (0, \ldots, 0)\,.$$

But

$$1 + v^t w \;=\; \begin{pmatrix} 1 + v'^t w' & O^t \\ u & 1 \end{pmatrix}$$

where $u = (u_1 \ldots, u_{r-1}) = v_r(w_1, \ldots, w_{r-1})$. Hence $1 + v^t w$ is in $E_r(A)$, by (2) of Remark 5.1.1.

If $w_r \neq 0$ and $w_i = 0$ for some $i < r$, then by (5) of Remark 5.1.1, there is an elementary matrix ϵ in $E_r(A)$ such that

$$w\epsilon = (w_1, w_2, \ldots, w_{i-1}, -w_r, w_{i+1}, \ldots, w_{r-1}, 0).$$

Now $1 + v^t w = \epsilon(I + v'w')\epsilon^{-1}$, where $v' = \epsilon^{-1} v^t$ and $w' = w\epsilon$ and the proof of Corollary 5.1.2 is complete.

Proposition 5.1.1 Let $v = (v_1, \ldots v_r)$ in A^r be a unimodular element. Let $\psi : A^r \longrightarrow A$ be the map such that $\phi(e_i) = v_i$, where $e_1, \ldots, e_r$ is the standard basis of A^r. Then the kernel of $\phi = K$ is generated by the set

$$\{v_j e_i - v_i e_j | 1 \leq i < j \leq r\}.$$

Proof. Let $u = (u_1, \ldots, u_r)$ be in A^r be such that $uv^t = 1$. If $w = (w_1, \ldots, w_r)$ is in the kernel of ψ then $w = \sum_{i=1}^{r} w_i e_i = \sum_{i=1}^{r} w_i(e_i - v_i u)$.
So, $e_1 - v_1 u, \ldots, e_r - v_r u$ generate the kernel of ψ.

Now we have $e_i - v_i u = \sum_{k=1, k\neq i}^{r} u_k(v_k e_i - v_i e_k)$. This completes the proof of Proposition 5.1.1.

Corollary 5.1.3 Let $v = (v_1, \ldots, v_r)$ be a unimodular row in A^r and $w = (w_1, \ldots, w_r)$ be in A^r such that $w_1 v_1 + \cdots + w_r v_r = 0$. If $r \geq 3$, then $I + v^t w$ is in $E_r(A)$.

Proof. By Proposition 5.1.1

$$w = \sum_{i<j} a_{ij}(v_i e_j - v_j e_i) = \sum_{i<j} w_{ij},$$

where $w_{ij} = a_{ij}(v_i e_j - v_j e_i)$. It follows from Corollary 5.1.2 that $I + v^t w_{ij}$ is in $E_r(A)$. But

$$I + v^t w = I + \sum_{i<j} v^t w_{ij} = \prod_{i<j} (I + v^t w_{ij}).$$

Hence the proof of Corollary 5.1.3 is complete.

Now we are ready to give the proof of Suslin's Theorem 5.1.1.

Proof of Suslin's Theorem 5.1.1. Since $E_n(A)$ is generated by

$$\{e_{ij}(\lambda) : 1 \leq i, j \leq n \text{ with } i \neq j \text{ and } \lambda \in A\},$$

it is enough to show that for any α in $GL_n(A)$ we have $\alpha e_{ij}(\lambda)\alpha^{-1}$ is in $E_n(A)$. One can check that $\alpha e_{ij}(\lambda)\alpha^{-1} = I + \lambda \alpha_i \beta_j$, where α_i is the ith column of α and β_j is the jth row of α^{-1}. Since $\beta_j \alpha_i = 0$ for $i \neq j$, we have $I + \lambda \alpha_i \beta_j$ is in $E_n(A)$ by Corollary 5.1.3. So, the proof of Theorem 5.1.1 is complete.

Remark 5.1.2 With $A = k[X, Y]$ where k is a field, $E_2(A)$ is not a normal subgroup of $GL_2(A)$. For a proof one can see Murthy's Notes ([GM]).

5.2 The Isotopy Subgroup $QL_n(A)$ of $GL_n(A)$

In this section we shall prove the "Isotopy analogue" of the following theorem of Suslin ([S]).

Theorem 5.2.1 (Suslin) *Let $R = k[X_1, \ldots, X_m]$ be a polynomial ring over k, where $k = \mathbb{Z}$ or a field. Then $SL_n(R) = E_n(R)$ for $n \geq 3$.*

For our purpose an "isotopy analogue" of this Theorem 5.2.1 will suffice. So, we omit the proof of Theorem 5.2.1. For a proof of Theorem 5.2.1 one can see Murthy's note ([GM]).

The following proposition gives some elementary facts about isotopy.

Proposition 5.2.1 Suppose A is a commutative ring.

(i) Suppose $\alpha_i : M \longrightarrow M'$ and $\beta_i : M' \longrightarrow M''$ are isomorphisms of A-modules, for $i = 0, 1$. If α_0 is isotopic to α_1 and β_0 is isotopic to β_1 then $\beta_0 o \alpha_0$ is isotopic to $\beta_1 o \alpha_1$.

(ii) If M is an A-module then

$$QL(M) = \{\alpha \in \ Aut(M) : \alpha \ \ is \ \ isotopic \ \ to \ \ Id_M\}$$

is a normal subgroup of $Aut(M)$.

(iii) Let M be an A-module and α, β, γ be in $Aut(M)$. If α is isotopic to β and β is isotopic to γ then α is isotopic to γ.

Proof. If $\alpha(T) : M[T] \longrightarrow M'[T]$ is an isotopy from α_0 to α_1 and if

$$\beta(T) : M'[T] \longrightarrow M''[T]$$

is an isotopy from β_0 to β_1 then $\beta(T) o \alpha(T)$ is an isotopy from $\beta_0 o \alpha_0$ to $\beta_1 o \alpha_1$. This establishes (i). Now it follows from (i) that $QL(M)$ is closed under composition. Again, if α is in $QL(M)$ and $\alpha(T)$ is an isotopy from Id_M to α then $\alpha(T)^{-1}$ is an isotopy from Id_M to α^{-1}. Hence $QL(M)$ is a subgroup of $Aut(M)$. If β is in $Aut(M)$ and $\beta_1 = \beta \otimes Id_{A[T]}$ and if $\alpha(T)$ is an isotopy from Id_M to α_1, then $\beta_1^{-1} o \alpha(T) o \beta_1$ is an isotopy from Id_M to $\beta^{-1}\alpha\beta$. Hence $QL(M)$ is a normal subgroup of $Aut(M)$. So, the proof of (ii) is complete. To see (iii), let $I_1(T)$ be an isotopy from α to β and $I_2(T)$ be an isotopy from β to γ and let $\beta_1(T) = \beta^{-1} \otimes A[T]$. Then $\gamma(X) = I_1(T) o \beta_1(T) o I_2(T)$ is an isotopy from α to γ. This establishes (iii).

Definition 5.2.1 Let A be a commutative ring and let

$$QL_n(A) = \{\alpha \ \ in \ \ GL_n(A) : \alpha \ \ is \ \ isotopic \ \ to \ \ identity\}.$$

It follows from Proposition 5.2.1 that $QL_n(A)$ is a normal subgroup of $GL_n(A)$. This subgroup $QL_n(A)$ will be called the *isotopy subgroup* of $GL_n(A)$. Note that if A is a reduced ring then $QL_n(A)$ is in fact a subgroup of $SL_n(A)$.

More generally, for an A-module M we let

$$QL(M) = \{\alpha : \alpha \ is \ an \ automorphism \ of \ M \ that \ is \ isotopic \ to \ Id_M\}.$$

It follows from Proposition 5.2.1 that $QL(M)$ is a normal subgroup of the group of automorphisms of M.

The following is the isotopy analogue of Theorem 5.2.1 of Suslin

Theorem 5.2.2 *Let $R = k[X_1, \ldots, X_n]$ be a polynomial ring over k, where $k = \mathbb{Z}$ or a field. Then $SL_n(R) = QL_n(R)$.*

Proof. Let $\alpha = \alpha(X_1, \dots, X_n)$ be in $SL_n(R)$, then $\alpha(X_1T, X_2T, \dots, X_nT)$ is in $SL_n(R[T])$ and is an isotopy from $\alpha_0 = \alpha(0, \dots, 0)$ to α. Since α_0 is in $SL_n(k) = E_n(k)$, we have α_0 is isotopic to identity, by (3) of Remark 5.1.1. Hence α is isotopic to identity by Proposition 5.2.1. This completes the proof of Theorem 5.2.2.

We shall need the following proposition in the latter sections.

Proposition 5.2.2 Let $R = \mathbb{Z}[X_1, X_1^{-1}, X_2, \dots, X_n]$ be a Laurent polynomial ring with one Laurent polynomial variable X_1 and $n-1$ polynomial variables $X_2, \dots, X_n$. Then $SL_n(R) = QL_n(R)$.

Proof. Let

$$\alpha = \alpha(X_1, X_1^{-1}, X_2, \dots, X_n)$$

be in $SL_n(R)$. Since $\alpha(X_1, X_1^{-1}, X_2T, \dots, X_nT) = \beta(T)$ gives an isotopy from $\alpha(X_1, X_1^{-1}, 0, \dots, 0)$ to α, we can assume $n = 1$ and $R = \mathbb{Z}[X, X^{-1}]$.

Now suppose that α is in $SL_n(\mathbb{Z}[X, X^{-1}])$. Then $\beta = X^p\alpha$ is in $\mathbf{M}_n(\mathbb{Z}[X])$ for some integer $p \geq 0$. Let $\det \beta = X^k$. We claim that $\beta = \epsilon\delta$ for some ϵ in $QL_n(\mathbb{Z}[X, X^{-1}])$ and a diagonal matrix δ in $GL_n(\mathbb{Z}[X, X^{-1}])$. We shall prove the claim by induction on k. If $k = 0$, then β is in $SL_n(\mathbb{Z}[X]) = QL_n(\mathbb{Z}[X])$ by Theorem 5.2.2 and hence in $QL_n(\mathbb{Z}[X, X^{-1}])$.

Now suppose $k > 0$ and let "bar" denote "(modulo X)". So, $det(\bar{\beta}) = 0$. By (6) of Remark 5.1.1, there are ϵ_1, ϵ_2 in $E_n(\mathbb{Z})$ such that $\epsilon_1\bar{\beta}\epsilon_2 = diag(d_1, \dots, d_n)$. Since $d_1d_2\dots d_n = det(\epsilon_1\bar{\beta}\epsilon_2) = 0$, we can assume that $d_n = 0$. Hence $\epsilon_1\bar{\beta}\epsilon_2 = diagonal(d_1, \dots, d_{n-1}, 0)$. For $i = 1, 2$ let E_i be in $E_n(\mathbb{Z}[X])$ such that $\bar{E}_i = \epsilon_i$. Since $\bar{E}_1\bar{\beta}\bar{E}_2 = diagonal(d_1, \dots, d_{n-1}, 0)$ the last column of $E_1\beta E_2$ is a multiple of X. Write $\beta_1 = E_1\beta E_2(digonal(1, 1, \dots, 1, X^{-1}))$. Then β_1 is in $\mathbf{M}_n(\mathbb{Z}[X])$ and $\det \beta_1 = X^{k-1}$. Therefore, by induction, $\beta_1 = \epsilon\delta$ for some ϵ in $QL_n(\mathbb{Z}[X, X^{-1}])$ and δ is a diagonal matrix in $GL_n(\mathbb{Z}(X, X^{-1}])$. So, $\beta = E_1^{-1}\beta_1 DE_2^{-1} = E_1^{-1}\epsilon\delta DE_2^{-1}$, where $D = diagonal(1, 1, \dots, 1, X)$. Let

$$\epsilon' = E_1^{-1}\epsilon(\delta DE_2^{-1}D^{-1}\delta^{-1}) \quad \text{and} \quad \delta' = \delta D.$$

Then $\beta = \epsilon'\delta'$. Since $E_n(\mathbb{Z}[X, X^{-1}]) \subseteq QL_n(\mathbb{Z}[X, X^{-1}])$ and $QL_n(\mathbb{Z}[X, X^{-1}])$ is a normal subgroup of $SL_n(\mathbb{Z}[X, X^{-1}])$, ϵ' is in $QL_n(\mathbb{Z}[X, X^{-1}])$ and also δ' is a diagonal matrix in $GL_n(\mathbb{Z}[X, X^{-1}])$. Hence the claim is established.

Therefore $\alpha = X^{-p}\beta = X^{-p}\epsilon\delta = \epsilon(X^{-p}\delta)$, where ϵ is in $QL_n(\mathbb{Z}[X, X^{-1}])$ and δ is a diagonal matrix in $GL_n(\mathbb{Z}[X, X^{-1}])$. Since $det(X^p\delta) = \det \alpha = 1$, it follows from Lemma 5.1.1 that $X^p\delta$ is in $E_n(\mathbb{Z}[X, X^{-1}]) \subseteq QL_n(\mathbb{Z}[X, X^{-1}])$. Hence the proof of Proposition 5.2.2 is complete.

5.3 The Theorem of Suslin

In this section we prove the following theorem of Suslin on completion of unimodular rows to an invertible matrix.

Theorem 5.3.1 (Suslin) *Let A be a commutative ring and $(x_0, x_1, \ldots, x_n)$ be a unimodular row in $U_{n+1}(A)$. Let $r_0, r_1, \ldots, r_n$ be positive integers such that the product $r_0 r_1 \cdots r_n$ is divisible by $n!$. Then $(x_0^{r_0}, x_1^{r_1}, \ldots, x_n^{r_n})$ is completable to an invertible matrix.*

To prove Theorem 5.3.1, first we prove the following proposition.

Proposition 5.3.1 Let $(x_0, x_1, \ldots, x_n)$ be a unimodular row in $U_{n+1}(A)$. Let $\bar{A} = A/(x_n)$ and let "–" bar denote the images in $\bar{A}$. If $(\bar{x}_0, \ldots, \bar{x}_{n-1})$ is completable to an invertible matrix in $\mathbf{M}_n(\bar{A})$, then $(x_0, \ldots, x_{n-1}, x_n^n)$ is completable to an invertible matrix in $\mathbf{M}_{n+1}(A)$.

Proof. Let ϕ be a matrix in $\mathbf{M}_n(A)$ such that the first column of ϕ is $(x_0, \ldots, x_{n-1})^t$ and $\bar{\phi}$ is in $GL_n(\bar{A})$. Let ψ be in $\mathbf{M}_n(A)$ be such that $\bar{\psi}$ is the inverse of $\bar{\phi}$. Hence $\phi\psi = I_n + x_n\alpha$ and $\psi\phi = I_n + x_n\beta$ for α, β in $\mathbf{M}_n(A)$.

Since

$$\begin{pmatrix} \phi & \alpha \\ x_n I_n & \psi \end{pmatrix} \begin{pmatrix} \psi & -\beta \\ -x_n I_n & \phi \end{pmatrix} = \begin{pmatrix} I_n & -\phi\beta + \alpha\phi \\ 0 & I_n \end{pmatrix},$$

we have

$$W_1 = \begin{pmatrix} \phi & \alpha \\ x_n I_n & \psi \end{pmatrix}$$

is in $GL_{2n}(A)$.

Note that $A(det(\phi)) + Ax_n = A$. Hence, by Lemma 5.1.1, there is an ϵ in $E_n(A)$ such that

$$w_1 = \begin{pmatrix} x_n^n & 0 \\ 0^t & I_{n-1} \end{pmatrix} \equiv x_n\epsilon \quad \text{modulo} \quad (det(\phi)A).$$

Hence $w_1 = x_n\epsilon + (det\phi)\mu$ for some μ in $\mathbf{M}_n(A)$. If $\gamma = \mu(adj\phi)$, then $\gamma\phi = (det\phi)\mu$.

Now

$$W_2 = \begin{pmatrix} I_n & 0 \\ \gamma & \epsilon \end{pmatrix} \begin{pmatrix} \phi & \alpha \\ x_n I_n & \psi \end{pmatrix} = \begin{pmatrix} \phi & \alpha \\ w_1 & \alpha\gamma + \epsilon\psi \end{pmatrix}$$

is in $GL_{2n}(A)$. It follows that

$$W_2 = \begin{pmatrix} \phi_2 & \alpha_2 \\ w_2 & \psi_2 \end{pmatrix} \qquad \text{where}$$

1. ϕ_2 is in $GL_{n+1}(A)$, with its first column $(x_0, x_1, \ldots, x_{n-1}, x_n^n)^t$,

2. α_2, ψ_2 are in $\mathbf{M}_{n-1}(A)$ and

3. $w_2 = (0^t, I_{n-1}, \lambda)$ is an $(n-1) \times (n+1)$-matrix, where $0^t = (0, 0, \ldots, 0)$ and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{n-1})^t$ for some $\lambda_1, \ldots, \lambda_{n-1}$ in A.

By repeated use of (5) of Remark 5.1.1, it follows that there are ϵ_1 and ϵ_2 in $E_{2n}(A)$ such that

$$\epsilon_1 W_2 \epsilon_2 = \begin{pmatrix} \phi_3 & 0 \\ 0 & I_{n-1} \end{pmatrix}$$

in $GL_{2n}(A)$, where ϕ_3 is in $GL_{n+1}(A)$, with $(x_0, \ldots, x_{n-1}, x_n^n)^t$ as its first column. This completes the proof of Proposition 5.3.1.

The following is an immediate Corollary to Proposition 5.3.1.

Corollary 5.3.1 Let $(x_0, x_1, \ldots, x_n)$ be a unimodular row in $U_{n+1}(A)$. Then $(x_0, x_1, x_2^2, \ldots, x_n^n)$ is completable to an invertible matrix.

Now the proof of Theorem 5.3.1 will follow from Corollary 5.3.1, by "shifting" the exponents as follows.

Lemma 5.3.1 Let $(x_0, x_1, \ldots, x_n)$ be a unimodular row in $U_{n+1}(A)$ and let r be a nonnegative integer. Then there is an invertible matrix α such that

$$(x_0^r, x_1, x_2, \ldots, x_n)\alpha = (x_0, x_1^r, x_2, \ldots, x_n).$$

To prove Lemma 5.3.1, we need the following lemma.

Lemma 5.3.2 Suppose $(x_0, x_1, \ldots, x_n)$ is a unimodular row in $U_{n+1}(A)$. Let T be an indeterminate and r be a nonnegative integer. Then the kernel P of the map

$$A[T]^{n+1} \to A[T],$$

defined by the unimodular row

$$(x_0^r, x_1 + x_0 T, x_2, \ldots, x_n)$$

is an extended projective module.

Proof. By Quillen's theorem, we can assume that A is local. Let m be the maximal ideal of A. If one of $x_0, x_2, \ldots, x_n$ does not belong to m then P is free by Exercise 2.6.1. So, we assume that $x_0, x_2, \ldots, x_n$ are in m and hence x_1 is a unit in A. Since x_1 is a unit, it follows that $(x_0^r, x_1 + x_0T)$ is a unimodular row in $A[T]^2$. Since unimodular rows of length two are completable to an invertible matrix, there is an invertible matrix α_0 in $SL_2(A[T]) \subseteq GL_{n+1}(A[T])$ such that $(x_0^r, x_1 + x_0T, x_2, \ldots, x_n)\alpha_0 = (1, 0, x_2, \ldots, x_n)$. Hence there is an α in $GL_{n+1}(A)$ such that $(x_0, x_1 + x_0T, x_2, \ldots, x_n)\alpha = (1, 0, \ldots, 0)$. So, P is free by Exercise 2.6.1. and the proof of Lemma 5.3.2 is complete.

Proof of Lemma 5.3.1. By Lemma 5.3.2, there is an exact sequence

$$0 \longrightarrow P_0[T] \longrightarrow A[T]^{n+1} \xrightarrow{\phi} A[T] \longrightarrow 0$$

of $A[T]$-modules, where T is an indeterminate, $\phi : A[T]^{n+1} \to A[T]$ is the map defined by the unimodular row $(x_0^r, x_1 + x_0T, x_2, \ldots, x_n)$ and P_0 is a projective A-module. Specializing, respectively, at $T = 0, -1$ we get the following two exact sequences

$$0 \longrightarrow P_0 \longrightarrow A^{n+1} \xrightarrow{\phi_0} A \longrightarrow 0$$

$$0 \longrightarrow P_0 \longrightarrow A^{n+1} \xrightarrow{\phi_{-1}} A \longrightarrow 0$$

where $\phi_0 : A^{n+1} \to A$ is the map defined by $(x_0^r, x_1, x_2, \ldots, x_n)$ and $\phi_{-1} : A^{n+1} \to A$ is the map defined by $(x_0^r, x_1 - x_0, x_2, \ldots, x_n)$.

It follows that there is an α_1 in $GL_{n+1}(A)$ such that

$$(x_0^r, x_1, x_2, \ldots, x_n)\alpha_1 = (x_0^r, x_1 - x_0, x_2, \ldots, x_n).$$

By adding $x_1^{r-1} + x_1^{r-2}x_0 + \cdots + x_0^{r-1}$-times the second column to the first column we get that there is an α_2 in $GL_{n+1}(A)$ such that

$$(x_0^r, x_1 - x_0, \ldots, x_n)\alpha_2 = (x_1^r, x_1 - x_0, \ldots, x_n).$$

Hence we have

$$(x_0^r, x_1, x_2, \ldots, x_n)\alpha_1\alpha_2 = (x_1^r, x_1 - x_0, x_2, \ldots, x_n).$$

Similarly, working with the unimodular row $(x_1, x_1 - x_0, x_2, \ldots, x_n)$, we get an α_3 in $GL_{n+1}(A)$ such that

$$(x_1^r, x_1 - x_0, \ldots, x_n)\alpha_3 = (x_1^r, -x_0, \ldots, x_n).$$

It is also easy to see that there is an α_4 in $GL_{n+1}(A)$ such that

$$(x_1^r, -x_0, \ldots, x_n)\alpha_4 = (x_0, x_1^r, \ldots, x_n).$$

Hence we have $(x_0^r, x_1, x_2, \ldots, x_n)\alpha_1\alpha_2\alpha_3\alpha_4 = (x_0, x_1^r, x_2, \ldots, x_n)$. This completes the proof of Lemma 5.3.1.

Chapter 6

Complete Intersections

In this Chapter we shall be concerned with the set theoretic and ideal theoretic number of generators of ideals in polynomial rings over commutative noetherian rings. As before, for a module M over a noetherian commutative ring A, $\mu(M)$ will denote the minimal number of generators of M.

6.1 The Foundations of Complete Intersections

Definition 6.1.1 Suppose I is an ideal in a noetherian commutative ring A. We say that I is *set theoretically generated by r elements* $f_1, \ldots, f_r$ in A if $\sqrt{(f_1, \ldots, f_r)} = \sqrt{I}$.

Remark 6.1.1 It follows from Theorem 4.2.4 that any ideal I in a noetherian commutative ring A of dimension d is set theoretically generated by $d+1$ elements.

The following is a theorem of Eisenbud and Evans ([EE3]).

Theorem 6.1.1 (Eisenbud-Evans) *Suppose $R = A[X]$ is a polynomial ring over a commutative noetherian ring A with* $\dim A = d$. *Then any ideal I of R is set theoretically generated by $d+1$ elements.*

Proof. Although the theorem can be derived from Theorem 4.3.2, we give the original proof of Eisenbud-Evans. We use induction on d. We can also assume that A is reduced.

If $d = 0$ then $A[X] = k_1[X] \times \ldots \times k_r[X]$, where $k_1, \ldots, k_r$ are fields. Hence $A[X]$ is a principal ideal ring. So, I is generated by one element.

Now assume $d > 0$. Let S be the set of all nonzero divisors of A. Then $S^{-1}R = (S^{-1}A)[X] \approx k_1[X] \times \ldots \times k_r[X]$, where $k_1, \ldots, k_r$ are fields. So, $S^{-1}I$ is one generated. So, there is f_0 in I and s in S such that $I_s = (f_0)A_s[X]$. Let $\bar{A} = A/(s)$ and J be the image of I in $\bar{A}[X]$. Since $\dim \bar{A} < d$, there are $f_1, \ldots, f_d$ in I such that $\sqrt{f_1\bar{A} + \cdots + f_d\bar{A}} = \sqrt{J}$. It is easy to check that $\sqrt{(f_0, f_1, \ldots, f_d)} = \sqrt{I}$. So, I is set theoretically generated by $d + 1$ elements. This completes the proof of Theorem 6.1.1.

Definition 6.1.2 Let A be a noetherian commutative ring.

(1) A sequence of elements $a_1, \ldots, a_r$ is called a *regular sequence* if a_i is a nonzero divisor on $A/(a_1, \ldots, a_{i-1})$, for $i = 0, \ldots, r - 1$.

(2) An ideal I is called a *complete intersection ideal* of height r if I is generated by a regular sequence $a_1, \ldots, a_r$ of length r.

(3) An ideal I of A is said to be a *locally complete intersection ideal* of height r if $I_\wp$ is a complete intersection ideal of height r for all $\wp$ in $V(I) = \{\wp$ in $Spec(A) : I \subseteq \wp\}$.

Exercise 6.1.1 Suppose I is an ideal in a ring A.

(1) If I is a complete intersection ideal of height r, then I/I^2 is a free A/I-module of rank r.

(2) If I is a locally complete intersection ideal of height r, then I/I^2 is a projective A/I-module of rank r.

Proof. To prove (1) let I be generated by a regular sequence $a_1, a_2, \ldots, a_r$. Then it is easy to check that the images of $a_1, \ldots, a_r$ form a basis of I/I^2. And (2) follows from (1).

Remark 6.1.2 Let A be a commutative noetherian ring.

(1) It follows from Forster-Swan Theorem 4.2.3 that a locally complete intersection ideal I of height r in A is generated by $\dim A + 1$ elements.

(2) It also follows from Theorem 4.3.3 that a locally complete intersection ideal I of height r in a polynomial ring $R = A[X]$ is generated by $\dim A + 1$ elements.

The following is a consequence of Nakayama's lemma.

Theorem 6.1.2 (see[MK1]) *Let I be an ideal in a noetherian commutative ring A. Then $\mu(I/I^2) \leq \mu(I) \leq \mu(I/I^2) + 1$. Further, if x is an element in A, then $\mu((I, x)) \leq \mu(I/I^2) + 1$.*

Also , if A is a local ring then $\mu(I) = \mu(I/I^2)$.

Proof. Let $\mu(I/I^2) = r$ and $I = (f_1, \ldots, f_r) + I^2$. By Nakayama's lemma, there is an s in I such that $(1 + s)I \subseteq (f_1, \ldots, f_r)$. It is easy to check that $(I, x) = (f_1, \ldots, f_r, s + (1 + s)x)$. The local case is easy. This completes the proof of Theorem 6.1.2.

Exercise 6.1.2 Let $R = k[X_1, \ldots, X_n]$ be a polynomial ring in n variables over a field k. Then any maximal ideal of R is generated by n elements.

Proof. One way to prove Exercise 6.1.2 will be to use the change of variables Theorem 6.1.5 below and induction.

The following is a theorem of Ferrand and Szpiro that is very central in this theory of set theoretic complete intersections.

Theorem 6.1.3 (Ferrand-Szpiro) *Let A be a noetherian commutative ring and let I be a locally complete intersection ideal of height $r \geq 2$ and $\dim A/I \leq 1$. Then there is a locally complete intersection ideal J of height n such that*

1. $\sqrt{I} = \sqrt{J}$ *and*
2. *J/J^2 is free A/J-module of rank r.*

Proof. Since rank $I/I^2 = r$ and $\dim(A/I) \leq 1$, by Theorem 4.2.1, $I/I^2 = F \oplus L$ where L is a projective A/I-module of rank one and F is a free A/I-module of rank $r - 1$. Since $I/I^2 \otimes L$ has a free direct summand (Theorem 4.2.1), there is a surjective map $\phi : I/I^2 \longrightarrow L^{-1}$, where $L^{-1} = Hom(L, A/I)$. Let J be the ideal of A such that $J/I^2 = \text{kernel }(\phi)$.

Since $I^2 \subseteq J \subseteq I$ we have $\sqrt{I} = \sqrt{J}$. We will see that J is a locally complete intersection ideal of height r. To see this let $\wp$ be a prime ideal in $V(I) = V(J)$.

Let f_r be an element in $I_\wp$ be such that image of f_r in $L_\wp^{-1}$, via ϕ, generates $L_\wp^{-1}$ and let $f_1, \ldots, f_{r-1}$ be in $J_\wp$ be such that their images generate $(J/I^2)_\wp$. So, $I_\wp = (f_1, f_2, \ldots, f_{n-1}, f_r) + I_\wp^2$ and hence $I_\wp = (f_1, f_2, \ldots, f_{r-1}, f_r)$. By induction, we shall prove that for $1 \leq i \leq r$ there are $g_1, \ldots, g_i$ in $I_\wp$ such that

1. $I_\wp = (g_1, \ldots, g_i, f_{i+1}, \ldots, f_r)$,
2. $g_1, \ldots, g_i$ is a regular sequence and

3. $g_j - f_j$ is in $J_\wp^2$ for $j = 1$ to i.

To do this we assume that the assertions holds for $i < r$ and prove the assertions for $i+1$. We write $A' = A_\wp, I' = I_\wp, J' = J_\wp$. Let $\wp_1, \ldots, \wp_k$ be the associated primes of $A'/(g_1, \ldots, g_i)A'$ and let $P_1, \ldots, P_t$ be maximal elements in the set $\{\wp_1 \ldots, \wp_k\}$. For $k = 1$ to t, since depth $A'_{P_k} = i < r$ and since I' is generated by a regular sequence of length r, it follows that I' is not contained in P_k. Hence J' is not contained in P_k for $k = 1$ to t. Assume that f_{i+1} is in $P_1, \ldots, P_{t_0}$ and not in $P_{t_0+1}, \ldots, P_t$ and let λ be in $J'^2 \bigcap P_{t_0+1} \bigcap \cdots \bigcap P_t \backslash P_1 \bigcup \cdots \bigcup P_{t_0}$ and let $g_{i+1} = f_{i+1} + \lambda$. So, $I' = (g_1, \ldots, g_i, g_{i+1}, f_{i+2}, \ldots, f_r) + I'^2$ and hence $I' = (g_1, \ldots, g_{i+1}, f_{i+2}, \ldots, f_r)$. This establishes the assertion.

Hence there are $g_1, \ldots, g_r$ such that

1. $I' = (g_1, \ldots, g_r)$,
2. $g_1, \ldots, g_r$ is a regular sequence,
3. the images of $g_1, \ldots, g_{r-1}$ generate J'/I'^2 and the image of g_r generates $L_\wp^{-1}$.

Note that g_r^2 is in J'. Now if g is in J' then $g - (\lambda_1 g_1 + \cdots + \lambda_{r-1} g_{r-1})$ is in $I'^2 = (g_1, \ldots, g_r)^2$ for some $\lambda_1, \ldots, \lambda_{r-1}$ in A'. Hence g is in $(g_1, \ldots, g_{r-1}, g_r^2)$. So, $J' = (g_1, \ldots, g_{r-1}, g_r^2)$ is generated by a regular sequence of length r. Therefore J is a locally complete intersection ideal of height r.

Now we shall prove that J/J^2 is a free A/J-module of rank r. Note that I/J is nilpotent in A/J and hence it is enough to prove that $J/J^2 \otimes A/I \approx J/IJ$ is a free A/I-module.

We have the following two exact sequences

$$0 \longrightarrow J/I^2 \longrightarrow I/I^2 \longrightarrow L^{-1} \longrightarrow 0$$

and

$$0 \longrightarrow I^2/IJ \longrightarrow J/IJ \longrightarrow J/I^2 \longrightarrow 0$$

of projective A/I-modules. Also note that

$$L^{-1} \approx I/J \quad \text{and} \quad L^{-2} \approx I/J \otimes I/J \approx I^2/IJ.$$

Again by Theorem 4.2.1 $J/IJ \approx F \oplus L_0$ for some projective A/I-module L_0 of rank one. It is enough to prove that $L_0 \approx A/I$. We have $J/I^2 \oplus L^{-1} \approx I/I^2 \approx F \oplus L$ and $L^{-2} \oplus J/I^2 \approx J/IJ \approx F \oplus L_0$. So,

$$L^{-2} \oplus (F \oplus L) \approx L^{-2} \oplus (J/I^2 \oplus L^{-1}) \approx F \oplus L_0 \oplus L^{-1}.$$

By Theorem 4.2.2 of Bass, $L^{-2} \oplus L \approx L_0 \oplus L^{-1}$. Now Theorem 6.1.3 follows from the cancellation property of rank one projective modules (Theorem 4.2.8) and the following Lemma 6.1.1. This completes the proof of Theorem 6.1.3.

Lemma 6.1.1 Let A be a noetherian commutative ring with $\dim A \leq 1$. Let L_1 and L_2 be two rank one projective A-modules. Then $L_1 \oplus L_2 \approx A \oplus L_1 L_2$.

Proof. Without loss of generality, we can assume that A is reduced. Let S be the set of all nonzero divisors of A. Since $S^{-1}L_1 \approx S^{-1}A$, we can assume that $L_1 = I_1$ is an ideal of A. By Theorem 4.1.1, there is an f is $Hom(L_2, A)$ that is basic on $V(I_1)$ and also at all the minimal primes of A. Let $f(L_2) = I_2$, then $L_2 \approx I_2$ and $I_1 + I_2 = A$. Hence we have an exact sequence

$$0 \longrightarrow I_1 \bigcap I_2 \longrightarrow I_1 \oplus I_2 \longrightarrow A \longrightarrow 0\,.$$

Since $I_1 \bigcap I_2 = I_1 I_2 \approx L_1 L_2$, the proof of Lemma 6.1.1 is complete.

Remark. The argument used in picking $g_1, \ldots, g_{n-1}$ in the proof of Theorem 6.1.3 is known as *the prime avoidance argument*. As an exercise, an interested reader can prove Theorem 4.2.4 by using such prime avoidance arguments.

The following is another central theorem in this theory, which is due to Boratynski.

Theorem 6.1.4 (Boratynski) *Let A be a commutative ring and let I be an ideal in A such that $I = (f_1, \ldots, f_n) + I^2$. Let $J = (f_1, \ldots, f_{n-1}) + I^{(n-1)!}$. Then J is the image of a projective A-module P of rank n.*

Proof. It follows from Nakayama's lemma that there is an s in I such that $I_{1+s} = (f_1, \ldots, f_n)$ and hence $J_{1+s} = (f_1, \ldots, f_{n-1}, f_n^{(n-1)!})$.

Let $\phi_1 : A_{1+s}^n \longrightarrow J_{1+s}$ be the map defined by $\phi(e_i) = f_i$ for $i = 1$ to $n-1$ and $\phi(e_n) = f_n^{(n-1)!}$, where $e_1, \ldots, e_n$ is the standard basis of A_{1+s}^n.

Let $\phi_2 : A_s^n \longrightarrow J_s = A_s$ be the map defined by $\phi(e_1) = 1$ and $\phi(e_i) = 0$ for $i = 2$ to n.

Since $(f_1, \ldots, f_{n-1}, f_n^{(n-1)!})$ is unimodular in $A_{s(1+s)}^n$, by Suslin's Theorem 5.3.1, there is an invertible matrix θ in $SL_n(A_{s(1+s)})$ such that first row of θ is $(f_1, \ldots, f_{n-1}, f_n^{(n-1)!})$.

Now consider the following fiber product diagram :

$$\begin{tikzcd}
Q \arrow[rr] \arrow[dr, "\phi"] \arrow[dd] & & A_s^n \arrow[dr, "\phi_2"] \arrow[dd] & \\
& J \arrow[rr] \arrow[dd] & & J_s \arrow[dd] \\
A_{1+s}^n \arrow[rr, "F"] \arrow[dr, "\phi_1"] & & A_{s(1+s)}^n \arrow[dr] & \\
& J_{1+s} \arrow[rr] & & J_{s(1+s)}
\end{tikzcd} .$$

Here $F : A_{1+s}^n \to A_{s(1+s)}^n$ is the composition map

$$A_{1+s}^n \to A_{s(1+s)}^n \xrightarrow{\theta} A^n s(1+s).$$

In this diagram Q is the fiber product of A_{1+s}^n and A_s^n via θ. The map $\phi : Q \longrightarrow J$ is got by the properties of fiber product diagrams and ϕ is surjective because so are ϕ_1 and ϕ_2. Also note that Q is a projective A-module of rank n because $Q_s \approx A_s^n$ and $Q_{1+s} \approx A_{1+s}^n$. This completes the proof of Theorem 6.1.4.

Before we close this section we prove the following *change of variables* theorem that we shall need in the later sections.

Theorem 6.1.5 (Suslin) *Let $R = A[X_1, \ldots, X_n]$ be a polynomial ring over a noetherian commutative ring A and let I be an ideal in R such that height$(I) > \dim A$. Let $\phi : R \longrightarrow R$ be the A-algebra automorphism such that*

$$\phi(X_i) = X_i + X_n^{r_i} \quad \textit{for} \quad i = 1, \ldots, n-1 \quad \textit{and}$$

$$\phi(X_n) = X_n,$$

where $r_1, \ldots, r_{n-1}$ are nonnegative integers. If $r_1, \ldots, r_{n-1}$ are large enough then $\phi(I)$ contains a monic polynomial in X_n with coefficients in $A[X_1, \ldots, X_{n-1}]$. In particular, if A is a field then for any nonzero polynomial f in R, $\phi(f)$ is a monic polynomial in X_n with coefficients in $A[X_1, \ldots, X_{n-1}]$.

We shall use induction on n to prove Theorem 6.1.5. First, we need the following lemma.

Lemma 6.1.2 Let $R = A[X]$ be a polynomial ring over a noetherian commutative ring A and I be an ideal in R. Let $\ell(I) =$

$$\{a \in A : \textit{ there is } f = aX^n + a_{n-1}X^{n-1} + \cdots + a_0 \in I \textit{ with } a_{n-1}, \ldots, a_0 \textit{ in } A\}.$$

Then $\ell(I)$ is an ideal in A and $height(\ell(I)) \geq height(I)$.

Proof. It is obvious that $\ell(I)$ is an ideal. Further, since $\ell(\sqrt{I}) \subseteq \sqrt{\ell(I)}$, we can assume that I is a reduced ideal. Let $I = \wp_1 \bigcap \ldots \bigcap \wp_k$ where $\wp_1, \ldots, \wp_k$ are minimal primes over I. As $\ell(\wp_1)\ell(\wp_2)\ldots\ell(\wp_k) \subseteq \ell(I)$, it is enough to prove the lemma for prime ideals I. If I is an extended prime ideal then $I = \ell(I)R$ and hence $height(I) = height(\ell(I))$. If I is not an extended ideal then let $\wp = I \bigcap A$. So, $\wp R \neq I$ and $height(\wp) = height(I) - 1$. Note that $\wp \subseteq \ell(I)$. Let $\wp'$ be a minimal prime ideal over $\ell(I)$. As $height(\wp') = height(\wp)$ would imply that $\ell(I) = \wp$ and $I = \wp R$ is extended, we have $height(\wp') > height(\wp) = height(I) - 1$. This completes the proof of Lemma 6.1.2.

Proof of Theorem 6.1.5. If $n = 1$ then $height(\ell(I)) \geq height(I) > \dim(A)$. So, $\ell(I) = A$ and hence I contains a monic polynomial.

Now assume that $n > 1$. Write $R = A'[X_1]$ where $A' = A[X_2, \ldots, X_n]$. Let $\ell(I) =$

$$\{a \in A' : \textit{there is } f = aX_1^n + a_{n-1}X_1^{n-1} + \cdots + a_0 \in I \textit{ with } a_{n-1}, \ldots, a_0 \textit{ in } A'\}.$$

Since $height\ (\ell(I)) > \dim(A)$, by induction, we can assume that $\ell(I)$ has a monic polynomial g in X_n. So, there is an f in I such that $f = gX_1^r + g_{r-1}X_1^{r-1} + \cdots + g_0$ where $g_{r-1}, \ldots, g_0$ are in A'. Now Theorem 6.1.5 follows easily.

The following definition will be convenient for the future discussions.

Definition 6.1.3 Let $R = A[X, X^{-1}]$ be a Laurent polynomial ring over a commutative ring A. A Laurent polynomial f is called a *doubly monic polynomial* if both the coefficients of the highest and lowest degree terms of f are units in A.

6.2 Complete Intersections in Polynomial Rings

In this section we shall prove many important results about the number of generators of ideals I in polynomial rings $R = k[X_1, \ldots, X_n]$ over fields k. Recall that, up to a change of variables (Theorem 6.1.5), any nonzero ideal I contains a monic polynomial in X_n, with coefficients in $A = k[X_1, \ldots, X_{n-1}]$. Interestingly, most of the results about ideals I in polynomial rings over fields extend to ideals I in polynomial rings $R = A[X]$, over commutative noetherian rings A, that contain monic polynomials. Our first theorem in this section is as follows.

Theorem 6.2.1 ([Ma1]) *Let $R = A[X]$ be a polynomial ring over a noetherian commutative ring A and let I be an ideal of R that contains a monic polynomial. If $\mu(I/I^2) \geq \dim(R/I) + 2$, then $\mu(I) = \mu(I/I^2)$.*

To prove Theorem 6.2.1, we need the following lemma on prime avoidance.

Lemma 6.2.1 Let A be a commutative noetherian ring and let $I,\ J$ be two ideals of A such that $J \subseteq I$. Let $n = \mu(I/I^2)$ and let $f_1, f_2, \ldots, f_r$ be elements of I with $r < n$. Assume that

1. $(f_1, f_2, \ldots, f_r, g_{r+1}, \ldots, g_n) + I^2 = I \quad$ for some $g_{r+1}, \ldots, g_n$ in I,
2. whenever a prime ideal $\wp$ contains $(f_1, \ldots, f_r) + J$ and does not contain I, the image of $\wp$ in $A/(f_1, J)$ has height at least u, for some fixed integer u.

Then we can find an element f_{r+1} in I such that

1. $(f_1, \ldots, f_r, f_{r+1}, g_{r+2}, \ldots, g_n) + I^2 = I \quad$ and
2. whenever a prime ideal $\wp$ contains $(f_1, \ldots, f_r, f_{r+1}) + J$ and I is not contained in $\wp$, then the image of $\wp$ in $A/(f_1, J)$ has height at least $u + 1$.

Proof. Let $\wp_1, \ldots, \wp_k$ be minimal primes over $(f_1, \ldots, f_r) + J$ that does not contain I. Note that the images of $\wp_i$ in $A/(f_1, J)$ has height at least u. Assume that g_{r+1} is in $\wp_1 \ldots, \wp_t$ and not in $\wp_{t+1}, \ldots, \wp_k$. Let λ be in

$$I^2 \bigcap \wp_{t+1} \bigcap \ldots \bigcap \wp_k \backslash (\wp_1 \bigcup \ldots \bigcup \wp_t).$$

Now Lemma 6.2.1 follows with $f_{r+1} = g_{r+1} + \lambda$.

Proof of Theorem 6.2.1. Let $J = A \bigcap I$. Let $n = \mu(I/I^2)$ and $I = (g_1, \ldots, g_n) + I^2$ for some $g_1, \ldots, g_n$ in I. Since I contains a monic polynomial f, for large enough integers p, $f_1 = g_1 + f^p$ is monic and $I = (f_1, g_2, \ldots, g_n) + I^2$.

Since $A/J \longrightarrow R/I$ and $A/J \longrightarrow R/(J, f_1)$ are integral extensions, we have

$$\dim(R/I) = \dim(A/J) = \dim(R/(J, f_1)).$$

By repeated application of Lemma 6.2.1, we can find $f_2, \ldots, f_n$ such that $I = (f_1, \ldots, f_n) + I^2$ and for any prime ideal $\wp$ in $Spec(R)$, if $(f_1, \ldots, f_n) + JR$ is contained in $\wp$ and I is not contained in $\wp$ then the image of $\wp$ in $R/(J, f_1)$ has height at least $n - 1$, which is impossible because $n - 1 > \dim R/(J, f_1)$. Hence for any prime $\wp$ in $Spec(R)$, if $(J, f_1, f_2, \ldots, f_n)$ is contained in $\wp$ then I is also contained in $\wp$.

Now let $R_1 = R[T, T^{-1}] = A[X, T, T^{-1}]$ be the Laurent polynomial ring in the variable T over R.

We define an A-automorphism $\psi : R_1 \longrightarrow R_1$ such that

$$\psi(X) = X + T + T^{-1} \quad \text{and} \quad \psi(T) = T.$$

We shall write $I_1 = \psi(IR_1), I' = I_1 \bigcap R[T]$ and $J' = I' \bigcap R = I_1 \bigcap R$. Since $\psi(J) = J$ is contained in J', it follows that

(i) $(\psi(f_1), \ldots, \psi(f_n)) + I_1^2 = I_1$,

(ii) if $(\psi(f_1), \ldots, \psi(f_n))R_1 + J'R_1$ is contained in a prime ideal $\wp$ in $Spec(R_1)$, then I_1 is also contained in $\wp$.

Since f_1 is monic in X, we have that $\psi(f_1)$ is a doubly monic polynomial in T over R. Hence there is an integer $r_1 \geq 0$ such that $a_1 = T^{r_1}\psi(f_1)$ is a monic polynomial in $R[T]$ with $a_1(0) = 1$. We can pick integers $r_2, \ldots, r_n$ such that $a_i = T^{r_i}\psi(f_i)$ is in $TR[T]$ for $i = 2$ to n.

Since $TR[T] + a_1R[T] = R[T]$, it follows from (i) and (ii) that

(iii) $I' = (a_1, \ldots, a_n)R[T] + I'^2$ and

(iv) if $(a_1, \ldots, a_n)R[T] + J'R[T]$ is contained in a prime ideal $\wp$ in $Spec(R[T])$, then I' is also contained in $\wp$.

We shall prove that I' is generated by n elements.

First we claim that $I'_{1+J'} = (a_1, \ldots, a_n)R_{1+J'}[T]$.

To see this let m be a maximal ideal in $Spec(R_{1+J'}[T])$ that contains the ideal $(a_1, \ldots, a_n)$. Since $R_{1+J'} \longrightarrow R_{1+J'}[T]/(a_1)$ is an integral extension, J' is in the radical of $R_{1+J'}/(a_1)$ and hence by (iv) it follows that $I'_{1+J'}$ is contained in m. So, by (iii) it follows that $(I'_{1+J'})_m = (a_1, \ldots, a_n)R_{1+J'}[T]_m$. Therefore $I'_{1+J'} = (a_1, a_2, \ldots, a_n)R_{1+J'}[T]$.

So there is an s in J' such that $I'_{1+s} = (a_1, \ldots, a_n)R_{1+s}[T]$. Let

$$\phi_1 : R_{1+s}[T]^n \longrightarrow I'_{1+s}$$

be the surjective map defined by $\phi_1(e_i) = a_i$, where $e_1, \ldots, e_n$ is the standard basis of $R^n_{1+s}[T]$. Also let

$$\phi_2 : R_s[T]^n \longrightarrow I'_s = R_s[T]$$

be the surjective map defined by $\phi_2(e_1) = 1$ and $\phi_i(e_i) = 0$ for $i = 2$ to n.

Now let $K = kernel(\phi_1)$ and $K' = kernel(\phi_2)$. Note that K' is free. Further, since K_s is projective and $(K_s)_{a_1}$ is free, by Theorem 3.2.2 of Quillen and Suslin, K_s is free.

Now let "bar" denote "(modulo T)". Since $\bar{a}_1 = 1$ and $\bar{a}_i = 0$ for $i = 2, \ldots, n$ it follows that $\overline{(\phi_1)}_s = (\overline{\phi_2})_{1+s}$. So, we have an isomorphism $\beta_0 : \bar{K}_s \longrightarrow \bar{K}'_{1+s}$ such that the following diagram of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bar{K}_s & \longrightarrow & R^n_{s(1+s)} & \longrightarrow & \bar{I}'_{s(1+s)} & \longrightarrow & 0 \\
 & & \downarrow \beta_0 & & \| & & \| & & \\
0 & \longrightarrow & K'_{1+s} & \longrightarrow & R^n_{s(1+s)} & \longrightarrow & \bar{I}'_{s(1+s)} & \longrightarrow & 0
\end{array}$$

commutes.

Since K_s and K'_{1+s} are extended modules, there is an isomorphism $\beta : K_s \longrightarrow K'_{1+s}$ such that $\bar{\beta} = \beta_0$. Using splittings of $(\phi_1)_s$ and $(\phi_2)_{1+s}$ which are equal "(modulo T)", we can find an isomorphism

$$\theta : R_{s(1+s)}[T]^n \longrightarrow R_{s(1+s)}[T]^n$$

such that $\bar{\theta} = Id$ and the following diagram of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_s & \longrightarrow & R_{s(1+s}[T]^n & \xrightarrow{\phi_1} & I'_{s(1+s)} & \longrightarrow & 0 \\
 & & \downarrow \beta & & \downarrow \theta & & \| & & \\
0 & \longrightarrow & K'_{1+s} & \longrightarrow & R_{s(1+s}[T]^n & \xrightarrow{\phi_2} & I'_{s(1+s)} & \longrightarrow & 0
\end{array}$$

commutes.
Since $sR + (1+s)R = R$, we have the following fiber product diagram :

$$\begin{tikzcd}
Q \arrow[rr] \arrow[dr, "\phi"] \arrow[dd] & & R_s[T]^n \arrow[dr, "\phi_2"] \arrow[dd] & \\
 & I' \arrow[rr] \arrow[dd] & & I'_s \arrow[dd] \\
R_{1+s}[T]^n \arrow[rr, "F"] \arrow[dr, "\phi_1"] & & R_{s(1+s)}[T]^n \arrow[dr] & \\
 & I'_{1+s} \arrow[rr] & & I'_{s(1+s)}
\end{tikzcd} .$$

Here $F : R_{1+s}[T]^n \to R_{s(1+s)}[T]^n$ is the composition map

$$R_{1+s}[T]^n \to R_{s(1+s)}[T]^n \xrightarrow{\theta} R_{s(1+s)}[T]^n.$$

In this diagram Q is the fiber product of $R_s[T]^n$ and $R_{1+s}[T]^n$ via θ. The map $\phi : Q \longrightarrow I'$ is got by the properties of fiber product. We have ϕ is surjective because ϕ_1 and ϕ_2 are surjective.

If $\theta_1 : Q_s \longrightarrow R_s[T]^n$ and $\theta_2 : Q_{1+s} \longrightarrow R_{1+s}[T]^n$ are the natural isomorphisms, then $(\theta_1)_{1+s} o (\theta_2^{-1})_s = \theta \equiv Id$ (modulo T). By Lemma 4.3.3, it follows that $Q \approx R[T]^n$ is free. Hence I' is generated by n elements.

Therefore, $\psi(IR[T, T^{-1}]) = I_1 = I'_T$ is also generated by n elements and hence so is $IR[T, T^{-1}]$. Now by substituting $T = 1$, it follows that I is generated by n elements. So, the proof of Theorem 6.2.1 is complete.

Remark. Under the hypothesis of Theorem 6.2.1, Mohan Kumar ([MK2]) proved that I is the image of a projective $R = A[X]$-module of rank $\mu(I/I^2)$.

Before we go into various consequences of Theorem 6.2.1, we prove the following extension of Theorem 6.1.4 of Boratynski.

Theorem 6.2.2 (Mandal-Roy) *Let $R = A[X]$ be a polynomial ring over a commutative noetherian ring A and I be an ideal of R that contains a monic polynomial. Suppose $I = (f_1, \ldots, f_n) + I^2$ and $J = (f_1, \ldots f_{n-1}) + I^{(n-1)!}$. Then J is generated by n elements. In particular,I is set theoretically generated by n elements.*

To prove Theorem 6.2.2 we need the following proposition.

Proposition 6.2.1 (Mandal-Roy,[MR]) Let A be a commutative ring and let $(f_1, \ldots, f_n)$ be a unimodular row with $n \geq 2$. Let $r_1, \ldots, r_n$ be nonnegative integers such that the product $r_1 r_2 \ldots r_n$ is divisible by $(n-1)!$. Then there is a matrix σ in $SL_n(A)$ such that

(i) $(f_1^{r_1}, \ldots, f_n^{r_n})$ is the first row of σ and

(ii) σ_{f_1} is isotopic to identity.

Proof. For $n = 2$ the proof is obvious. Assume $n \geq 3$ and let $B =$

$\mathbb{Z}[X_1, \ldots, X_n, Y_1, \ldots, Y_n]/(X_1Y_1 + \cdots + X_nY_n - 1) = \mathbb{Z}[x_1, \ldots, x_n, y_1, \ldots, y_n]$.

Suppose $f_1g_1 + \cdots + f_ng_n = 1$ for some $g_1, \ldots, g_n$ in A. There is a homomorphism $\psi : B \longrightarrow A$ that sends x_i to f_i and y_i to g_i.

By Suslin's Theorem 5.3.1, we can find a matrix τ in $SL_n(B)$ such that $(x_1^{r_1}, \ldots, x_n^{r_n})$ is the first row of τ. Since

$$B_{x_1} \approx \mathbb{Z}[X_1, X_1^{-1}, X_2, \ldots, X_n, Y_2, \ldots, Y_n],$$

by Proposition 5.2.2., τ_{X_1} is isotopic to identity. Now the proof of Proposition 6.2.1 is completed by taking the image of τ in $SL_n(A)$ as σ.

Now we are ready to prove Theorem 6.2.2.

Proof of Theorem 6.2.2. In case $n = 1$, it is easy to see that I is a projective ideal. Since I contains a monic polynomial, I is in fact free by Theorem 3.2.2.

Assume $n \geq 2$. We can assume that f_1 is monic. As in the proof of Theorem 6.1.4, there is an s in I such that $I_{1+s} = (f_1, \ldots, f_n)$ and hence $J_{1+s} = (f_1, \ldots, f_{n-1}, f_n^{(n-1)!})$. Since $(f_1, \ldots, f_n)$ is unimodular in $R^n_{s(1+s)}$, by Proposition 6.2.1, there is a θ in $SL_n(R_{s(1+s)})$ such that θ_{f_1} is isotopic to identity and the first row of θ is $(f_1, \ldots, f_{n-1}, f_n^{(n-1)!})$. As in Theorem 6.1.4, there is a projective R-module Q of rank n that maps onto J, where Q is given by the fiber product diagram

$$\begin{array}{ccccc} Q & & \longrightarrow & & R^n_s \\ \downarrow & & & & \downarrow \\ R^n_{1+s} & \longrightarrow & R^n_{s(1+s)} & \xrightarrow{\theta} & R^n_{s(1+s)}. \end{array}$$

We want to prove Q is free. By Theorem 3.2.2, it is enough to prove Q_{f_1} is free. But Q_{f_1} is given by the fiber product diagram

$$\begin{array}{ccccc} Q_{f_1} & & \longrightarrow & & R^n_{sf_1} \\ \downarrow & & & & \downarrow \\ R^n_{(1+s)f_1} & \longrightarrow & R^n_{s(1+s)f_1} & \xrightarrow{\theta_f} & R^n_{s(1+s)f_1}. \end{array}$$

Let $\theta_1 : (Q_{f_1})_s \longrightarrow (R^n_{f_1})_s$ and $\theta_2 : (Q_{f_1})_{1+s} \longrightarrow (R^n_{f_1})_{(1+s)}$ be the natural isomorphisms. Then $(\theta_1)_{1+s}(\theta_2^{-1})_s = \theta_{f_1}$ is isotopic to identity. So, $(\theta_1)_{1+s}$ is isotopic to $(\theta_2)_s$. Hence by Lemma 4.3.2, $Q_{f_1} \approx R^n_{f_1}$ is free. Therefore Q is free by Theorem 3.2.2. So, the proof of Theorem 6.2.2 is complete.

Now we shall deduce some of the well known results about the number of generators of ideals in polynomial rings over fields, as applications of Theorem 6.2.1 and Theorem 6.2.2.

Theorem 6.2.3 (Mohan Kumar,[MK2]) *Suppose $R = k[X_1, \ldots, X_n]$ is a polynomial ring over a field k and I is an ideal of R. If $\mu(I/I^2) \geq \dim(R/I) + 2$ then $\mu(I) = \mu(I/I^2)$.*

Proof. It is an immediate consequence of Theorem 6.2.1 and the change of variables Theorem 6.1.5.

Theorem 6.2.4 (Ferrand-Szpiro, Mohan Kumar) *Let $R = k[X_1, \ldots, X_n]$ be a polynomial ring over a field k and I be a locally complete intersection ideal of R with height$(I) = n-1$. Then there are elements $f_1, \ldots, f_{n-1}$ in I such that $\sqrt{I} = \sqrt{(f_1, \ldots, f_{n-1})}$. That is, I is set theoretically generated by $n-1$ elements.*

Proof. If $n = 2$, then I is a principal ideal because $height(I) = 1$. So, we assume that $n \geq 3$. By Theorem 6.1.3, there is a locally complete intersection ideal J such that (1) $\sqrt{J} = \sqrt{I}$ and (2) J/J^2 is free A/J-module of rank $n-1$. Hence by Theorem 6.2.3 $\mu(J) = \mu(J/J^2) = n-1$. This completes the proof of Theorem 6.2.4.

The same proofs give the following extensions of Theorems 6.2.3 and 6.2.4.

Theorem 6.2.5 ([Ma1]) *Suppose $R = A[X_1, \ldots, X_n]$ is a polynomial ring over a noetherian commutative ring A and I is an ideal of R with height$(I) > \dim A$.*

1. *If $\mu(I/I^2) \geq \dim(R/I) + 2$ then $\mu(I) = \mu(I/I^2)$.*
2. *If I is a locally complete intersection ideal of height $\dim A + (n-1) \geq 3$ then I is set theoretically generated by $\dim A + n - 1$ elements.*

We remark that Theorems 6.2.3 and 6.2.4 give partial answers, respectively, to the following two long standing open questions.

Open Problems 6.2.1 Suppose $R = k[X_1, \ldots, X_n]$ is a polynomial ring over a field k and I is an ideal of R.

Question 1(Murthy (Mu1])). Is $\mu(I) = \mu(I/I^2)$?

Question 2. Suppose I has pure height $n-1$, then whether or not I is set theoretically generated by $n-1$ elements?

In case when k has positive characteristic, the Question 2 is a theorem of Cowsik and Nori ([CN]). In the next section, we shall give a proof of this theorem of Cowsik and Nori.

The following is an extension Theorem 6.2.4.

Theorem 6.2.6 ([Lu],[Ma3]) *Suppose $R = A[X]$ is a polynomial ring over a noetherian commutative ring A and I is a locally complete intersection ideal of R, with $\dim(R/I) \geq 1$. If I contains a monic polynomial, then I is set theoretically generated by d elements where $d = \dim A$.*

Proof. We shall give a proof by induction on $\dim(R/I)$. Let $r = height(I)$.

Suppose $\dim(R/I) = 1$. If $height(I) = r = 1$, then it is easy to see that I is a principal ideal. If $r \geq 2$, then by Theorem 6.1.3, there is a locally complete intersection ideal J of height r such that $\sqrt{I} = \sqrt{J}$ and $\mu(J/J^2) = r$. By Theorem 6.2.2, J is set theoretically generated by $r \leq d$ elements. Hence I is also set theoretically generated by d elements.

Now assume that $\dim(R/I) > 1$. Let $\wp_1, \ldots, \wp_k$ be the associated primes of A and let $Q_1, \ldots, Q_t$ in $Spec(R)$ be associated to R/I. Write $P_i = Q_i \bigcap A$ for $i = 1$ to t and $S = A \backslash \{\wp_1 \bigcup \ldots \bigcup \wp_k \bigcup P_1 \bigcup \ldots \bigcup P_t\}$. Let $A' = S^{-1}A$, $R' = S^{-1}R$

and $I' = S^{-1}I$. Since A' is semilocal and R'/I' is integral over $A'/I' \bigcap A'$, R'/I' is also a semilocal ring. As I'/I'^2 is projective R'/I'-module of rank r we have I'/I'^2 is also free of rank r. So, $\mu(I'/I'^2) = r$ and by Theorem 6.2.2, I' is set theoretically generated by r elements. So, there is an a in S such that I_a is set theoretically generated by $r \leq d-1$ elements.

Since a is a nonzero divisor on R/I we have $I_1 = I + aR$ is a locally complete intersection ideal of height $r+1$, and also that $\bar{I} = I_1/aR$ is a locally complete intersection ideal of height r in R/aR. Since $\dim R/I_1 < \dim R/I$, by induction $\bar{I}$ is set theoretically generated by $\dim(A/aA)$ elements. Hence $\bar{I}$ is set theoretically generated by $d-1$ elements. It follows from the following Lemma 6.2.2 that I is set theoretically generated by d elements. So, the proof of Theorem 6.2.6 is complete.

Lemma 6.2.2 ([Lu]) Let A be a commutative noetherian ring and let a be a nonzero divisor of A. Suppose that I is an ideal of A and $r \geq 0$ is an integer such that

1. I_a is set theoretically generated by r elements in A_a and
2. $I + aA/aA$ is set theoretically generated by r elements in A/aA.

Then I is set theoretically generated by $r+1$ elements.

Proof. Let $f_1, \ldots, f_r,\ g_1, \ldots, g_r$ be in I be such that $\sqrt{(f_1, \ldots, f_r)A_a} = \sqrt{I_a}$ and the images of $g_1, \ldots, g_r$ in $\bar{I} = I + aA/aA$ generate $\bar{I}$ set theoretically. Write $J = (f_1, \ldots, f_r)$. We can assume that g_i is in J_a^2 for $i = 1$ to r. Write $h_i = af_i + g_i$. Then $J_a = (h_1, \ldots, h_r)A_a + J_a^2$. By Theorem 6.1.2, there is an element h_{r+1} in J_a such that $(h_1, \ldots, h_{r+1})A_a = J_a$. We can also assume that h_{r+1} is in J. Now it is easy to see that $\sqrt{I} = \sqrt{(h_1, \ldots, h_{r+1})}$ and the proof of Lemma 6.2.2 is complete.

Corollary 6.2.1 ([Ma3]) *Suppose I is a locally complete intersection ideal in a noetherian commutative ring A with $\dim A/I \geq 1$. Then I is set theoretically generated by $\dim A$ elements.*

Proof. Write $J = IA[X] + XA[X]$. It follows from Theorem 6.2.6 that J is set theoretically generated by $\dim A$ elements. By substituting $X = 0$, we see that I is set theoretically generated by $\dim A$ elements.

Remark. In ([Fa2]), Forster asked if there is a smooth affine algebra A over a field k with $\dim(A) > 1$ and a locally complete intersection ideal I of A of height one , such that I needs at least $\dim A + 1$ set theoretic generators. We see here (Corollary 6.2.1) that this will not be possible.

6.3 The Theorem of Cowsik-Nori on Curves

In this section we give a proof of the theorem of Cowsik-Nori that settles Question 2 of the Open Problems 6.2.1 for fields of positive characteristic.

Theorem 6.3.1 (Cowsik-Nori) *Suppose k is a field of positive characteristic p. Let I be an ideal of pure height $n-1$ in the polynomial ring $R = k[X_1, \ldots, X_n]$. Then I is set theoretically generated by $n-1$ elements.*

Proof. The proof we give here is due to Moh ([Mo], unpublished). Let $K = k^{1/\infty}$. Let $R' = K[X_1, \ldots, X_n]$ and $I' = IR'$. If I' is set theoretically generated by $f_1, \ldots, f_{n-1}$ then for some large enough N, we have $f_1^{p^N}, \ldots, f_{n-1}^{p^N}$ are in I and generate I set theoretically. So, we can assume that k is a perfect field. We can also assume that I is a reduced ideal.

By Theorem 6.3.2 below, after a change of variables, we can assume that

$$k[X_1, X_2]/I \cap k[X_1, X_2] \to R/I$$

is an integral and birational extension.

Now write $A_0 = k[X_1, X_2]/I \cap k[X_1, X_2]$ and $A = k[X_1, \ldots, X_n]/I$ and let $C = \{t \ in \ A_0 \ : tA \subseteq A_0\}$ be the conductor of this extension.

We have $S^{-1}A_0 = S^{-1}A$ where S is the set of all nonzero divisors of A_0. So, C has height one. Hence A/C is Artinian and $dim_k A$ is finite. Let x_i be the image of X_i in A/C and let V_{ir} be the k-linear subspace of A/C generated by $\{x_i^{p^j} : j = r, r+1, \ldots\}$. Note that for a fixed $i = 3$ to n we have V_{ir} is a decreasing sequence of subspaces of a finite dimensional space. Hence there is an integer N such that $V_{ir} = V_{iN}$ for $i = 3, \ldots, n$ and $r = N, N+1, \ldots$.

As x_i^N is in $V_{i(N+1)}$, there are $\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{it}$ in k and $c_3, \ldots, c_n$ in $k[X_1, X_2]$ such that $f_i = X_i^{p^N} + \lambda_{i1}X_i^{p^{N+1}} + \cdots + \lambda_{it}X_i^{p^{N+t}} + c_i$ is in I for $i = 3, \ldots, n$.

For $i = 3, \ldots, n$ write $Y_i = X_i^{p^N}$ and also write $R_0 = k[X_1, X_2, Y_3, \ldots, Y_n]$ and $I_0 = I \cap R_0$. Since $\sqrt{I} = \sqrt{I_0 R}$, we shall prove that I_0 is set theoretically generated by $n-1$ elements in R_0.

Since the matrix $(\partial f_i/\partial Y_j)_{i,j=3,\ldots n}$ is identity, $R_1 = R_0/(f_3, \ldots, f_n)$ is a regular ring of dimension 2. Hence $I_0/(f_3, \ldots, f_n)$ is an invertible ideal and hence I_0 is a locally complete intersection ideal of height $n-1$. Therefore, by Theorem 6.2.4, I_0 is set theoretically generated by $n-1$ elements. So, the proof of Theorem 6.3.1 is complete.

Now we state and prove Theorem 6.3.2 on birational projection to plane that was used in the proof of Theorem 6.3.1. I learned the proof of this Theorem 6.3.2 from some notes of Balwant Singh([Si]).

Theorem 6.3.2 *Let $A = k[X_1, \ldots, X_n]$ be a polynomial ring over a perfect field k and I be a reduced ideal of pure height $n-1$. Then, after a change of variables $\phi : A \to A$, we have $k[X_1, X_2]/I \cap k[X_1, X_2] \to A/I$ is an integral and birational extension.*

We need the following lemmas to prove the theorem.

Lemma 6.3.1 Let $k[X, Y]$ be a polynomial ring in two variables over a perfect field k with *characteristic* $(k) = p$ and f be an irreducible polynomial. Then either $\partial f/\partial X \neq 0$ or $\partial f/\partial Y \neq 0$.

Proof. We assume that $p \geq 1$. If $\partial f/\partial X = \partial f/\partial Y = 0$ then $f = g^p$ for some g in $k[X, Y]$. But since f is irreducible the proof of Lemma 6.3.1 is complete.

Lemma 6.3.2 Let $k[X, Y]$ and f be as in Lemma 6.3.1. Then for large enough m if p does not divide m then $f(X + Y^m, Y)$ is monic in Y and

$$\partial f(X + Y^m, Y)/\partial Y \neq 0.$$

Proof. Write $F = F_m = f(X + Y^m, Y)$. By Theorem 6.1.5, F is monic in Y for large enough m. We also have

$$\frac{\partial F}{\partial Y} = mY^{m-1}\frac{\partial f}{\partial X}(X + Y^m, Y) + \frac{\partial f}{\partial Y}(X + Y^m, Y).$$

Now if $\frac{\partial f}{\partial Y} \neq 0$ then there is an integer m_0 such that $\partial f/\partial Y$ is not in (Y^{m_0}). Hence for any nonnegative integer m, we have that $\frac{\partial f}{\partial Y}(X + Y^m, Y)$ is not in (Y^{m_0}). In this case for $m \geq m_0 + 1$, it follows that $\partial F/\partial Y \neq 0$. If $\partial f/\partial Y = 0$, then by Lemma 6.3.1 we have $\partial f/\partial X \neq 0$ and hence

$$\partial F/\partial Y = mY^{m-1}\frac{\partial f}{\partial X}(X + Y^m, Y) \neq 0$$

if p does not divide m. This completes the proof of Lemma 6.3.2.

The following lemma is easy to prove and we omit the proof.

Lemma 6.3.3 Let $k \to K$ be a finite field extension and $K = k(y, z)$ where y is separable over k. Then $L = k(ay + z)$ for all but finitely many a in k.

Lemma 6.3.4 Let $A = k[Y,Z]$ be a polynomial ring over a field k and $\wp_1, \wp_2$ be two distinct maximal ideals in A. Then $\wp_1 \cap k[aY+Z] \neq \wp_2 \cap k[aY+Z]$ for all but finitely many a in k.

Proof. Let L be the algebraic closer of k and for $i = 1,2$ let $L_i = A/\wp_i$ be the quotient fields . Since $k \to L_i$ is finite, we can fix two k-embeddings $L_i \to L$. For $i = 1,2$ let y_i, z_i denote the images of Y, Z, respectively, in L_i. Let E be the set of all k-embeddings $\sigma : L_1 \to L$. Then E is finite. Write

$$S = \{a \ in \ k : \ a(\sigma(y_1) - y_2) = z_2 - \sigma(z_1) \ for \ some \ \sigma \ in \ E\}.$$

If $\sigma(y_1) = y_2$ and $\sigma(z_1) = z_2$ for some σ in E, then it follows that $\wp_1 = \wp_2$, which is impossible. So , we have that S is a finite set.

Let a be in k and not in S. We will see that $\wp_1 \cap k[aY+Z] \neq \wp_2 \cap k[aY+Z]$. For $i = 1,2$ since $\wp_i$ is the kernel of the map $k[Y,Z] \to L_i$, note that for σ in E, $(\sigma(y_1), \sigma(z_1)) \neq (y_2, z_2)$. Hence it follows that $\sigma(ay_1 + z_1) \neq ay_2 + z_2$ for all σ in E. So, $ay_1 + z_1$ and $ay_2 + z_2$ are not conjugates and therefore they have distinct minimal monic polynomials over k. For $i = 1,2$ let $f_i(T)$ be the minimal monic polynomial of $ay_i + z_i$. It is easy to see that $f_i(aY+Z)$ are distinct irreducible elements in $k[aY+Z]$ and are in $\wp_i \cap k[aY+Z]$. Hence $\wp_1 \cap k[aY+Z] \neq \wp_2 \cap k[aY+Z]$. This completes the proof of Lemma 6.3.4.

Now we are ready to prove the projection Theorem 6.3.2.

Proof of Theorem 6.3.2. By Theorem 6.1.5, after a change of variables,

$$k \to k[X_1] \to k[X_1, X_2]/I_2 \to \cdots \to k[X_1, \ldots, X_n]/I_n$$

are all integral extensions, where $I_r = I \cap k[X_1, \ldots, X_r]$. Assume for the moment that $k[X_1, X_2]/I_2 \to k[X_1, X_2, X_3]/I_3$ is integral and birational. Hence $k[X_1, X_2, X_4, \ldots, X_n]/J \to k[X_1, \ldots, X_n]/I_n$ is integral and birational, where $J = I \cap k[X_1, X_2, X_4, \ldots, X_n]$. Hence, by induction, again after a change of variables, $k[X_1, X_2]/I \cap k[X_1, X_2] \to k[X_1, \ldots, X_n]/I$ is integral and birational. So, it is enough to prove the theorem for $n = 3$.

We write $X_1 = X, X_2 = Y, X_3 = Z$. Also write $I = \wp_1 \cap \ldots \cap \wp_r$,where $\wp_1, \ldots, \wp_r$ are in $Spec(k[X,Y,Z])$. Again by Theorem 6.1.5, we can assume that

$$k[X,Y]/I \cap k[X,Y] \to k[X,Y,Z]/I$$

is an integral extension.

For $i = 1$ to r, we have $\wp_i \cap k[X,Y] = f_i k[X,Y]$ for some irreducible polynomial f_i in $k[X,Y]$. By Lemma 6.3.2, for large enough m that is not divisible by p, $g_i = f_i(X + Y^m, Y)$ is monic and $\partial g_i/\partial Y \neq 0$ for $i = 1, \ldots, r$. Hence after the change of variables $X \to X + Y^m, \ Y \to Y$, and replacing f_i by $f_i(X+Y^m, Y)$ we can assume that $\partial f_i/\partial Y \neq 0$.

For $i = 1, \ldots, r$ let K_i be the fraction field of $k[X, Y, Z]/\wp_i$ and x_i, y_i, z_i be the images of X, Y, Z in K_i. Let $k(X)$ be the field of fractions of $k[X]$. It follows that $k(X) \to K_i$ is integral and y_i is separable over $k(X)$. Since $k(X)$ is infinite, by Lemma 6.3.3 and Lemma 6.3.4 there is an a in $k(X)$ such that $K_i = k(X)(ay_i + z)$ for all $i = 1, \ldots, r$ and $k(X)[Y, Z] \cap \wp_i$ are distinct. Hence it follows that $k(X)[Y, Z]/I$ is generated by $aY + Z$ over $k(X)$.

Write $a = c/d$ with c, d in $k[X]$ and with $\beta c - \lambda d = 1$ for some λ and β in $k[X]$. Then $k(X)[Y, Z]/I$ is generated by $cY + dZ$ over $k(X)$.

Now

$$\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & d \\ 0 & \lambda & \beta \end{pmatrix}$$

is an invertible matrix. After the change of variables that sends $(X, Y, Z)^t$ to $\alpha^{-1}(X, Y, Z)^t$, we have that $k(X)[Y, Z]/I$ is generated by Y over $k(X)$. Hence

$$k[X, Y]/I \cap k[X, Y] \to k[X, Y, Z]/I$$

is an integral and birational extension. So, the proof of Theorem 6.3.2 is complete.

Chapter 7

The Techniques of Lindel

In this Chapter we prove some of the theorems about projective modules over polynomial rings in several variables. First, in section 7.1, we deal with the Bass-Quillen Conjecture. Later we prove the analogues of the theorems of Plumstead on the existence of unimodular elements (Theorem 4.3.2) and the Cancellation property (Theorem 4.3.1) for projective modules over polynomial rings in several variables. Techniques used in this chapter are almost entirely due to Lindel.

7.1 The Bass-Quillen Conjecture

Recall that for a field k, a ring A containing k is called *essentially of finite type* over k if A is the localization of an affine algebra over k.

In this section we prove Lindel's theorem that deals with the following conjecture.

Conjecture 7.1.1 (Bass-Quillen) Let $R = A[T_1, \ldots, T_n]$ be a polynomial ring over a regular ring A. Suppose P is a finitely generated projective R-module. Whether P is extended from A or not?

Lindel affirmatively settled this conjecture when A is essentially of finite type over a field. Before we state Lindel's theorem, we do some preparatory work.

Definition 7.1.1 Let R_1 be a subring of a commutative ring R and h be an element of R_1. We say that $R_1 \to R$ is an *analytic isomorphism* along h, if the induced map $R_1/hR_1 \to R/hR$ is an isomorphism or equivalently if $R = R_1 + hR$ and $hR_1 = hR \cap R_1$.

It follows that if $R_1 \to R$ is an analytic isomorphism along h then it is also so along h^r.

The following is a theorem of Nashier that puts some of the main arguments of Lindel in a very nice form.

Theorem 7.1.1 ([N1]) *Let (A, m) be a regular local ring, essentially of finite type over a perfect field k of dimension d. Let a be a nonzero element of m^2. Then there exists a regular local subring S of A such that*

1. *S is the localization of a polynomial ring $C = k[X_1, \ldots, X_d]$ at a maximal ideal $M = (f(X_1), X_2, \ldots, X_d)$, where $f(X_1)$ is an irreducible polynomial in X_1;*

2. *there is an element h in $S \cap aA$ such that $S \to A$ is an analytic isomorphism along h.*

Proof. Let $m = (Z, X_2, \ldots, X_d)$. By usual prime avoidance arguments, we can assume that $(a, X_2, \ldots, X_d)$ is a regular system of parameters. To see this let us assume that $(a, X_2, \ldots, X_r)$ is a regular sequence and let $\wp_1, \ldots, \wp_i$ be the mimimal prime ideals over $(a, X_2, \ldots, X_r)$. Assume that $r \leq d - 1$ and that X_{r+1} is in $\wp_1, \ldots, \wp_j$ and not in $\wp_{j+1}, \ldots, \wp_i$. Pick λ in $m^2 \cap \wp_{j+1} \cap \ldots \cap \wp_i$ that is not in $\wp_1 \cup \ldots \cup \wp_j$. Now replacing X_{r+1} by $X_{r+1} + \lambda$ we can assume that $m = (Z, X_2, \ldots, X_d)$ and $(a, X_2, \ldots, X_{r+1})$ is a regular sequence.

Now since $a, X_2, \ldots, X_d$ is a regular sequence, $k[a, X_2, \ldots, X_d]$ is a polynomial ring. Let B be the integral closure of $k[a, X_2, \ldots, X_d]$ in A and $m_1 = m \cap B$. As $m_1 \cap k[a, X_2, \ldots, X_d] = (a, X_2, \ldots, X_d)$, it follows that m_1 is a maximal ideal in B.

Since the fields of fractions $Q(k[a, X_2, \ldots, X_d])$ and $Q(A)$ have same transcendence degree over k, the extension $Q(k[a, X_2, \ldots, X_d]) \to Q(A)$ is algebraic. As B is integral over $k[a, X_2, \ldots, X_d]$, it follows that $Q(B) = Q(A)$. Also since $Q(k[a, X_2, \ldots, X_d]) \to Q(A)$ is finite, B is a finite $k[a, X_2, \ldots, X_d]$-module. As the completion of A is an integral domain and contains the completion of B_{m_1}, we have B_{m_1} is analytically irreducible. Also note that B_{m_1} is normal and that A/m_1A is finite B/m_1B-module because m_1A is m-primary and A/m is finite over k. Hence by Zariski' main theorem, we have $A = B_{m_1}$.

As k is perfect, $L = B/m_1 = k(\bar{\alpha})$ for some α in B, where bar "-" means "(modulo m_1)". Let f be the minimal monic polynomial of $\bar{\alpha}$ over k. Then $f(\alpha)$ is in m_1 and $f'(\alpha)$ is not in m_1. For y in m_1, we have $f(\alpha + y) \equiv f(\alpha) + f'(\alpha)y$ (modulo m_1^2). We claim that for a suitable choice of y in m_1 we have

$$m_1 = (f(y + \alpha), X_2, \ldots, X_d) + m_1^2.$$

Since $A = B_{m_1}$, we have $m_1/m_1^2 = L\bar{Z} + L\bar{X}_2 + \cdots + L\bar{X}_d$. Let $f(\bar{\alpha}) = c_1\bar{Z} + c_2\bar{X}_2 + \cdots + c_d\bar{X}_d$ where $c_1, \ldots, c_d$ are in L. If $c_1 \neq 0$ then take $y = 0$ and if $c_1 = 0$ then take $y = Z$. The claim is established with this choice of y. So, by replacing α by $y + \alpha$ we can assume that $m_1 = (f(\alpha), X_2, \ldots, X_d) + m_1^2$.

Let $m_1, m_2, \ldots, m_r$ be the maximal ideals in B over $(a, X_2, \ldots, X_d)$. Pick X_1 in B such that $X_1 \equiv \alpha$ (modulo m_1^2) and $X_1 \equiv 0$ (modulo m_i) for $i = 2, \ldots, r$. We claim that $m_1 = (a, f(X_1), X_2, \ldots, X_d)$. To see this note that $f(X_1) \equiv f(\alpha)$ (modulo m_1^2), and hence $m_1 = (f(X_1), X_2, \ldots, X_d) + m_1^2$. Also note that the only maximal ideal that contains $(a, f(X_1), X_2, \ldots, X_d)$ is m_1. So, the claim is established.

Since X_1 is integral over $k[X_1, \ldots, X_d]$, by replacing X_1 by $X_1 + a^j$ for some suitable j, we can assume that a is integral over $k[X_1, \ldots, X_d]$.

Write $C = k[X_1, \ldots, X_d]$, $M = (f(X_1), X_2, \ldots, X_d)$ and $S = C_M$. We shall prove that $S \to A$ is an analytic isomorphism along some h in aS.

Note that $m \cap C = M$ and that $(M, a)A = m$. We also have that

$$L \approx C[a]/(M, a) \approx B/m_1.$$

Since B is a finite $C[a]$-module, by Nakayama's lemma we have $A = B_{m_1} = C[a]_{(M,a)}$.

So, we have $A = S + aA$ and hence $S/S \cap aA = A/aA$. Let $F = T^n + \lambda_{n-1}T^{n-1} + \cdots + \lambda_0$ be the minimal monic polynomial of a over C. We claim that λ_1 is not in M. To see this first note that the maximal ideal m of $A = C[a]_{(M,a)}$ is generated by M. As a is in m^2 there are μ_i and η_i in C for $i = 0, 1, \ldots, r$ such that

$$(\eta_0 + \eta_1 a + \cdots + \eta_r a^r)a = (\mu_0 + \mu_1 a + \cdots + \mu_r a^r)$$

with μ_0, μ_1 in M and η_0 not in M. Hence there is a polynomial $H(T) = \alpha_0 + \alpha_1 T + \cdots + \alpha_r T^r$ in $C[T]$ such that $H(a) = 0$ and α_1 is not in M. Now since $F(T)$ divides $H(T)$, it follows that λ_1 is not in M.

Now write $h = \lambda_0$ and claim that $S \to A$ is an analytic isomorphism along h. First, since $h = \lambda_0 = -a(\lambda_1 + \lambda_2 a + \cdots + \lambda_{n-1}a^{n-2} + a^{n-1})$ and since the last factor is a unit in A, we have $A = S + aA = S + hA$. Also since $S[a]$ is a free S-module we have $A = S[a]_{(M,a)}$ is a flat S-module. Since S is local, $S \to A$ is a faithfully flat extension. Hence we have $hA \cap S = hS$. Therefore, $S \to A$ is an analytic isomorphism along h. This completes the proof of Theorem 7.1.1.

Now we are ready to state and prove Lindel's theorem.

Theorem 7.1.2 (Lindel,[L1]) *Let A be a regular ring of dimension d , essentially of finite type over a field k and let $R = A[T_1, \ldots, T_n]$ be a polynomial ring. Then any finitely generated projective R-module P is extended from A.*

Proof. First note that by Quillen's Theorem 3.1.1, we can assume that A is local. So, we write $A = C_\wp$ where C is an affine algebra over k and $\wp$ is a prime ideal.

Now we want to reduce the problem to the case when k is perfect. Let k_0 be the primefield of k. Write $C = k[X_1, \ldots, X_m]/(f_1, \ldots, f_r)$ where $X_1, \ldots, X_m$ are variables and $f_1, \ldots, f_r$ are in $k[X_1, \ldots, X_m]$. Since P is projective, it is the image of an idempotent endomorphism α of a free module. Let k' be the subfield of k generated by coefficients of $f_1, \ldots, f_r$ and that of the entries of α. Write $C' = k'[X_1, \ldots, X_m]/(f_1, \ldots, f_r)$, $\wp' = \wp \cap C'$, $A' = C'_{\wp'}$ and $R' = A'[T_1, \ldots, T_n]$. Since α is defined over R', it follows that P is an extension of a finitely generated R'-module P'. Also note that $A = A' \otimes_{k'} k$ is a faithfully flat extension of A'. So, A' is also regular. Since k' is a finite extension of k_0, A' is essentially of finite type over k_0. Hence by replacing A by A', we can assume that k is perfect.

Now we prove the theorem by induction on dim $A = d$. If $d \leq 1$, then P is free by Quillen-Suslin Theorem 3.2.4. Now we assume that $d \geq 2$. By Theorem 7.1.1, we can find a subring S of A such that

1. $S = k[X_1, \ldots, X_d]_M$ where $X_1, \ldots, X_d$ are variables and $M = (f(X_1), X_2, \ldots, X_d)$ is a maximal ideal with $f(X_1)$ in $k[X_1]$,
2. there is a nonzero element h in MS such that $S \to A$ is an analytic isomorphism along h.

Note that

$$\begin{array}{ccc} S & \longrightarrow & A \\ \downarrow & & \downarrow \\ S_h & \longrightarrow & A_h \end{array}$$

is a fiber product diagram.

Since dim $A_h <$ dim A, by induction P_h is extended from A_h. Hence $P_h \approx P_0 \otimes A_h[T_1 \ldots, T_n]$ for some projective A_h-module P_0. Since

$$P_0 \approx P_h/(T_1, \ldots, T_n)P_h \approx (P/(T_1, \ldots, T_n)P)_h$$

is free, P_h is also free. Let rank$(P) = r$ and F be the free S_h-module of rank r. By patching P and F via an isomorphism $F \otimes A_h \approx P_h$ we get a projective S-module P' such that $P' \otimes R \approx P$. So, by replacing A by S, we assume that $A = k[X_1, \ldots, X_d]_M$ where $M = (f(X_1), X_2, \ldots, X_d)$ is a maximal ideal in the polynomial ring $k[X_1, \ldots, X_d]$.

Write $A_0 = k[X_1, \ldots, X_{d-1}]_{(f(X_1), X_2, \ldots, X_{d-1})}$. Then $A_0[X_d] \to A$ is an analytic isomorphism along X_d. By the same argument as above,

$$P \approx P' \otimes A[T_1, \ldots, T_n]$$

for some projective $A_0[X_d, T_1, \ldots, T_n]$-module P'. But since $dim\ A_0 \leq d-1$, by induction, P' is free and hence P is also free. This completes the proof of Theorem 7.1.2.

7.2 The Unimodular Element Theorems

In this section and the next one, one of our main emphasis is on some of the techniques developed by Lindel.

In this section we prove some theorems about the existence of unimodular elements for projective modules over polynomial rings. The main existence theorem in this section is due to Bhatwadekar and Roy. First we have to set up some preliminaries and notations to start our discussion on Lindel's techniques. In the core of Lindel's methods is the *set up* achieved in the following Proposition 7.2.1.

Proposition 7.2.1 Let A be a noetherian commutative ring and M be a finitely generated A-module. Let S be a multiplicative subset of A such that $S^{-1}M$ is A_S-free of rank r. Then there are elements s in S and $e_1, \ldots, e_r$ in M and $g_1, \ldots, g_r$ in $M^* = Hom(M, A)$ such that

1. $ann\ (s) = ann\ (s^2)$,
2. with $F = Ae_1 + \cdots + Ae_r$ and $G = Ag_1 + \cdots + Ag_r$ we have $sM \subseteq F$ and $sM^* \subseteq M^*$,
3. the matrix $(g_i(e_j) : i, j = 1, \ldots, r) = diagonal(s, s, \ldots, s)$.

Further, if S consists only of nonzero divisors , then F and G are free.

Proof. There is a t in S such that M_t is free. Since $ann\ (t^k)$ is an increasing sequence, by replacing t by a power of t we can assume that $ann\ (t) = ann\ (t^2)$. We can find $e_1, \ldots, e_r$ in M such that their images form a basis of M_t. Write $F = Ae_1 + \cdots + Ae_r$. Since M is finitely generated we have $t^l M \subseteq F$ for some nonnegative integer l. Let $\phi_1, \ldots, \phi_r$ in $Hom(M_t, A_t)$ be the dual basis of $e_1, \ldots, e_r$ and hence the matrix $(\phi_i(e_j) : i, j = 1, \ldots, r) = diagonal\ (1, 1, \ldots, 1)$. There are $g_1, \ldots, g_r$ in M^* and a positive integer $l' \geq l$ such that $\phi_i = g_i/t^{l'}$ for $i = 1, \ldots, r$. We can modify g_i by multiplying it by a suitable power of t and assume that $(g_i(e_j) : i, j = 1, \ldots, r) = diagonal\ (t^{l'}, \ldots, t^{l'})$ and also that for all g in M^* , $t^{l'}g = g(e_1)g_1 + \cdots + g(e_r)g_r$. Now taking $s = t^{l'}$ the assertion follows. This completes the proof of Proposition 7.2.1.

Notations 7.2.1 Let A be a commutative ring.

1) For an A-module M and an element m in M recall that

$$O(m) = O(m, M) = \{g(m) : g \ is \ in \ M^*\}.$$

Also recall that $Um(M)$ denotes the set of all unimodular elements of M.

2) For m in M and g in M^* we shall also use the notation that

$$< g, m >=< m, g >= g(m).$$

3) Let $A = \oplus_{k \geq 0} A_k$ be a graded ring and let b be an element of A_0. Then the *substitution map* $h_b : A \to A$ is defined by $h_b(x) = b^k x$ for x in A_k.

Note that for r in A_0, we have $h_b(r) = r$ and for x in A we have $h_b(x) - x$ is in $(1 - b)A_+$ where $A_+ = A_1 \oplus A_2 \oplus \ldots$.

The lemma below is also very central in the methods of Lindel.

Lemma 7.2.1 ([L2]) Let $A = \oplus_{k\geq 0} A_k$ be a graded ring and M be a finitely generated A-module. Let S be a multiplicative subset of A_0 such that $S^{-1}M$ is free of rank r. Let s in S, $e_1, \ldots, e_r$ in M, $g_1, \ldots, g_r$ in M^*, F, G be as in Proposition 7.2.1. Suppose $h = h_{1+\lambda s^2} : A \to A$ is a substitution map for some λ in A_0. Then there are two additive maps $\chi : M \to M$ and $\chi^* : M^* \to M^*$ such that for a in A, p in M and g in M^* we have

1. $\chi(p) - p$ is in sA_+F and $\chi^*(g) - g$ is in sA_+G,
2. $\chi(ap) = h(a)\chi(p), \quad \chi^*(ag) = h(a)\chi^*(g)$,
3. $< \chi^*(g), \chi(p) >= h(g(p))$ and
4. $Ah(O(p)) \subseteq O(\chi(p))$.

Proof. For simplicity, first assume that S consists only of nonzero divisors of A. Let $sp = a_1e_1 + \cdots + a_re_r$ for some $a_1, \ldots, a_r$ in A. For $i = 1, \ldots, r$ we have $h(a_i) = a_i + s^2c_i$ for some c_i in A_+. Let $q = sc_1e_1 + \cdots + sc_re_r$ and define

$$\chi(p) = p + q.$$

Clearly, $\chi(p) - p = q$ is in sA_+F. For a in A, $sap = aa_1e_1 + \cdots + aa_re_r$ and $h(a) = a + s^2c$ for some c in A_+. So,

$$h(a_ia) = h(a_i)h(a) = a_ia + s^2(a_ic + c_ia + s^2c_ic) \quad \text{and} \quad \text{hence}$$

$$\chi(ap) = ap + \sum s(a_ic + c_ia + s^2c_ic)e_i.$$

Also $h(a)\chi(p) = (a + s^2c)(p + \sum sc_ie_i)$. Hence $h(a)\chi(p) = \chi(ap)$.

Similarly, for g in M^*, let $sg = b_1g_1 + \cdots + b_rg_r$ and $h(b_i) = b_i + s^2d_i$ for some $b_1, \ldots, b_r$ in A and $d_1, \ldots, d_r$ in A_+. Let $f = sd_1g_1 + \cdots + sd_rg_r$ and define

$$\chi^*(g) = g + f.$$

We shall check that $< \chi^*(g), \chi(p) >= h(< g, p >)$. It is easy to see that

$$s\chi(p) = h(a_1)e_1 + \cdots + h(a_r)e_r \quad \text{and} \quad s\chi^*(g) = h(b_1)g_1 + \cdots + h(b_r)g_r.$$

So,

$$s^2 < \chi^*(g), \chi(p) >=< s\chi^*(g), s\chi(p) >= s(h(a_1)h(b_1) + \cdots + h(a_r)h(b_r))$$

$$= h(s(a_1b_1 + \cdots + a_rb_r)) = h(< sg, sp >) = s^2h(< g, p >).$$

Hence $< \chi^*(g), \chi(p) >= h(< g, p >)$ and the assertion is established in this case.

In the general case, we define χ and χ^* exactly the same way. Using the fact that $ann\ (s) = ann\ (s^2)$ we check the rest to finish the proof. This completes the proof of Lemma 7.2.1.

Now we are ready to prove the theorem of Bhatwadekar and Roy on the existence of unimodular elements.

Theorem 7.2.1 (Bhatwadekar-Roy,[BR]) *Suppose $R = A[T_1, \ldots, T_n]$ is a polynomial ring over a noetherian commutative ring A with $\dim(A) = d$ finite. Let P be a finitely generated projective R-module with rank($P_\wp$) $> d$ for all $\wp$ in $Spec(R)$. Then*

1. *P has a unimodular element and*
2. *if P_t is free for some t in A then the map $Um(P) \to Um(P/tT_nP)$ is surjective.*

Proof. First we prove (2). We also write $T = T_n$. We can assume that $\mathrm{rank}(P) = r > d$ is constant. Let x be an element in $Um(P/tTP)$ and let p be an element in P whose image is x. As (p, tT) is unimodular in $P \oplus R$, by Theorem 4.1.1, there is a p' in P, such that $p + tTp'$ is basic at all prime ideals of height less than r. Hence by replacing p by $p + tTp'$ we can assume that $height(O(p)) \geq r > d$. Hence, after a change of variables (Theorem 6.1.5), we can also assume that $O(p)$ contains a monic polynomial in T. As image of $p = x$ is unimodular, it follows that $O(p)$ contains a polynomial f of the form $f = 1 + tTf_1$ where f_1 is in R.

As P_t is free, we can find $s = t^k, e_1, \ldots, e_r$ in P and $g_1, \ldots, g_r$ in P^* and F, G as in Proposition 7.2.1.

We write $R = A'[T] = A' \oplus A'T \oplus A'T^2 \oplus \ldots$, where $A' = A[T_1, \ldots, T_{n-1}]$ and look at it as a graded ring. As $A'/O(p) \cap A' \to R/O(p)$ is integral and $O(p) + s^2R = R$, we have $O(p) \cap A' + s^2A' = A'$. So, $b = 1 + s^2b'$ is in $O(p) \cap A'$ for some b' in A'. Let $h = h_b : R \to R$ be the substitution map that sends T to bT.

By Lemma 7.2.1, there are maps $\chi : P \to P$, $\chi^* : P^* \to P^*$ such that for p in P and g in P^*, we have $\chi(p) - p$ is in $sTF \subseteq sTP$ and $< \chi*(g), \chi(p) >= h(g(p))$.

Write $p' = \chi(p)$. Then the image of p' is x in P/tTP and $h(f) = 1 + tbTf_1(bT)$ and $h(b) = b$ are in $O(p')$. Hence $O(p') = R$ and p' is a unimodular element. This completes the proof of the second part of Theorem 7.2.1.

To prove (1) of Theorem 7.2.1, we assume that A is reduced and P has constant rank r. We use induction on $\dim R = d + n$. If $d + n = 0$ then R is a product of fields and the assertion is obvious. Now assume $d + n \geq 1$. If $\dim\ A \geq 1$ then let S be the set of all nonzero divisors of A. Since $\dim(S^{-1}R) < d + n$ and since $\dim\ (S^{-1}A) = 0$, by repeated application of the induction hypothesis we see that $S^{-1}P$ is free of rank r. So, we can find a t in S so that P_t is free. Again, as $R/tTR \approx (A[T]/tTA[T])[T_1, \ldots, T_{n-1}]$, by induction $Um(P/tTP)$ is nonempty. Now from (2) it follows that P has a unimodular element.

Now we are left with the case $\dim A = 0$. Since we can assume that A is reduced and $Spec(A)$ is connected, we can assume that A is a field. If $n = 1$ then $R = A[T]$ is a principal ideal domain and hence P is free. So, we assume that $n \geq 2$. Write $R = A_0[T_2, \ldots, T_n]$ where $A_0 = A[T_1]$. If the rank $r = 1$ then P is isomorphic to an invertible ideal and hence free because R is a unique factorization domain. Now, we assume that $r \geq 2$. In this case, we have $1 \leq$ dim $A_0 < r$ and the assertion follows from the previous case . This completes the proof of Theorem 7.2.1.

Remark. The proof of Theorem 7.2.1 that we gave here is due to Lindel. One of the main features of this proof is that we never used the theorems of Quillen and Suslin (e. g. Theorems 3.1.1, 3.2.3) that we have discussed before. So, this also produces an independent proof of the Quillen-Suslin Theorem 3.2.3.

The following is an easy lemma.

Lemma 7.2.2 Let R be a commutative ring and $I,\ J$ be two ideals. Suppose P is a finitely generated projective R-module. Then the map

$$Um(P/IJP) \to Um(P/IP)$$

is surjective if the map

$$Um(P/JP) \to Um(P/(I+J)P))$$

is surjective.

Proof. The proof follows by chasing the following fiber product diagram:

$$\begin{array}{ccc} P/(I\cap J)P & \longrightarrow & P/IP \\ \downarrow & & \downarrow \\ P/JP & \longrightarrow & P/(I+J)P. \end{array}$$

This completes the proof of Lemma 7.2.2.

Theorem 7.2.2 (Lindel,[L2]) *Let $R = A[X_1, \ldots, X_n]$ be a polynomial ring over a noetherian commutative ring A with* $\dim(A) = d$. *Suppose P is a finitely generated projective R-module and I is an ideal in A. If rank$(P_\wp) > d$ for all $\wp$ in Spec(R) then*

1. *the map $Um(P) \to Um(P/IP)$ is surjective and*

2. the map $Um(P) \to Um(P/X_nP)$ is surjective.

Proof. We assume that A has no nontrivial idempotent element and hence P has constant a rank. We use induction on $\dim(R) = d+n$. In the case $n = 0$ 1) follows from Theorem 4.1.1. Also if $d = 0$ then I is a nilpotent ideal and P is free by Theorem 3.2.3. So, both the assertions follow in this case.

Now assume that $d \geq 1$ and $n \geq 1$. Pick a nonzero divisor s in A such that P_s is free. By Theorem 7.2.1,

$$Um(P) \to Um(P/sX_nP)$$

is surjective. Also by induction hypothesis

$$Um(P/sP) \to Um(P/(s, X_n)P)$$

is surjective. By Lemma 7.2.2,

$$Um(P/sX_nP) \to Um(P/X_nP)$$

is surjective. Therefore $Um(P) \to Um(P/X_nP)$ is surjective and 2) is established.

Now we prove 1). Note that for an element p in P, if the image of p in P/IP is unimodular then there is an element a in I such that $1+a$ is in $O(p)$. So, the image of p in P/aP is also unimodular. Hence we can assume that $I = (a)$ is generated by one element a in A. Also let s be as above. Again by induction

$$Um(P/sP) \to Um(P/((a, s)P) \quad \text{and} \quad Um(P/X_nP) \to Um(P/(as, X_n)P)$$

are surjective. It follows, by Lemma 7.2.2, that

$$Um(P/asP) \to Um(P/aP) \quad \text{and} \quad Um(P/asX_nP) \to Um(P/asP)$$

are surjective. Also by Theorem 7.2.1, $Um(P) \to Um(P/asX_nP)$ is surjective and hence the proof of Theorem 7.2.2 is complete.

7.3 The Action of Transvections

In this section we shall prove the theorem of Lindel about the action of the group of *transvections* on unimodular elements of projective modules over polynomial rings and derive two important theorems([Ra], [BM]). First we give the definition and some elementary properties of transvections.

Definition 7.3.1 Let M be a finitely generated module over a commutative ring R and $M^* = Hom(M, R)$.

1) For an element p in M and g in M^* by pg we mean the endomorphism of M that sends m to $g(m)p$. If we consider p as a map from R to M then this notation is consistent with that of the composition of maps. Note that if $g(p) = 0$ then $(pg)^2 = 0$ and hence $Id_M + pg$ (henceforth we write $1 + pg$) is an automorphism of M.

2) An automorphism of M of the form $1 + pg$ where p is in M and g is in M^* with $g(p) = 0$ will be called a *transvection* of M if either p is in $Um(M)$ or if g is in $Um(M^*)$.

3) As usual $Aut(M)$ will denote the group of all R-linear automorphisms of M. For an ideal I of R, $\mathcal{E}l(M, I)$ will denote the subgroup of $Aut(M)$ generated by transvections of the type $1 + pg$ such that $pg \equiv 0$ (modulo I). If M is projective then, as p or g is unimodular, $pg \equiv 0$ (modulo I) if and only if p is in IM or g is in IM^*. We write $\mathcal{E}l(M)$ for $\mathcal{E}l(M, R)$. This subgroup $\mathcal{E}l(M)$ will be called the *group of transvections* of M.

4) For an ideal I, the group of all $r \times r$-matrices with determinant one that are identity (modulo I) will be denoted by $SL_r(R, I)$.

5) For R-modules M and N an element of $M \oplus N$ will be denoted by a "row" (m, n). For this reason, for an $r \times r$- matrix α and $(f_1, \ldots, f_r)$ in R^r we may write $\alpha(f_1, \ldots, f_r)$ for *what would be denoted in the matrix notation,* by $\alpha(f_1 \ldots, f_r)^t$.

The following are some of the basic facts about the group of transvections.

Lemma 7.3.1 Let R be a noetherian commutative ring and I be an ideal of R.

1) Suppose M is a finitely generated R-module. Then $\mathcal{E}l(M, I)$ is a normal subgroup of $Aut(M)$.

2) The subgroup $E_r(R)$ (see Definition 5.1.1) of $GL_r(R)$ of the elementary matrices is contained in $\mathcal{E}l(R^r)$. If projective R-modules are free then $E_r(R) = \mathcal{E}l(R^r)$.

3) ([BR]) Let P be a finitely generated projective R-module such that the map $Um(P) \to Um(P/IP)$ is surjective. Then $\mathcal{E}l(P) \to \mathcal{E}l(P/IP)$ is surjective. Further, if $R = A[X]$ is a polynomial ring and I is an ideal of A then $\mathcal{E}l(P, X) \to \mathcal{E}l(P/IP, X)$ is surjective.

Proof. The proof of 1) follows from the identity that

$$u(1 + pg)u^{-1} = 1 + u(p)(gu^{-1}).$$

Let $E_{(i,j)}(\lambda)$, for λ in R, be the $r \times r$-matrix whose only nonzero entry is at the (i,j)-th place and is equal to λ. Then the generator $e_{(i,j)} = 1 + E_{(i,j)}(\lambda)$ of $E_r(R)$ is the map $1 + \lambda e_i p_j$ where $e_1, \ldots, e_r$ is the standard basis of R^r and p_j is the j-th projection $R^r \to R$. Hence $E_r(R) \subseteq \mathcal{E}l(R^r)$. Now to see the last part of 2) observe that the definition of $E_r(R)$ is independent of the basis and a generator of $\mathcal{E}l(R^r)$ is in $E_r(R)$ with respect to some suitable basis.

Now we prove 3). We shall only prove the last part of 3). Since the map $Um(P) \to Um(P/IP)$ is surjective we shall derive that the map

$$Um(P^*) \to Um(P^*/IP^*)$$

is also surjective. To see this let ϕ be in $Um(P^*/IP^*)$ and let f in P^* be a lift of ϕ. There is an element x in $Um(P/IP)$ such that $\phi(x) = 1$. So, by hypothesis, we can find a y in $Um(P)$ whose image is x. So, $f(y) = 1 + a$ for some a in I and $g(y) = 1$ for some g in P^*. If $\psi = f - ag$ then $\psi(y) = 1$. Hence ψ is in $Um(P^*)$ and is a lift of ϕ. So, $Um(P^*) \to Um(P^*/IP^*)$ is surjective.

Let $1 + pg$ be a generator of $\mathcal{E}l(P/IP, X)$ for some p in P/IP and g in $(P/IP)^*$. Assume p is unimodular. Since $pg \equiv 0$ (modulo X), we have $g = Xg'$ for some g' in $(P/IP)^*$.

Let q in $Um(P)$ be a lift of p and $f_1,\ f_2$ in P^* be such that f_1 lifts of g' and $f_2(q) = 1$. As $Xg'(p) = 0$, we have $g'(p) = 0$ and hence $f_1(q) = b$ is in IR. Write $f = Xf_1 - bXf_2$. Then f is in XP^* and $f(q) = 0$. Also we see that $1 + qf$ lifts $1 + pg$. Similarly, we see that if g is unimodular then $1 + pg$ lifts in $\mathcal{E}l(P, X)$. This completes the proof of Lemma 7.3.1.

The following is a version of a *key lemma* in the work of Lindel that extends the corresponding theorem of Suslin (see Corollary 7.3.1) for unimodular rows in polynomial rings.

Lemma 7.3.2 Let $R = A[X]$ be a polynomial ring over a noetherian commutative ring A and let M be a finitely generated R-module. Assume that s is in A and X is a nonzero divisor on M. Assume that $e_1, \ldots, e_r$ (*resp.* $g_1, \ldots, g_r$) are elements of M (*resp.* M^*) such that the matrix

$$(< g_i, e_j >: i, j = 1, \ldots, r) = \textit{diagonal}\ (1, 1, s, \ldots, s).$$

Let $p(X) = f_1(X)e_1 + f_2(X)e_2 + \cdots + f_r(X)e_r$ be in M for some $f_1, \ldots, f_r$ in R, such that

1. $f_1 \equiv 1$ (modulo sX),
2. f_2 is a monic polynomial,
3. $f_i(0) = 0$ for $i = 2, \ldots, r$ and
4. $(f_1, f_2, \ldots, f_r) = R$.

Then for all $h(X)$ in R, we have $p(h(X))$ is unimodular. Further, for $h,\ h'$ in R with $h(0) = h'(0) = 0$, whenever $h - h'$ is in (sX), there is an u in $SL_2(R, sX)\mathcal{E}l(M, X)$ such that

$$u(p(h(X))) = p(h'(X)).$$

Remark 7.3.1 Before we go into the proof of Lemma 7.3.2 the following clarifications are in order.

1. Note that the substitution $p(h)$ in the statement of Lemma 7.3.2 has obvious meaning.

2. Note that in the statement of Lemma 7.3.2, $Re_1 + Re_2$ is identified with R^2. Under this identification we have $M = R^2 \oplus N$ where

   $$N = \{m \ in \ M : g_1(m) = g_2(m) = 0\}.$$

   Because of this $SL_2(R)$ can be identified as a subgroup of $Aut(M)$.

3. Also note that $\mathcal{E}l(M, X)$ is a normal subgroup of $Aut(M)$. Therefore it follows that $SL_2(R, sX)\mathcal{E}l(M, X)$ is a subgroup of $Aut(M)$.

The following is an important lemma of Suslin that we need to prove Lemma 7.3.2.

Lemma 7.3.3 (Suslin) Let $R = A[X]$ be a polynomial ring over a commutative ring A and let c be in $A \cap (f_1, f_2)$ for some $f_1,\ f_2$ in R. Then for any ideal I of R and $b,\ b'$ in R with $b - b'$ in cI, there is a matrix u in $SL_2(R, I)$ such that

$$u(f_1(b), f_2(b)) = (f_1(b'), f_2(b')).$$

Proof. We write $c = f_1(X)g_1(X) + f_2(X)g_2(X)$ for some $g_1,\ g_2$ in R and $b' = b + cy$ for some y in I.

First we assume that c is a nonzero divisor. Write

$$\alpha = \begin{pmatrix} f_1(b') & -g_2(b') \\ f_2(b') & g_1(b') \end{pmatrix} \begin{pmatrix} g_1(b) & g_2(b) \\ -f_2(b) & f_1(b) \end{pmatrix}.$$

Then $\det\alpha = c^2$ and $\alpha \equiv 0$ (modulo c). Hence $\alpha = cu$ for some u. Since $c^2 detu = det\alpha = c^2$ and c is nonzero divisor in A, it follows that u is in $SL_2(R)$. Also note that there are $\lambda, \mu, \gamma, \delta$ in R such that

$$\alpha = \begin{pmatrix} f_1(b) + cy\lambda & -g_2(b) + cy\mu \\ f_2(b) + cy\gamma & g_1(b) + cy\delta \end{pmatrix} \begin{pmatrix} g_1(b) & g_2(b) \\ -f_2(b) & f_1(b) \end{pmatrix}$$

and hence
$$u = \begin{pmatrix} 1+y\lambda' & y\mu' \\ y\gamma' & 1+y\delta' \end{pmatrix}$$
for some $\lambda', \mu', \gamma', \delta'$ in R. Therefore u is in $SL_2(R, I)$.

Also since $\alpha(f_1(b), f_2(b)) = c((f_1(b'), f_2(b'))$ we have
$$u((f_1(b), f_2(b))) = (f_1(b'), f_2(b'))$$
and the proof is complete in this case.

In the general case when c is possibly a zero divisor we proceed as follows. Let $Y,\ Z$ be two other variables and for $i = 1, 2$ let $\phi_i,\ \psi_i$ in $A[X, Y, Z]$ be defined by identities

1. $f_i(X + YZ) = f_i(X) + YZ\phi_i(X, Y, Z)$ and
2. $g_i(X + YZ) = g_i(X) + YZ\psi_i(X, Y, Z)$.

Write $\phi'_i = \phi_i(b, y, c)$ and $\psi'_i = \psi_i(b, y, c)$ and let
$$u = \begin{pmatrix} 1 + y\phi'_1 g_1(b) + y\psi'_2 f_2(b) & y\phi'_1 - y\psi'_2 f_1(b) \\ y\phi'_2 g_1(b) - y\psi'_1 f_2(b) & 1 + y\phi'_2 g_2(b) + y\psi'_1 f_1(b) \end{pmatrix}.$$

Clearly, $u \equiv Id$ (modulo I). Now using the identity
$$c = f_1(X)g_1(X) + f_2(X)g_2(X)$$
$$= f_1(X + YZ)g_1(X + YZ) + f_2(X + YZ)g_2(X + YZ),$$
the proof of Lemma 7.3.3 is finished by direct computations.

Now we are ready to prove Lindel's Lemma 7.3.2.

Proof of Lemma 7.3.2. We write $M = R^2 \oplus N$ where $N = \{m\ in\ M : g_1(m) = g_2(m) = 0\}$. Write $G = SL_2(R, sX)\mathcal{E}l(M, X)$. Let $J =$
$$\{b \in A : for\ h,\ h' \in XR\ with\ h - h'\ in\ (bsX),\ u(p(h)) = p(h') for\ some\ u \in G\}.$$
Clearly, J is an ideal. We shall prove that $J = A$.

First we claim that $A \cap (f_1, f_2)$ is contained in J. To prove this let $b = d_1 f_1 + d_2 f_2$ be in $A \cap (f_1, f_2)$. Let $h,\ h'$ be in R such that $h(0) = h'(0) = 0$ and $h - h'$ is in (bsX). By Lemma 7.3.3, there is an u in $SL_2(R, sX)$ such that $u(f_1(h), f_2(h)) = (f_1(h'), f_2(h'))$.

As $h - h'$ is in (bsX), for any polynomial f in R we have $f(h) - f(h')$ is in (bsX). Therefore
$$p(h) = f_1(h)e_1 + f_2(h)e_2 + f_3(h')e_3 + \cdots + f_r(h') - bw$$
for some w in sXN. Write $u_1 = (1 + d_1(h)wg_1)(1 + d_2(h)wg_2)$. Then u_1 is in $\mathcal{E}l(M, X)$ and
$$u_1(p(h)) = p(h) + bw = f_1(h)e_1 + f_2(h)e_2 + f_3(h')e_3 + \cdots + f_r(h')e_r.$$

Hence $uu_1(p(h)) = p(h')$ and hence b is in J. So, the claim is established.

Now to prove $J = A$, assume the contrary that J is contained in a maximal ideal m of A. Since f_2 is monic, $A/A \cap (f_1, f_2) \to R/(f_1, f_2)$ is integral. Also since $(f_1, f_2) + sR = R$, it follows that $A \cap (f_1, f_2) + sA = A$. Since $(f_1, f_2) \cap A$ is contained in m, we have s is not in m.

As $f_1 \equiv 1$ (modulo sX), it follows that $(f_1, f_2, sXf_3, \ldots, sXf_r)$ is unimodular. Since $\dim(R/(m, f_2)) = 0$, by usual prime avoidance argument, $(c, f_2) + mR = R$ for some $c = f_1 + sXc_3f_3 + \cdots + sXc_rf_r$ with $c_3, \ldots, c_r$ in R. Again since $A/(c, f_2) \cap A \to R/(c, f_2)$ is integral it follows that $(c, f_2) \cap A + m = A$.

Our next claim is that $(c, f_2) \cap A$ is contained in J and hence in m. So, let b be in $(c, f_2) \cap A$ and $h,\ h'$ be in R with $h(0) = h'(0) = 0$ and $h - h'$ in (sbX). Write $u_1 = (1 + e_1hc_3(h)g_3)(1 + e_1hc_4(h)g_4) \ldots (1 + e_1hc_r(h)g_r)$. Then u_1 is in $\mathcal{E}l(M, X)$ and

$$u_1(p(h)) = p(h) + (hc_3(h)sf_3(h) + \cdots + hc_r(h)sf_r(h))e_1$$

$$= c(h)e_1 + f_2(h)e_2 + \cdots + f_r(h)e_r.$$

Similarly, there is u_2 in $\mathcal{E}l(M, X)$ such that

$$u_2(p(h')) = c(h')e_1 + f_2(h')e_2 + \cdots + f_r(h')e_r.$$

By Lemma 7.3.3, there is u_3 in $SL_2(R, sX)$ such that

$$u_3(c(h)e_1 + f_2(h)e_2 + f_3(h)e_3 + \cdots + f_r(h)e_r)$$

$$= c(h')e_1 + f_2(h')e_2 + f_3(h)e_3 + \cdots + f_r(h)e_r.$$

As in the first claim there is an u_4 in $\mathcal{E}l(M, X)$ such that

$$u_4(c(h')e_1 + f_2(h')e_2 + f_3(h)e_3 + \cdots + f_r(h)e_r)$$

$$= c(h')e_1 + f_2(h')e_2 + f_3(h')e_3 + \cdots + f_r(h')e_r.$$

So, if we let $u = u_2^{-1}u_4u_3u_1$ then u is in G and $u(p(h)) = p(h')$. Therefore, as was claimed $(c, f_2) \cap A$ is contained in J and hence in m. But this contradicts the fact that $(c, f_2) \cap A + m = A$. So, Lemma 7.3.2 is established.

The following is a version of a theorem of Suslin that will be useful later and as well be helpful to understand the proof of our main Theorem 7.3.2 in this section.

Corollary 7.3.1 Let $R = A[X]$ be a polynomial ring over a commutative ring A and let $(f_1, f_2, \ldots, f_r)$ be a unimodular row with $r \geq 3$, and f_2 be monic. If $(f_1(0), f_2(0), \ldots, f_r(0)) = (1, 0, \ldots, 0)$ then there is an u in $\mathcal{E}l(R^r, X)$ such that $u((f_1, \ldots, f_r)) = (1, 0, \ldots, 0)$.

Proof. We apply Lemma 7.3.2 with $s = 1$ and the standard basis $e_1, \ldots, e_r$ of R^r. So,there is an u in $\mathcal{E}l(R^r, X)$ and v in $SL_2(R, X)$ such that

$$vu((f_1, \ldots, f_r)) = (f_1(0), \ldots, f_r(0)) = (1, 0, \ldots, 0).$$

Hence $u((f_1, \ldots, f_r)) = (1 + Xg_1, Xg_2, 0, \ldots, 0)$ for some $g_1,\ g_2$ in R.

Since the arguments will not be different, we assume $r = 3$ for notational conveniences. There are $\lambda_1,\ \lambda_2$ in R such that $(1+Xg_1)\lambda_1 + Xg_2\lambda_2 = 1$. Write

$$w = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ X\lambda_1 & X\lambda_2 & 1 \end{pmatrix},$$

then w is in $\mathcal{E}l(R^3, X)$(see Lemma 7.3.1 part 2) and

$$w((1 + Xg_1, Xg_2, 0)) = (1 + Xg_1, Xg_2, X).$$

Now write

$$U_1 = \begin{pmatrix} 1 & 0 & -g_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$U_2 = \begin{pmatrix} 1 & 0 & 0 \\ -Xg_2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$U_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -X & 0 & 1 \end{pmatrix}.$$

By normality (Lemma 7.3.1), we have $U = U_1^{-1}U_3U_2U_1$ is in $\mathcal{E}l(R^3, X)$ and

$$U((1 + Xg_1, Xg_2, X)) = (1, 0, 0).$$

So, the proof of Corollary 7.3.1 is complete.

Corollary 7.3.2 Let $R = A[X_1, \ldots, X_n]$ be a polynomial ring over a noetherian commutative ring A with $\dim(A) = d$. Let $(f_1, \ldots, f_r)$ be a unimodular row in R^r with $r \geq max(3, d + 2)$ and $(f_1, \ldots, f_r) \equiv (1, 0, \ldots, 0)$ (modulo X_n). Then there is an u in $\mathcal{E}l(R^r, X_n)$ such that

$$u((f_1, \ldots, f_r)) = (1, 0, \ldots, 0).$$

Proof. By Theorem 4.1.1, there are $a_1, \ldots, a_{r-1}$ in R such that with $f'_i = f_i + a_i f_r$ for $i = 1, \ldots, r-1$ we have height$(f'_1, \ldots, f'_{r-1}) \geq r - 1 > d$. Since f_r is in (X_n), as in the proof of Corollary 7.3.1 there is an u in $\mathcal{E}l(R^r, X_n)$ such that

$$u((f_1, f_2, \ldots, f_r)) = (f'_1, \ldots, f'_{r-1}, f_r).$$

Replacing $(f_1, \ldots, f_r)$ by $(f'_1, \ldots, f'_{r-1}, f_r)$ we assume that the height of the ideal $(f_1, \ldots, f_{r-1})$ is strictly bigger than d. By a change of variables (Theorem 6.1.5) that sends X_i to $X_i + X_n^N$ for $i = 1, \ldots, n-1$ and X_n to X_n, for a large enough N, we can assume that there is a polynomial $f = \lambda_1 f_1 + \cdots + \lambda_{r-1} f_{r-1}$ in R that is monic in X_n. So, $f_r + X^k f$ is monic in X_n for some suitable integer $k \geq 1$. Again as in the proof of Corollary 7.3.1, there is a v in $\mathcal{E}l(R^r, X)$ such that

$$v((f_1, \ldots, f_r)) = (f_1, \ldots, f_{r-1}, f_r + X^k f).$$

Since $f_r + X^k f$ is monic in X_n, and $(f_1, \ldots, f_{r-1}, f_r + X^k f) \equiv (1, 0, \ldots, 0)$ (modulo X_n), it follows from Corollary 7.3.1 that there is a w in $\mathcal{E}l(R^r, X)$ such that $wv((f_1, \ldots, f_r)) = (1, 0, \ldots, 0)$. This completes the proof of Corollary 7.3.2.

Now we are ready to state the main theorem of Lindel in this section.

Theorem 7.3.1 (Lindel,[L2]) *Suppose $R = A[X_1, \ldots, X_n]$ is a polynomial ring over a noetherian commutative ring A with $\dim(A) = d$. Suppose P is a finitely generated projective R-module with $rank(P_\wp) \geq max(2, d+1)$ for all $\wp$ in $Spec(R)$. Let (a, p) be a unimodular element in $R \oplus P$. Then there is an u in $\mathcal{E}l(R \oplus P)$ such that $u((a, p)) = (1, 0)$.*

Further, if $(a, p) \equiv (1, 0)$ (modulo X_n) then there is an u in $\mathcal{E}l(R \oplus P, X_n)$ such that $u((a, p)) = (1, 0)$.

Proof. First we prove the last part. Note also that we can assume that $Spec(A)$ is connected. Hence $\text{rank}(P) = r_0$ is constant. Since for the nilradical N of A, $\mathcal{E}l(R \oplus P, X_n) \to \mathcal{E}l((R \oplus P)/N(R \oplus P), X_n)$ is surjective (see Lemma 7.3.1), we can assume that A is reduced. We shall use induction on $\dim(A) = d$.

If $d = 0$ then P is free by Quillen-Suslin theorem (Theorem 3.2.2 or 7.2.1). In this case the assertion follows from Corollary 7.3.2.

Now we assume that $d > 0$. By Theorem 7.2.1, we can write $P = R \oplus P'$ for some projective R-module P'. We write $Q = R \oplus P = R^2 \oplus P'$ and $\text{rank}Q = r = r_0 + 1$.

Let S be the set of all nonzero divisors of A. So, $S^{-1}P'$ is free of rank $r - 2$. As in Proposition 7.2.1, we can find a nonzero divisor s of A and a free submodule F of P' with basis $e_3, \ldots, e_r$ and a free submodule G of P'^* with basis $g_3, \ldots, g_r$ such that $sP' \subseteq F$, $sP^* \subseteq G$ and the matrix

$$(< g_i, e_j >: 1, j = 3, \ldots, r) = diagonal\ (s, \ldots, s).$$

Let e_1, e_2, respectively, denote the elements $(1, 0, 0)$ and $(0, 1, 0)$ in $Q = R^2 \oplus P'$. For $i = 3, \ldots, r$ we extend g_i to Q by defining $g_i(e_1) = g_i(e_2) = 0$. We define g_1, g_2 in Q^* such that $g_{1|P'} = g_{2|P'} = 0$ and the matrix

$$(< g_i, e_j >: i, j = 1, 2) = diagonal\ (1, 1).$$

Thus we have that the matrix

$$(< g_i, e_j >: i, j = 1, \ldots, r) = diagonal\ (1, 1, s, \ldots, s).$$

We shall write $X_n = X$ and $a = f_1$. So, $(a, p) = f_1e_1 + f_2e_2 + p'$ for some f_1, f_2 in R and p' in P'. As $(f_1, p) \equiv (1, 0)$ (modulo X), we have $f_1(0) = 1,\ f_2(0) = 0$ and p' is in XP'. (*For a polynomial f in R, $f(0)$ denotes $f(X_1, \ldots, X_{n-1}, 0)$.*)

Since $\dim(A/sA) < d$, by induction, there is an u' in $\mathcal{E}l(Q/sQ, X)$ such that $u'(f_1q_1 + f_2q_2 + q') = q_1$, where q_1, q_2, q' are ,respectively, the images of $e_1,\ e_2,\ p'$ in Q/sQ. Since $Um(Q) \to Um(Q/sQ)$ is surjective (see Theorem 7.2.2), there is an u in $\mathcal{E}l(Q, X)$ that lifts u' (Lemma 7.3.1). Hence replacing $f_1e_1 + f_2e_2 + p'$ by $u(f_1e_1 + f_2e_2 + p')$ we assume that $f_1 \equiv 1$ (modulo sX), f_2 is in (sX) and p' is in sXP'.

Since $f_1e_1 + f_2e_2 + p'$ is unimodular and $f_1 \equiv 1$ (modulo sX), we have $f_1e_1 + sXf_2e_2 + p'$ is also unimodular. Hence by Theorem 4.1.1, there is an h_1 in R and a p'' in P' such that the ideal

$$\mathcal{I} = R(f_1 + h_1sXf_2) + O(p' + sXf_2p'')$$

has height at least $r - 1 \geq d + 1$. Therefore, after a change of variables (Theorem 6.1.5), that sends X_i to $X_i + X^N$ for $i = 1, \ldots, n-1$ and X to X, where N is large enough, we can assume that $\mathcal{I}$ contains a monic polynomial h in X with coefficients in $A[X_1, \ldots, X_{n-1}]$.

Now we write $R = B[X]$ where $B = A[X_1, \ldots, X_{n-1}]$. We also write $h = (f_1 + sXf_2h_1)h' + g(p' + sXf_2p'')$ for some h' in R and g in P^*. Let k be a positive integer such that $f_2 + X^kh$ is a monic polynomial in X. We shall regard g as an element in Q^* by putting $g(e_1) = g(e_2) = 0$.

For $i = 1, \ldots, 4$ define u_i in $\mathcal{E}l(Q, X)$ as follows:

$$u_1 = 1 + e_1sXh_1g_2,$$

$$u_2 = 1 + sXp''g_2,$$

$$u_3 = 1 + e_2X^kh'g_1,$$

$$u_4 = 1 + e_2X^kg.$$

By replacing $f_1e_1 + f_2e_2 + p'$ by $u_4u_3u_2u_1(f_1e_1 + f_2e_2 + p')$ we assume that $f_1 \equiv 1$ (modulo sX), $f_2(0) = 0$ and that p' is in sXP'.

Since $sP' \subseteq F$, we have $p' = X(f_3e_3 + \cdots + f_re_r)$ for some $f_3, \ldots, f_r$ in R. Thus

$$f_1e_1 + f_2e_2 + p' = f_1e_1 + f_2e_2 + Xf_3e_3 + \cdots + Xf_re_r.$$

As $f_1 \equiv 1$ (modulo sX) and f_2 is monic, we have $(f_1, f_2) \cap B + sB = B$. So, there is an element b in B such that $1 - sb$ is in (f_1, f_2). Moreover $f_1e_1 + f_2e_2 + Xf_3e_3 + \cdots + Xf_re_r$ is unimodular. Since Q_s is free and $f_1 \equiv 1$ (modulo sX) we have $(f_1, f_2, Xf_3, \ldots, Xf_r)$ is a unimodular row.

By an application of Lemma 7.3.2, with $h = X$ and $h' = (1 - sb)X$, we get an u_5 in $\mathcal{E}l(Q, X)$ and an u_6 in $SL_2(R, sX)$ such that

$$u_5(f_1e_1 + f_2e_2 + Xf_3e_3 + \cdots + Xf_re_r) =$$

$$f_1'e_1 + f_2'e_2 + (1 - sb)Xf_3((1 - sb)X)e_3 + \cdots + (1 - sb)Xf_r((1 - sb)X)e_r,$$

where $u_6((f_1', f_2')) = (f_1((1 - sb)X), f_2((1 - sb)X))$.

Note that $f_i(X) - f_i((1 - sb)X)$ is in (sX) for $i = 1, 2$ and also since u_6 is in $SL_2(R, sX)$, we have $f_i'((1 - sb)X) - f_i'(X)$ is in (sX) for $i = 1, 2$. Hence $f_1' \equiv f_1 \equiv 1$ (modulo sX) and $f_2'(0) = f_2(0) = 0$. It also follows that $1 - sb$ is in $(f_1', f_2')R = (f_1((1 - sb)X), f_2((1 - sb)X))R$. As

$$f_1'e_1 + f_2'e_2 + (1 - sb)Xf_3((1 - sb)X)e_3 + \cdots + (1 - sb)Xf_r((1 - sb)X)e_r$$

is unimodular and since $1-sb$ is in (f_1', f_2'), it follows that $f_1'e_1+f_2'e_2$ is unimodular in Q and hence in R^2. Therefore, there is g in Q^* such that $g(f_1'e_1+f_2'e_2) = 1$ and $g_{|P'} = 0$. Let

$$v_3 = 1 + (X + (sb - 1)Xf_3((1 - sb)X))e_3g \quad \text{and}$$

$$v_i = 1 + e_i(sb - 1)Xf_i((1 - sb)X)g$$

for $i = 4, \ldots, r$ and $v = v_3v_4 \ldots v_r$. Then v is in $\mathcal{E}l(Q, X)$ and

$$vu_5(f_1e_1 + f_2e_2 + Xf_3e_3 + \cdots + Xf_re_r)$$

$$= v(f_1'e_1 + f_2'e_2 + (1 - sb)Xf_3((1 - sb)X)e_3 + \cdots + (1 - sb)Xf_r((1 - sb)X)e_r)$$

$$= f_1'e_1 + f_2'e_2 + Xe_3.$$

Now we can write $f_1' = 1 + sXf_1''$, $f_2' = Xf_2''$ for some f_1'', f_2'' in R. Let $U_1 = 1-e_1f_1''g_3$, $U_2 = 1-e_2Xf_2''g_1$, $U_3 = 1-Xe_3g_1$ and write $U = U_1^{-1}U_3U_2U_1$. Then, since U_2, U_3 are in $\mathcal{E}l(Q, X)$, we have U is in $\mathcal{E}l(Q, X)$. Finally,

$$U(f_1'e_1 + f_2'e_2 + Xe_3) = e_1.$$

This completes the proof of the later part of Theorem 7.3.1.

To prove the first part of Theorem 7.3.1, we proceed by induction on the number of variables n. (*Or we could repeat the above arguments with an appropriate version of Lindel's Lemma 7.3.2*). If $n = 0$ then the assertion follows from Theorem 4.1.1. If $n \geq 1$, then since $\mathcal{E}l(R \oplus P) \to \mathcal{E}l((R \oplus P)/X_n(R \oplus P))$ is surjective, we can assume that $(a, p) \equiv (1, 0)$ (modulo X_n). Now the assertion follows from the later part of the theorem. This complete the proof of Theorem 7.3.1.

The following theorem of Suslin is an immediate consequence of Theorem 7.3.1 and Lemma 7.3.1.

Theorem 7.3.2 (Suslin) *Let $R = k[X_1, \dots, X_n]$ be a polynomial ring over a field k. Let $(f_1, \dots, f_r)$ be a unimodular row in R^r with $r \geq 3$. Then there is an elementary matrix u in $E_r(R)$ such that $u((f_1, \dots, f_r)) = (1, 0, \dots, 0)$.*

The following theorem of Rao on Cancellation of projective modules is also an immediate consequence of Theorem 7.3.1.

Theorem 7.3.3 (Rao) *Let $R = A[X_1, \dots, X_n]$ be polynomial ring over a commutative noetherian ring A with $\dim(A) = d$. Suppose that P is a finitely generated projective R-module with rank$(P_\wp) \geq d + 1$ for all $\wp$ in Spec(R). Then P has cancellative property, i.e. $P \oplus Q \approx P' \oplus Q$ for some finitely generated projective R-modules Q and P' implies that $P \approx P'$.*

Proof. It suffices to prove the theorem when $Q = R$. Let $\varphi : P' \oplus R \to P \oplus R$ be an isomorphism and $\varphi((0, 1)) = (p, a)$. By Theorem 7.3.1, there is an u in $\mathcal{E}l(P \oplus R)$ such that $u((p, a)) = (0, 1)$ and hence $u\varphi((0, 1)) = (0, 1)$. Therefore $u\varphi$ induces an isomorphism $P' \approx P$ and the proof of Theorem 7.3.3 is complete.

The following conjecture about lifting of automorphisms of projective modules was considered in ([BM]).

Conjecture 7.3.1 ([BM]) *Suppose $R = A[X]$ is a polynomial ring over a noetherian commutative ring A with $\dim(A) = d$. Let P be a finitely generated projective R-module. Then, whether the natural map*

$$Aut_R(P) \to Aut_A(P/XP)$$

is surjective?

The following partial answer on the conjecture 7.3.1 was obtained in ([BM]).

Theorem 7.3.4 (Bhatwadekar-Mandal) *Suppose $R = A[X_1, \dots, X_n]$ is a polynomial ring over a noetherian commutative ring A with $\dim(A) = d$. Let P be a finitely generated projective R-module with rank$(P_\wp) \geq d + 1$ for all $\wp$ in Spec(R).Then the map $Aut_R(P) \to Aut_{A'}(P/X_nP)$ is surjective, where $A' = A[X_1, \dots, X_{n-1}]$.*

Proof. Without loss of generality we can assume that rank$(P) = r$ is constant. If $r = 1$ then $Aut(P)$ is isomorphic to the group of units of R and $Aut(P/X_nP)$

is isomorphic to the group of units of A'. Hence $Aut(P) \to Aut(P/X_nP)$ is surjective. So, we assume $r \geq 2$. Also note that if P is free of rank r then $Aut(P) \approx GL_r(R) \to Aut(P/X_nP) \approx GL_r(A')$ is surjective.

Now let Q be a finitely generated R-module such that $F = P \oplus Q$ is free. Let "overbar" denote (modulo X_n).Let g be an automorphism of $\bar{P}$. So, $g \oplus Id_{\bar{Q}}$ can be lifted to an automorphism of $P \oplus Q = F$. Hence $g \oplus Id_{\bar{Q}} \oplus Id_{\bar{P}}$ can also be lifted to an automorphism of $P \oplus Q \oplus P$. This means that there is an automorphism $H : P \oplus F \to P \oplus F$ such that $\bar{H} = g \oplus Id_{\bar{F}}$.

By downward induction we can assume that there is an automorphism

$$H : P \oplus R \to P \oplus R$$

such that $\bar{H} = g \oplus Id_{\bar{R}}$. Let $H((0,1) = (p,a)$. Then, since $\bar{H} = g \oplus Id_{\bar{R}}$, we have $(p,a) \equiv (0,1)$ (modulo X_n). Hence, by Theorem 7.3.1 there is an u in $\mathcal{E}l(P \oplus R, X)$ such that $u((p,a)) = (0,1)$. Let $\varphi = uH$. Since $\varphi((0,1)) = (0,1)$, φ induces an automorphism h of P. Since $\bar{u} = Id$, we have $\bar{h} = g$ and the proof of Theorem 7.3.4 is complete.

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