Witt Groups of Cohen-Macaulay Rings

Satya Mandal, University Kansas
Based on a joint research project with
Sarang Sane, University of Kansas

ATM Workshop-Conference on
Classical and Non-Stable Algebraic $K-$theory
July 15-24, 2013

Objective and Notations

- To extend and apply Triangulated Witt groups (due to Balmer and Others) to Cohen-Macaulay schemes. The existing theory, mainly, applies to regular schemes.
- Let A denote any noetherian commutative ring.
  - The category of finitely generated $A-$modules will be denoted by $\mathcal{M}(A)$.
  - The category of finitely generated $A-$modules with finite projective dimensions will be denoted by $\mathcal{B} = \mathcal{M}FPD(A)$.

Continued

- $\mathcal{A} := \mathcal{M}FPDfl(A)$ will denote the subcategory of objects in $\mathcal{B}$ with finite projective dimension.
- First, we proceed to justify that, for non-regular rings A, right category to work with is $\mathcal{B} := \mathcal{M}FPD(A)$ or its subcategory $\mathcal{A} := \mathcal{M}FPDfl(A)$

Continued: Chain Complexes

In terms of chain complexes, we have the following:

- As usual $Ch^b(\mathcal{M}(A))$, $Ch^b(\mathcal{B})$, $Ch^b(\mathcal{A})$ will denote the categories of bounded chain complexes of objectcts in the respective categories. Also, $K^b(\mathcal{M}(A)), K^b(\mathcal{B})$, $K^b(\mathcal{A})$ will denote the categories of bounded chain complexes and morphisms of chain homotopic maps.
- Similarly, $K^b\left(\mathcal{P}(A)\right)$ denotes the category of chain complexes of objects in $\mathcal{P}(A)$ and morphisms of chain homotopic maps. ($\mathcal{P}(A)$ *denote the category of finitely generated projective* $A-$*modules*.)

Choice of the Exact Category

- When A is regular, there is a functor

$$\mathcal{M}(A) \longrightarrow K^b(\mathcal{P}(A)) \quad M \mapsto P_\bullet$$

where $P_\bullet$ is given by a projective resolution of M, with $H_0(P_\bullet) = M$. (use axiom of choice).

- If A is not regular, we ONLY have functor

$$\mathcal{B} \longrightarrow K^b(\mathcal{P}(A)) \quad M \mapsto P_\bullet$$

- So, we work with $\mathcal{B}, \mathcal{A}$.

Continued

- Given our choice the exact categories, we restrict ourselves in the full subcategories of complexes with homologies in $\mathcal{B}$ *or* $\mathcal{A}$.
- In this lecture, we work mostly with and denote them by

$$K^b_{\mathcal{A}}(\mathcal{A}), \quad K^b_{\mathcal{A}}(\mathcal{P}(A))$$

and comment on other similar categories.

In the regular case all these K^b-categories and the corresponding derived categories are triangulated.

- Do, we have such luxury, in the non-regular case?
- Questions:
  - What kind of dualities these categories may have?
  - Are these categories closed under cone construction?

Examples

Sankar P. Dutta gave the following example to demonstrate that $K^b_{\mathcal{B}}(\mathcal{P}(A))$ is not closed under usual duality induced by $Hom(-, A)$.

Example(Dutta). Let $(A, \mathfrak{m}, k)$ be any non-regular Cohen-Macaulay local ring, $\dim A = d$.

- Let

$$\cdots \longrightarrow P_d \xrightarrow{\partial_d} P_{d-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow k \longrightarrow 0$$

be an (infinite) projective resolution of k.

Continued

- Let $M = cokernel(\partial_d^*)$. Since $Ext^r(k, A) = 0 \ \forall \ r < d$,

$$0 \longrightarrow P_0^* \longrightarrow \cdots \longrightarrow P_d^* \longrightarrow M \longrightarrow 0 \quad is \ exact.$$

  So, $M \in \mathcal{B}$.
- Dualizing this sequence, it follows that $Ext_A^d(M, A) \cong k$, which does not have finite projective dimension.
- In particular, $K_{\mathcal{B}}^b(\mathcal{P}(A))$ is not closed under duality.

On Duality

Theorem. Let A be a Cohen-Macaulay ring and $\mathcal{A} = \mathcal{M}FPDfl(A)$.

- The functor $Ext^d(*, A) : \mathcal{A} \longrightarrow \mathcal{A}$ is a duality on $\mathcal{A}$.

- Theorem (-,Sane)

  $K^b_{\mathcal{A}}(\mathcal{P}(A))$ is closed under duality, indused by $Hom(*, A)$.

Failure of Cone Construction

The follwoing example shows that $Ch^b_{\mathcal{A}}(\mathcal{P}(A))$ is not closed under cone construction.

Example. Let $(A, \mathfrak{m})$ be a non-regular Cohen-Macauly ring with $\dim A = d$, such that $\mathfrak{m} = (f_1, f_2, \ldots, f_d, z)$. We can assume, using prime avoidance, that $f_1, f_2, \ldots, f_d$ is a regular sequence. Let $U_\bullet = Kos_\bullet(f_1, f_2, \ldots, f_d)$ be the Koszul complex.

- The only nonzero homology of $U_\bullet$ is $H_0(U_\bullet) = \frac{A}{(f_1, f_2, \ldots, f_d)} \in \mathcal{A}$,
- So, $U_\bullet$ and all its translates are objects are in $K^b_{\mathcal{A}}(\mathcal{P}(A))$.
- Let $C(z)$ be the cone of the chain map $z : U_\bullet \longrightarrow U_\bullet$.

Continued

- Considering the exact sequence of chain complexes

$$0 \longrightarrow U_\bullet \longrightarrow C(z) \longrightarrow U_\bullet[1] \longrightarrow 0$$

it follows that $H_0(Cone(z)) \cong$

$$\operatorname{coker}\left(\frac{A}{(f_1, f_2, \ldots, f_d)} \stackrel{\cdot z}{\to} \frac{A}{(f_1, f_2, \ldots, f_d)}\right) \cong \frac{A}{\mathfrak{m}} \notin \mathcal{A}.$$

So, $C(z)$ is not an object of $K^b_{\mathcal{A}}(\mathcal{P}(A))$.

The Derived Categories

- Recall, the derived category $D^b(\mathcal{E})$, of bounded complexes of objects in an exact category $\mathcal{E}$, was defined by inverting the quasi-isomorphisms in $K^b(\mathcal{E})$.
- For this lecture, A is a Cohen-Macaulay ring with $\dim A_m = d$ for all maximum ideals m of A.
  - As usual, the Derived category $D^b(\mathcal{P}(A))$ is a triangulated category with dulity. The duality is induced by $Hom(-A)$
  - Similarly, $D^b(\mathcal{A})$ is a triangulated category with duality. The duality is indued by $Ext^d(-, A)$

The Paradigm

- Today, we study, the "derived categories"

$$D^b_{\mathcal{A}}(\mathcal{A}), \quad D^b_{\mathcal{A}}(\mathcal{P}(A)), \quad \textit{obtained by inverting}$$

quasi-isomorphisms, repectively in, $K^b_{\mathcal{A}}(\mathcal{A})$, $K^b_{\mathcal{A}}(\mathcal{P}(A))$.
- $D^b_{\mathcal{A}}(\mathcal{A}), D^b_{\mathcal{A}}(\mathcal{P}(A))$ are not necessarily triangulated.

The Paradigm: Continued

- However,

$$D^b_{\mathcal{A}}(\mathcal{A}) \hookrightarrow D^b(\mathcal{A}) \quad \textit{is a full subcategory}.$$

- Likewise,

$$D^b_{\mathcal{A}}(\mathcal{P}(A)) \hookrightarrow D^b(\mathcal{P}(A) \quad \textit{is a full subcategory}.$$

- Note $D^b(\mathcal{A})$, $D^b(\mathcal{P}(A)$ are triangulated with duality.
- We will define Witt groups of subcategories of triangulated with duality.

Definition

Let $\delta = \pm 1$. Suppose $K := (K, \#, \delta, \varpi)$ is a triangulated category with translation T and $\delta-$duality $\#$. Suppose K_0 is a full subcategory of K that is closed under isomorphism, translation, orthogonal sum and duality.

- Define the Witt monoid of $MW(K_0)$ to be the submonoid

$$MW(K_0) = \{(P, \varphi) \in MW(K) : P \in Ob(K_0)\}.$$

- A symmetric space $(P, \varphi) \in MW(K_0)$ will be called a neutral space in $MW(K_0)$ if it has a Lagrangian (L, α, w) in $MW(K)$ such that $L, L^{\#} \in Ob(K_0)$.

Continued

- Let $NW(K_0)$ be the submonoid of $MW(K_0)$ generated by the isometry classes of neutral spaces in K_0.
- Define the Witt group

$$W(K_0) := \frac{MW(K_0)}{NW(K_0)}. \quad W(K_0) \textit{ has a group structure.}$$

Continued

- Accordingly, the shifted Witt groups

$$W^n\left(D^b_{\mathcal{A}}(\mathcal{P}(A))\right) := W\left(T^n\left(D^b_{\mathcal{A}}(\mathcal{P}(A)), *, 1, \varpi\right)\right), \quad and$$

$$W^n\left(D^b_{\mathcal{A}}(\mathcal{A})\right) := W\left(T^n\left(D^b_{\mathcal{A}}(\mathcal{A}), ^\vee, 1, \tilde{\varpi}\right)\right)$$

are defined, where

$$*, ^\vee \quad are\ induced\ by \quad Hom(-, A), Ext^d(-, A), \quad respectively.$$

The Serre Category Theorem

- Theorem (-,Sane)

  We have the diagram of isomorphisms

$$\begin{array}{ccc} W(\mathcal{A}) & \xrightarrow{\sim} & W\left(D^b_{\mathcal{A}}(\mathcal{A})\right) \\ & \searrow^{\sim} & \downarrow{\wr} \\ & & W\left(D^b(\mathcal{A})\right) \end{array}$$

  - The diagonal isomorphism is a theorem of Balmer.
  - The theorem holds for any Serre Category.

The Dévissage Theorem

Theorem (-,Sane)

- *0−Shift:*

$$W(\mathcal{A}) \stackrel{\sim}{\to} W^d(D^b_{\mathcal{A}}(\mathcal{P}(A)))$$

- *2−Shift:*

$$W^-(\mathcal{A}) \stackrel{\sim}{\to} W^{d+2}(D^b_{\mathcal{A}}(\mathcal{P}(A)))$$

- *Odd-Shift:*

$$W^{d+1}(D^b_{\mathcal{A}}(\mathcal{P}(A))) \cong W^{d-1}(D^b_{\mathcal{A}}(\mathcal{P}(A))) = 0.$$

- *4-periodicity describes all the shifted Witt groups.*

Diagramatically Stated:

Here is a commutative diagram of some of the isomorphisms:

$$\begin{array}{ccc} & & W\left(D^b\left(\mathcal{A}\right)\right) \\ & \overset{Balmer}{\nearrow} & \uparrow \wr \\ W\left(\mathcal{A}\right) & \xrightarrow{\sim} & W\left(D^b_{\mathcal{A}}\left(\mathcal{A}\right)\right) \\ & \underset{\sim}{\searrow} & \downarrow \wr \\ & & W^d\left(D^b_{\mathcal{A}}\left(\mathcal{P}(A)\right)\right) \end{array}$$

Comments

- When (A, m) is regular local, the theorem above is a result of Balmer-Walter. In that case,
  - $W(\mathcal{A}) = W(A/m) = W^d\left(D^b_{\mathcal{A}}(\mathcal{P}(A))\right)$
  - All the other three groups are zero.
- Our theorems assumes neither regular nor local.
- Our methods are fairly elementary.

The Theorem of Balmer

We borrow a good deal of methods from the work of Balmer, including following theorem:

Theorem.(Balmer) Let $(K, \#)$ be a triangulated category with duality containing $1/2$. Suppose K satisfies $(TR4^+)$.

- Let $(P_\bullet, \varphi)$ be a a symmetric space and
- let $\nu_1 : L_\bullet \longrightarrow P_\bullet$ be a morphism such that $\nu_1^{\#} \varphi \nu = 0$ (we say ν_1 is a sublagrangian).
- Choose any triangle over ν_1, as in the top line
- The second line is the dual of the first line.

Continued: The Theorem of Balmer

$$
\begin{array}{ccccccc}
T^{-1}N_\bullet & \longrightarrow & L_\bullet & \xrightarrow{\nu_1} & P_\bullet & \longrightarrow & N_\bullet \\
{\scriptstyle T^{-1}\mu_0}\downarrow & & \downarrow{\scriptstyle \mu_0^\#} & & \wr\downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \mu_0^\#} \\
T^{-1}L^\#_\bullet & \longrightarrow & N^\#_\bullet & \longrightarrow & P^\#_\bullet & \xrightarrow[\nu_1^\#]{} & L^\#_\bullet \\
 & & \downarrow & & & & \\
 & & R_\bullet & & & & \\
 & & \downarrow & & & & \\
 & & T^{-1}L_\bullet & & & &
\end{array}
$$

Continued: The Theorem of Balmer

- By choice μ_0 very good morphism (see Balmer).
- The verticle line is a triangle on μ_0.

Then, there there exists a symmetric form

$$\psi : R_\bullet \stackrel{\sim}{\to} R_\bullet^\# \qquad \ni \qquad [(P_\bullet, \varphi)] = [(R_\bullet, \psi)] \in W(K, \#).$$

The Map

We give a sketch of the proof that

$$\pi : W(\mathcal{A}) \stackrel{\sim}{\to} W^d\left(D^b\left(\mathcal{P}(A)\right)\right)$$

- First, there is a narural homomorphism

$$\pi([(M, \varphi_0)] = [(P_\bullet, \varphi)]$$

where $P_\bullet$ is a finite projective resolution of $H_0(P_\bullet) = M$.

The Surjectivity

- Let $(P_\bullet, \varphi)$ be a symmetric form in

$$T^d D^b_{\mathcal{A}}(\mathcal{P}(A)) \subseteq T^d D^b(\mathcal{P}(A))$$

- Upto Quasi-isomorphism the form looks like:

$$\begin{array}{ccccccc} P_{n+d} & \longrightarrow \cdots \longrightarrow & P_0 & \longrightarrow \cdots \longrightarrow & P_{-n} \\ \downarrow \varphi & & \downarrow \varphi & & \downarrow \varphi \\ P^*_{-n} & \longrightarrow \cdots \longrightarrow & P^*_d & \longrightarrow \cdots \longrightarrow & P^*_{n+d} \end{array}$$

Continued: The Surjectivity

- with $H_{-n}(P_\bullet) \neq 0$. Assume $N \geq 1$.
- By definition $\forall\ i \quad H_i(P_\bullet) \stackrel{\sim}{\to} H_i(P_\bullet^{\#})$.
- By a simple lemma, $H_i(P_\bullet) = 0 \quad \forall\ i > n$.
- We have

$$0 \neq H_{-n}(P_\bullet) \approx Ext^d\left(\frac{P_n}{\ker(\partial_n)}, A\right) \approx Ext^d\left(H_n(P_\bullet), A\right)$$

- Take a projective resolution $L_\bullet \twoheadrightarrow H_{-n}(P_\bullet)$, and complete the following diagram.

The Sublagrangian Construction: The Surjectivity

$$
\begin{array}{ccccccccccccc}
L_{n+d} & \longrightarrow & \cdots & \longrightarrow & L_n & & & & & & & & \\
\downarrow{\scriptstyle \nu} & & & & \downarrow{\scriptstyle \nu} & & & & & & & & \\
P_{n+d} & \longrightarrow & \cdots & \longrightarrow & P_n & \longrightarrow & \cdots & & P_0 & \longrightarrow & \cdots & \longrightarrow & P_{-n} \\
\downarrow{\scriptstyle \varphi} & & & & \downarrow{\scriptstyle \varphi} & & & & \downarrow & & & & \downarrow{\scriptstyle \varphi} \\
P^*_{-n} & \longrightarrow & \cdots & \longrightarrow & P^*_{-n+d} & \longrightarrow & \cdots & & P^*_d & \longrightarrow & \cdots & \longrightarrow & P^*_{n+d} \\
 & & & & & & & & & & & & \downarrow{\scriptstyle \nu^{\#}} \\
 & & \cdots & \longrightarrow & L^*_n & \longrightarrow & \cdots & & & \longrightarrow & \cdots & \longrightarrow & L^*_{n+d}
\end{array}
$$

The Sublagrangian Construction: The Surjectivity

- It follows $\forall\ i \quad H_i(\nu^{\#}\varphi\nu) = 0$
- $L^{\#}$ is "exact enough" to prove that $\nu^{\#}\varphi\nu \sim 0$ in K^b-category, hence in $D^b(\mathcal{P}(A))$.
- We apply Balmer's theorem, in $D^b(\mathcal{P}(A))$. He proved, there is a Lagrangian

$$\eta : N^{\#} \longrightarrow (P_\bullet, \varphi) \perp (R_\bullet, -\psi) \quad in \quad D^b(\mathcal{P}(A))$$

Final Steps: The Surjectivity

- Since $N^{\#}$ is in $D^b_{\mathcal{A}}(\mathcal{P}(A))$, η is a Lagrangian in $D^b_{\mathcal{A}}(\mathcal{P}(A))$.
- Hence
$$[(P_\bullet, \varphi)] = [(R_\bullet, \psi)] \textit{in} \quad D^b_{\mathcal{A}}(\mathcal{P}(A))$$
- Chasing the homology sequences, it folows
$$H_i(R_\bullet) = 0 \quad \textit{unless} \quad n-1 \leq i \leq -(n-1).$$
- Since $H_{-n}(R_\bullet) = 0$, it splits ar degree $-n$.
- Upto quasi-isomorphism, $R_\bullet$ is supported on $[(n-1)+d, -(n-1)]$.

Final Steps: The Surjectivity

- By induction,

$$[(P_\bullet, \varphi)] = [(Q_\bullet, \psi)] \textit{in} \quad D^b_{\mathcal{A}}(\mathcal{P}(A))$$

  where $H_i(Q_\bullet) = 0 \quad \forall\ i \neq 0$
- So, surjectivity is established.