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Witt, GW, K-theory of quasi-projective schemes

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ABSTRACT

In this article, we prove some results on Witt, Grothendieck–Witt (GW) and K-theory of noetherian quasi-projective schemes X, over affine schemes Spec (A). For integers $k \geq 0$, let $C\mathbb{M}^k(X)$ denote the category of coherent $\mathcal{O}_X$-modules $\mathcal{F}$, with locally free dimension $\dim_{\mathscr{V}(X)}(\mathcal{F}) = k = grade(\mathcal{F})$. We prove that there is an equivalence $\mathcal{D}^b\left(C\mathbb{M}^k(X)\right) \to \mathscr{D}^k\left(\mathscr{V}(X)\right)$ of the derived categories. It follows that there is a sequence of zig-zag maps $\mathbb{K}\left(C\mathbb{M}^{k+1}(X)\right) \longrightarrow \mathbb{K}\left(C\mathbb{M}^k(X)\right) \longrightarrow \coprod_{x \in X^{(k)}} \mathbb{K}\left(C\mathbb{M}^k(X_x)\right)$ of the $\mathbb{K}$-theory spectra that is a homotopy fibration. In fact, this is analogous to the homotopy fiber sequence of the G-theory spaces of Quillen (see proof of [16, Theorem 5.4]). We also establish similar homotopy fibrations of **GW**-spectra and $\mathbb{G}W$-bispectra, by application of the same equivalence theorem.

1. Introduction

In [16], Quillen established the foundation of K-theory of regular schemes X in a complete manner. In fact, for any scheme X, Quillen provides a complete foundation of K-theory of the category $Coh(X)$ of the coherent sheaves on X, along with that of the filtration of $Coh(X)$ by co-dimension of support of the objects $\mathcal{F} \in Coh(X)$. This relates to Gersten complexes and spectral sequences associated to any such scheme X (see [16, §5]). The K-theory of $Coh(X)$ is also known as G-theory. For regular schemes X, the K-theory of the category $\mathscr{V}(X)$ of locally free sheaves agrees fully with that of $Coh(X)$. Consequently, the K-theory of regular schemes appears very complete. However, the K-theory of non-regular schemes never reached the completeness and harmony that the K-theory of regular schemes had achieved. Work of Waldhausen [23] and Thomason–Trobaugh [22] would be milestones in this respect, most notably for their introduction of derived invariance theorems and localization theorems, applicable to non-regular schemes. Further, while developments in Grothendieck–Witt theory (GW-theory) and Witt theory followed the foot prints of K-theory [20,1], due to the lack of any natural duality on $Coh(X)$, the situation in these two areas appears even less complete. When X is non-regular, the category $\mathbb{M}(X)$ of coherent sheaves with finite

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$\mathscr{V}(X)$-dimension differs from $Coh(X)$. There appears to be a gap in the literature of K-theory, GW-theory, and Witt theory, with respect to the place of the category $\mathbb{M}(X)$. One can speculate, whether this lack of completeness is attributable to this gap. The goal of this one and the related articles is to work on this gap and attempt to establish the said literature on non-regular schemes at the same pedestal as that of regular schemes. For quasi-projective schemes over noetherian affine schemes, this goal is accomplished up to some degree of satisfaction. The special place of the full subcategory $C\mathbb{M}^k(X) \subseteq \mathbb{M}(X)$ would also be clear subsequently, where for integers $k \geq 0$, $C\mathbb{M}^k(X)$ will denote the full subcategory of objects $\mathcal{F}$ in $\mathbb{M}(X)$, with $\dim_{\mathscr{V}(X)}(\mathcal{F}) = grade\,(\mathcal{F}) = k$.

With respect to certain facets of Algebraic K-theory, Grothendieck–Witt (GW) theory and Witt theory, a common thread among them is their invariance properties with respect to equivalences of the associated Derived categories. We review some of the results on such invariances. For example, recall the theorem of Thomason–Trobaugh [22, Theorem 1.9.8]: suppose $\mathbf{A} \to \mathbf{B}$ is a functor of complicial exact categories with weak equivalences. Assume that the associated functor of the triangulated categories $\mathscr{T}\mathbf{A} \to \mathscr{T}\mathbf{B}$ is an equivalence. Then, the induced map $\mathbf{K}(\mathbf{A}) \to \mathbf{K}(\mathbf{B})$ of the $\mathbf{K}$-theory spaces is a homotopy equivalence (see [18, 3.2.24]). The non-connective version of this theorem was given by Schlichting ([19, Theorem 9], also see [18, 3.2.29]) which states, under the relaxed hypothesis, that: if $\mathscr{T}\mathbf{A} \to \mathscr{T}\mathbf{B}$ is an equivalence up to factors, then it induces a homotopy equivalence $\mathbb{K}(\mathbf{A}) \to \mathbb{K}(\mathbf{B})$ of the $\mathbb{K}$-theory spectra. While K-theory is defined for complicial exact categories with weak equivalences, Schlichting defined Grothendieck Witt (GW) spectra and bispectra ([20], also see Appendix A) of pointed dg categories with weak equivalences and dualities. Invariance theorems of $\mathbf{GW}$-spectra and $\mathbb{G}W$-bispectra, similar to that of K-theory, were established in [20, Theorems 6.5, 8.9]. Contrary to K-theory and GW-theory, Balmer defined Witt theory for Triangulated categories with dualities [1], which encompasses the Derived categories with dualities. Therefore, the shifted Witt groups are invariant with respect to equivalences of derived categories [1, Theorem 6.2]. Another common thread among these three areas is the exactness properties of the associated triangulated categories. In particular, the renowned Gersten complexes in K-theory, GW-theory and Witt theory, are obtained by routine manipulation (see Remark 4.5) of the respective invariants, by such derived equivalences and exactness properties of the associated triangulated categories. For our purpose, some of the existing exactness theorems [2,4] of derived categories would suffice. Therefore, we first consider equivalences of certain derived categories, over quasi-projective schemes, which we state subsequently.

The readers are referred to Notations 2.1 for clarifications regarding notations and the definition of grade. Other than the notations explained above, for integers $k \geq 0$, $\mathbb{M}^k(X)$ will denote the category of coherent $\mathcal{O}_X$-modules $\mathcal{F}$ with finite locally free dimension, and $grade\,(\mathcal{F}) \geq k$. We prove that, for a noetherian quasi-projective scheme X over an affine scheme $\operatorname{Spec}(A)$, and integers $k \geq 0$, the functor of the derived categories

$$\zeta : \mathcal{D}^b\left(C\mathbb{M}^k(X)\right) \to \mathcal{D}^b\left(\mathbb{M}^k(X)\right) \quad \text{is an equivalence}$$

(see Theorem 3.1). We also prove that the functor of the derived categories

$$\beta : \mathcal{D}^b\left(\mathbb{M}^{k+1}(X)\right) \to \mathcal{D}^b\left(\mathbb{M}^k(X)\right) \quad \text{is faithfully full}$$

(see Theorem 3.2). Consequently, the functor $\mathcal{D}^b\left(C\mathbb{M}^{k+1}(X)\right) \to \mathcal{D}^b\left(\mathbb{M}^k(X)\right)$ is faithfully full. Combining the results in [13], we have the following summary of results. Consider the commutative diagram

$$\begin{tikzcd}
\mathcal{D}^b(C\mathbb{M}^{k+1}(X)) \arrow[r, "\zeta", "\sim"'] \arrow[d, "\alpha"] & \mathcal{D}^b(\mathbb{M}^{k+1}(X)) \arrow[r, "\iota", "\sim"'] \arrow[d, "\beta"] & \mathscr{D}^{k+1}(\mathbb{M}(X)) \arrow[d, "\gamma"] & \mathscr{D}^{k+1}(\mathscr{V}(X)) \arrow[l, "\iota'"', "\sim"] \arrow[d, "\eta"] \\
\mathcal{D}^b(C\mathbb{M}^k(X)) \arrow[r, "\sim", "\zeta"'] & \mathcal{D}^b(\mathbb{M}^k(X)) \arrow[r, "\sim", "\iota"'] & \mathscr{D}^k(\mathbb{M}(X)) & \mathscr{D}^k(\mathscr{V}(X)) \arrow[l, "\sim"', "\iota'"]
\end{tikzcd} \tag{1}$$

of functors of derived categories. Then, all the horizontal functors are equivalences and, all the vertical functors are fully faithful (see Theorem 3.4).

Having stated the equivalence theorem (3.4), we first turn our attention to its consequences to Algebraic K-theory of quasi-projective schemes X over an affine scheme $\mathrm{Spec}\,(A)$. Note that $C\mathbb{M}^k(X)$ is an exact category. Quillen [16] defined $\mathbf{K}$-theory space $\mathbf{K}\,(\mathscr{E})$ of any exact category $\mathscr{E}$. To incorporate negative K-groups, following Bass, Karoubi and others, Schlichting formally introduced [18,19] $\mathbb{K}$-theory spectrum $\mathbb{K}\,(\mathscr{E})$ for such exact categories $\mathscr{E}$, and also of complicial exact categories with weak equivalences. By agreement theorems ([22, Theorem 1.11.17], [18, 3.2.30]), there are homotopy equivalences

$$\begin{cases} \mathbf{K}\,(\mathscr{E}) \xrightarrow{\sim} \mathbf{K}\big(Ch^b\,(\mathscr{E})\big) & \text{of the } \mathbf{K}\text{-theory spaces,} \\ \mathbb{K}\,(\mathscr{E}) \xrightarrow{\sim} \mathbb{K}\big(Ch^b\,(\mathscr{E})\big) & \text{of the } \mathbb{K}\text{-theory spectra,} \end{cases}$$

where the right hand sides correspond to the K-theory space/spectrum of the category $Ch^b\,(\mathscr{E})$ of chain complexes. Since $C\mathbb{M}^k(X)$ is an exact category, the agreement theorems apply. Now, by application of the Derived Equivalence Theorem 3.4, for a noetherian quasi-projective scheme X, over an affine scheme $\mathrm{Spec}\,(A)$, we obtain zig-zag homotopy equivalences (see 2.1 for notations)

$$\begin{cases} \mathbf{K}\left(C\mathbb{M}^k(X)\right) \xrightarrow{\sim} \mathbf{K}\left(\mathscr{C}h^k\left(\mathscr{V}(X)\right)\right) & \text{of the } \mathbf{K}\text{-theory spaces,} \\ \mathbb{K}\left(C\mathbb{M}^k(X)\right) \xrightarrow{\sim} \mathbb{K}\left(\mathscr{C}h^k\left(\mathscr{V}(X)\right)\right) & \text{of the } \mathbb{K}\text{-theory spectra.} \end{cases} \tag{2}$$

Now assume that X is Cohen–Macaulay. In this case, the Thomason–Waldhausen Localization theorem [18, 3.2.27] applies to the inclusion $\mathscr{C}h^{k+1}\left(\mathscr{V}(X)\right) \hookrightarrow \mathscr{C}h^k\left(\mathscr{V}(X)\right)$, and using the identifications (2), we obtain a sequence

$$\mathbb{K}\left(C\mathbb{M}^{k+1}(X)\right) \longrightarrow \mathbb{K}\left(C\mathbb{M}^k(X)\right) \longrightarrow \textstyle\coprod_{x\in X^{(k)}} \mathbb{K}\left(C\mathbb{M}^k(X_x)\right)$$

of zig-zag maps (via *homotopy equivalences*) of $\mathbb{K}$-theory spectra, that is a homotopy fibration (see Theorem 4.2), where $X_x := \mathrm{Spec}\,(\mathcal{O}_{X,x})$. This is an analogue of the homotopy fiber sequence of the G-theory spaces, due to Quillen (see proof of [16, Theorem 5.4]). Accordingly, for all integers $n, k \in \mathbb{Z}$ with $k \geq 0$, there is an exact sequence

$$\cdots \longrightarrow \mathbb{K}_n\left(C\mathbb{M}^{k+1}(X)\right) \longrightarrow \mathbb{K}_n\left(C\mathbb{M}^k(X)\right) \longrightarrow \oplus_{x\in X^{(k)}} \mathbb{K}_n\left(C\mathbb{M}^k(X_x)\right)$$

$$\longrightarrow \mathbb{K}_{n-1}\left(C\mathbb{M}^{k+1}(X)\right) \longrightarrow \cdots$$

of $\mathbb{K}$-groups (see Corollary 4.3). We remark (4.4) that, if X is regular, similar statements regarding $\mathbf{K}$-theory spaces and groups would also be valid. While these results allow us to rewrite the Gersten $\mathbb{K}$-theory complexes in terms of the $\mathbb{K}$-groups of the "local categories" $C\mathbb{M}^k(X_x)$ (see Remark 4.5), they provide further insight in to the same in terms of the $\mathbb{K}$-groups of $C\mathbb{M}^k(X)$. When, A is a Cohen–Macaulay local ring with $\dim A = d$ and $X = \mathrm{Spec}\,(A)$, it is a result of Roberts and Srinivas [17, Proposition 2] that the map $K_0\left(C\mathbb{M}^d(X)\right) \xrightarrow{\sim} K_0\left(\mathscr{C}h^d\left(\mathscr{V}(X)\right)\right)$ is an isomorphism, which would be a consequence of the above homotopy equivalence (2).

Results on GW-theory would be fairly similar. Note that in the diagram (1) of equivalences, the dg category $dg\mathbb{M}^k(X)$, associated to $\mathbb{M}^k(X)$, does not have a natural duality structure. Remedy for this was obtained by embedding this category in the respective category of perfect complexes. We assume that X is a quasi-projective scheme over an affine scheme $\mathrm{Spec}\,(A)$, with $1/2 \in A$. Then, for integers $r = 0, 1, 2, 3$ and $k \geq 0$, we obtain zig-zag homotopy equivalences

$$\begin{cases} \mathbf{GW}^{[r]}\left(dgC\mathbb{M}^k(X)\right) \xrightarrow{\sim} \mathbf{GW}^{[k+r]}\left(dg^k\mathscr{V}(X)\right) & \text{of the } \mathbf{GW}\text{-spectra,} \\ \mathbb{G}W^{[r]}\left(dgC\mathbb{M}^k(X)\right) \xrightarrow{\sim} \mathbb{G}W^{[k+r]}\left(dg^k\mathscr{V}(X)\right) & \text{of the } \mathbb{G}W\text{-bispectra.} \end{cases} \tag{3}$$

When X is Cohen–Macaulay, there is sequence of zig-zag maps

$$\mathbb{G}W^{[-1+r]}\left(dgC\mathbb{M}^{k+1}(X)\right) \longrightarrow \mathbb{G}W^{[r]}\left(dgC\mathbb{M}^k(X)\right) \longrightarrow \coprod_{x\in X^{(k)}} \mathbb{G}W^{[r]}\left(dgC\mathbb{M}^k\left(X_x\right)\right)$$

that is a homotopy fibration of $\mathbb{G}W$-bispectra. When X is regular, similar homotopy fibration of $\mathbf{GW}$-spectra would also be valid. Again, this is a $\mathbb{G}W$-analogue of the homotopy fiber sequence of the G-theory spaces of Quillen (see the proof of [16, Theorem 5.4]). Note that such statements about Grothendieck Witt theory would not make sense for $dg\mathbb{M}^k(X)$ or $dg\,Coh^k(X)$, because of non-existence of any natural duality in the respective categories.

With respect to implications to the derived Witt theory, assume that X is regular quasi-projective scheme over an affine scheme $\operatorname{Spec}(A)$, with $1/2 \in A$. For an integer $k \geq 0$, consider the exact sequence of the derived categories [4]:

$$\mathscr{D}^{k+1}\left(\mathscr{V}(X)\right) \longrightarrow \mathscr{D}^{k}\left(\mathscr{V}(X)\right) \longrightarrow \coprod_{x\in X^{(k)}} \mathscr{D}^{k}\left(\mathscr{V}(X)\right).$$

Then, the twelve term exact sequence of Witt groups, due to Balmer [1, Corollary 6.6], corresponding to this sequence, reduces to two five term exact sequences (see Theorem 4.16), one of them being the following:

$$0 \longrightarrow W^{-1}\left(\mathcal{D}^b\left(C\mathbb{M}^{k+1}(X)\right)\right) \longrightarrow W\left(C\mathbb{M}^k(X)\right) \longrightarrow \oplus_{x\in X^{(k)}} W\left(C\mathbb{M}^k(X_x)\right)$$

$$\longrightarrow W\left(C\mathbb{M}^{k+1}(X)\right) \longrightarrow W^1\left(\mathcal{D}^b\left(C\mathbb{M}^k(X)\right)\right) \longrightarrow 0$$

We point out that, contrary to the usual filtration by co-dimension of the support, in this article, for a scheme X we consider the filtrations $\mathbb{M}^k(X) \subseteq \mathbb{M}(X)$ and $Coh^k(X) \subseteq Coh(X)$ by grade (see Notations 2.1). When X is Cohen–Macaulay, these filtrations coincide with the filtration by co-dimension of the support. This article is a culmination of an initiative [9–11,15,12,13] to place the category $C\mathbb{M}^k(X)$ at its rightful place in the Algebraic K-theory, GW-theory and Witt theory of schemes X and the respective Gersten complexes. This category $C\mathbb{M}^k(X)$ behaves like the category of modules of finite length and finite projective dimension, at co-dimension $k \leq \dim X$.

Before we close this introduction, we comment on the layout of this article. In §2 we recall or prove some preliminaries that we need. The Derived Equivalence Theorem 3.4 is established in §3. In §4.1 we establish the implications in K-theory. We deal with GW-theory in §4.2. In §4.3 we discuss Derived Witt theory. In Appendix A, we give some background information on $\mathbf{GW}$-spectrum and $\mathbb{G}W$-bispectrum.

2. Preliminaries

First, we set up some notations.

Notations 2.1. Throughout this article, X will denote a noetherian scheme, with finite dimension $d := \dim X$. In most cases, X will be a quasi-projective scheme over a noetherian affine scheme $Spec(A)$. We introduce further notations.

1. For $x \in X$, denote $X_x := \operatorname{Spec}(\mathcal{O}_{X,x})$.
2. Throughout, $Coh(X)$ will denote the category of coherent $\mathcal{O}_X$-modules and $\mathscr{V}(X)$ will denote the category of all locally free sheaves on X.

3. (*Readers are referred to* [11, Def. 3.1] *for a definition of resolving subcategories of abelian categories.*) For a resolving subcategory $\mathscr{A}$ of an abelian category $\mathcal{C}$, for objects $\mathcal{F} \in \mathcal{C}$, $\dim_{\mathscr{A}}(\mathcal{F})$ will denote the minimum of the length of resolutions of $\mathcal{F}$ by objects in $\mathscr{A}$.
Denote
$$\mathbb{M}(\mathscr{A}) = \{\mathcal{F} \in \mathcal{C} : \dim_{\mathscr{A}}(\mathcal{F}) < \infty\}$$
With $\mathcal{C} = Coh(X)$ we denote
$$\mathbb{M}(X) := \mathbb{M}(\mathscr{V}(X)) = \{\mathcal{F} \in Coh(X) : \dim_{\mathscr{V}(X)}(\mathcal{F}) < \infty\}$$
4. In this article, we consider filtration of $Coh(X)$ and $\mathbb{M}(X)$ by grade, as opposed to usual filtration by co-dimension of the support.
   (a) Recall, for $\mathcal{F} \in Coh(X)$, $grade(\mathcal{F}) := \min\{t : \mathcal{E}xt^t(\mathcal{F}, \mathcal{O}_X) \neq 0\}$. We remark that, if X is Cohen–Macaulay, then $grade(\mathcal{F}) = co\dim(Supp(\mathcal{F}))$ (see [14]).
   (b) For integers $k \geq 0$, denote
$$\begin{cases} Coh^k(X) := Coh^k_g(X) := \{\mathcal{F} \in Coh(X) : grade(\mathcal{F}) \geq k\} \\ \mathbb{M}^k(X) := \mathbb{M}^k_g(\mathscr{A}) := \{\mathcal{F} \in \mathbb{M}(X) : grade(\mathcal{F}) \geq k\} \\ C\mathbb{M}^k(X) := \{\mathcal{F} \in \mathbb{M}(X) : grade(\mathcal{F}) = k = \dim_{\mathscr{V}(X)}(\mathcal{F})\} \end{cases}$$
   So, we have a filtration, by grade $\mathbb{M}(\mathscr{A}) = \mathbb{M}^0(\mathscr{A}) \supseteq \mathbb{M}^1(\mathscr{A}) \supseteq \cdots \supseteq \mathbb{M}^d(\mathscr{A}) \supseteq 0$. We will strictly be using this filtration by grade and the notation without the subscript g will be the norm. Note that $\mathbb{M}^k(\mathscr{A})$ is a Serre subcategory of $\mathbb{M}(\mathscr{A})$ (meaning, it has the "2 out of 3" property). Clearly, when X is Cohen–Macaulay, this filtration coincides with the filtration by co-dimension of the support. Also note that $Coh^k(X)$ is a Serre abelian subcategory of $Coh(X)$.
5. For an exact category $\mathscr{E}$, $Ch^b(\mathscr{E})$ will denote the category of chain complexes. The bounded derived category of $\mathscr{E}$ will be denoted by $\mathcal{D}^b(\mathscr{E})$.
6. For a complex $\mathcal{F}_\bullet \in Ch^b(Coh(X))$, the homologies will be denoted by $\mathcal{H}_i(\mathcal{F}_\bullet)$.
7. Also, for $\mathscr{E} = \mathscr{V}(X), \mathbb{M}(X)$, and integers $k \geq 0$,
$$\begin{cases} \mathscr{C}h^k(\mathscr{E}) := \left\{\mathcal{F}_\bullet \in Ch^b(\mathscr{E}) : \forall\ i\ \mathcal{H}_i(\mathcal{F}_\bullet) \in Coh^k(X)\right\} \\ \mathscr{D}^k(\mathscr{E}) := \left\{\mathcal{F}_\bullet \in \mathcal{D}^b(\mathscr{E}) : \forall\ i\ \mathcal{H}_i(\mathcal{F}_\bullet) \in Coh^k(X)\right\} \end{cases}$$
   would denote the full subcategory of such objects. (*Note the difference between two fonts* $\mathcal{D}$, $\mathscr{D}$.) We remark:
   (a) $\mathscr{C}h^k(\mathscr{E})$ is a complicial exact category (see [18] for definition). In fact, $\left(\mathscr{C}h^k(\mathscr{E}), \mathscr{Q}\right)$ is a complicial exact category with weak equivalences, where weak equivalences are the set $\mathscr{Q}$ of all quasi-isomorphisms.
   (b) Also, $\mathscr{D}^k(\mathscr{E})$ is a triangulated subcategory of $\mathcal{D}^b(\mathscr{E})$.

We recall the following lemma from [13, Lemma 2.1].

Lemma 2.2. *Suppose X is a quasi-projective noetherian scheme over* $\operatorname{Spec}(A)$*, with $\dim X = d$. Then, X is an open subset of $\tilde{X} := Proj(S)$, for some noetherian graded ring $S = \oplus_{i=0}^\infty S_i$, with $S_0 = A$.*

Let $Y \subseteq X$ be a closed subset of X, with $grade(\mathcal{O}_Y) \geq k$. Let $V(I) = \overline{Y}$ be the closure of Y, where I is the homogeneous ideal of S, defining $\overline{Y}$. Then, there is a sequence of homogeneous elements $f_1, \ldots, f_k \in I$ such that $f_{i_1}, \ldots, f_{i_j}$ induce regular $S_{(\wp)}$-sequences $\forall\ \wp \in Y \subseteq X$, and $\forall\ 1 \leq i_1 < i_2 < \cdots < i_j \leq k$. In particular, with $Z = V(f_1, \ldots, f_k) \cap X$, and $\mathcal{F}_n = \mathcal{O}_Z^n$, we have

1. *$\mathcal{F}_n \in \mathbb{CM}^k(X)$. In fact, $\oplus_{i=1}^n \mathcal{O}_Z \otimes \mathcal{L}_i \in \mathbb{CM}^k(X)$ for any locally free sheaves $\mathcal{L}_i$ of rank one.*
2. *Further, if $\mathcal{G} \in Coh^k(X)$, with $Y = Supp(\mathcal{G})$ and Z as above, there is a surjective map $\mathcal{F} \twoheadrightarrow \mathcal{G}$ where $\mathcal{F} := \oplus_{i=1}^n \mathcal{O}_Z \otimes \mathcal{O}_x(n_i)$, for some integers n_i. Note that $\mathcal{F} \in \mathbb{CM}^k(X)$.*

Remark 2.3. In this, one had the choices of the sequence $f_1, \dots, f_k$, as required above. We may exploit this flexibility later.

Lemma 2.4. *Suppose X is a quasi-projective noetherian scheme over* $\operatorname{Spec}(A)$*, with* $\dim X = d$*. Consider an exact sequence*

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0 \quad \text{where} \quad \mathcal{F} \in \mathbb{M}^k(X),\ \mathcal{E} \in \mathbb{CM}^k(X).$$

Then $\dim_{\mathscr{V}(X)}(\mathcal{K}) \leq \max\{k, \dim_{\mathscr{V}(X)}(\mathcal{F}) - 1\}$*. In fact,* $\dim_{\mathscr{V}(X)}(\mathcal{F}) \geq k + 1 \implies \dim_{\mathscr{V}(X)}(\mathcal{K}) = \dim_{\mathscr{V}(X)}(\mathcal{F}) - 1$.

Proof. If $\dim_{\mathscr{V}(X)}(\mathcal{F}) = k$ then there is nothing to prove. Assume, $\dim_{\mathscr{V}(X)}(\mathcal{F}) = m \geq k + 1$. Arguing locally, a simple Tor-argument establishes the lemma. □

Lemma 2.5. *Suppose X is a quasi-projective noetherian scheme over* $\operatorname{Spec}(A)$*, with* $\dim X = d$*. Suppose $\mathcal{F} \in \mathbb{M}^k(X)$. Then, there is a resolution*

$$0 \longrightarrow \mathcal{E}_n \longrightarrow \mathcal{E}_{n-1} \xrightarrow{\partial_{n-1}} \cdots \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_0 \longrightarrow \mathcal{F} \longrightarrow 0 \quad \text{with } \mathcal{E}_i \in \mathbb{CM}^k(X). \tag{4}$$

In fact, $n = \dim_{\mathscr{V}(X)}(\mathcal{F}) - k$.

Proof. By Lemma 2.2, there is a surjective map $\partial_0 : \mathcal{E}_0 \twoheadrightarrow \mathcal{F}$, where $\mathcal{E}_0 \in \mathbb{CM}^k(X)$. Now, let $\mathcal{F}_0 = \ker(\partial_0)$. If $\dim_{\mathscr{V}(X)}(\mathcal{F}) \geq k+1$, then by Lemma 2.4, $\dim_{\mathscr{V}(X)}(\mathcal{F}_0) = \dim_{\mathscr{V}(X)}(\mathcal{F}) - 1$. By repeating this process, we get an exact sequence, as in diagram (4), with $\mathcal{E}_n = \ker(\partial_{n-1})$ and $\dim_{\mathscr{V}(X)}(\mathcal{E}_n) = k$. Since $grade(\mathcal{E}_n) \geq k$, it follows $\mathcal{E}_n \in \mathbb{CM}^k(X)$. The proof is complete. □

Proposition 2.6. *Suppose X is a quasi-projective noetherian scheme over* $\operatorname{Spec}(A)$*, with* $\dim X = d$*. Suppose*

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0 \qquad \textit{be exact in} \qquad \mathbb{M}^k(X)$$

Then:

1. *First,*

$$\mathcal{K}, \mathcal{G} \in \mathbb{CM}^k(X) \implies \quad \mathcal{F} \in \mathbb{CM}^k(X)$$

2. *Then,*

$$\mathcal{F}, \mathcal{G} \in \mathbb{CM}^k(X) \implies \quad \mathcal{K} \in \mathbb{CM}^k(X)$$

Proof. The proof follows by routine chasing the long exact sequence of the $\mathcal{E}xt$-modules. □

Corollary 2.7. *Let X be a quasi-projective scheme over an affine scheme* $\operatorname{Spec}(A)$*. Then, $\mathbb{CM}^k(X) \subseteq Coh^k(X)$ is a resolving subcategory. Further,*

$$\mathbb{M}^k(X) = \{\mathcal{F} \in Coh^k(X) : \dim_{\mathbb{CM}^k(X)}(\mathcal{F}) < \infty\}$$

Proof. Let $\mathcal{G} \in Coh^k(X)$ be an object. By Lemma 2.2, there is a surjective morphism $\mathcal{F} \twoheadrightarrow \mathcal{G}$ with $\mathcal{F} \in C\mathbb{M}^k(X)$. Now it follows from Lemma 2.6 that $C\mathbb{M}^k(X)$ is a resolving subcategory of $Coh^k(X)$.

Now, suppose $\mathcal{F} \in \mathbb{M}^k(X)$. By Lemma 2.5, $\dim_{C\mathbb{M}^k(X)}(\mathcal{F}) < \infty$. Conversely, suppose $\mathcal{F} \in Coh^k(X)$ and $\dim_{C\mathbb{M}^k(X)}(\mathcal{F}) < \infty$. In particular, $\dim_{\mathscr{V}(X)}(\mathcal{F}) < \infty$. So, $\mathcal{F} \in \mathbb{M}(X)$ and hence $\mathcal{F} \in \mathbb{M}^k(X)$. The proof is complete. □

3. The equivalence theorems

In this section we state and prove the key equivalence theorems.

Theorem 3.1. *Let X be a noetherian quasi-projective scheme over an affine scheme* $\operatorname{Spec}(A)$ *and $k \geq 0$ be a fixed integer. Consider the inclusion functor $C\mathbb{M}^k(X) \hookrightarrow \mathbb{M}^k(X)$ and let*

$$\zeta : \mathcal{D}^b\left(C\mathbb{M}^k(X)\right) \longrightarrow \mathcal{D}^b\left(\mathbb{M}^k(X)\right) \quad \text{denote the induced functor}$$

of the derived categories. Then ζ is an equivalence of derived categories.

Proof. The proof is obtained by an application of the statement in [8, Last paragraph of §1.5], to the inclusion functor $C\mathbb{M}^k(X) \hookrightarrow \mathbb{M}^k(X)$, as follows. Consider the diagram:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F}_1 \arrow[r, dashed] \arrow[d, dashed] & \mathcal{F}_0 \arrow[r, dashed] \arrow[d, dashed, "f_0"] & \mathcal{F} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathcal{G}_1 \arrow[r] & \mathcal{G}_0 \arrow[r, "\iota_0"'] & \mathcal{F} \arrow[r] & 0 \end{tikzcd} \tag{5}$$

where the second line is a given exact sequence in $\mathbb{M}^k(X)$, with $\mathcal{F} \in C\mathbb{M}^k(X)$. By Lemma 2.2, there is a surjective map $f_0 : \mathcal{F}_0 \to \mathcal{G}_0$, where $\mathcal{F}_0 \in C\mathbb{M}^k(X)$, and let $\mathcal{F}_1 = \ker(\iota_0 f_0)$. By Lemma 2.6, $\mathcal{F}_1 \in C\mathbb{M}^k(X)$ and hence the top line is an exact sequence in $C\mathbb{M}^k(X)$. This establishes that (the dual of) the condition (b) of [8, Last paragraph of §1.5] is satisfied. The condition (a) of [8, Last paragraph of §1.5] is also satisfied by Lemma 2.5. It follows from the statement in [8, Last paragraph of §1.5] that ζ is an equivalence. The proof is complete. □

Using a very similar proof as above we obtain the following.

Theorem 3.2. *Let X be a noetherian quasi-projective scheme as in Theorem 3.1 and $k \geq 0$ be a fixed integer. Consider the inclusion functor $\mathbb{M}^{k+1}(X) \to \mathbb{M}^k(X)$. Then, the induced functor $\beta : \mathcal{D}^b\left(\mathbb{M}^{k+1}(X)\right) \longrightarrow \mathcal{D}^b\left(\mathbb{M}^k(X)\right)$ is fully faithful. Consequently, so is the functor $\mathcal{D}^b\left(C\mathbb{M}^{k+1}(X)\right) \longrightarrow \mathcal{D}^b\left(\mathbb{M}^k(X)\right)$.*

Proof. Consider the following commutative diagram of triangle functors:

$$\begin{tikzcd} \mathcal{D}^b\left(\mathbb{M}^{k+1}(X)\right) \arrow[r, "\psi_{k+1}"] \arrow[d, "\beta"'] & \mathcal{D}^b\left(Coh^{k+1}(X)\right) \arrow[d, "\beta'"] \\ \mathcal{D}^b\left(\mathbb{M}^k(X)\right) \arrow[r, "\psi_k"'] & \mathcal{D}^b\left(Coh^k(X)\right) \end{tikzcd}$$

Replicating the diagram (5), both ψ_k, ψ_{k+1} are fully faithful, by Keller's criterion [8, §1.5]. Further, β' is also fully faithful. Now β is fully faithful, since so are ψ_k, ψ_{k+1}, β'.

By Theorem 3.1 the functor $\zeta : \mathcal{D}^b\left(C\mathbb{M}^{k+1}(X)\right) \xrightarrow{\sim} \mathcal{D}^b\left(\mathbb{M}^{k+1}(X)\right)$ is an equivalence. Therefore, the second statement follows from the former. The proof is complete. $\square$

Remark 3.3. The author is thankful to the referee for pointing out the above shortened proofs of Theorems 3.1 and 3.2. Some readers may benefit from the original, more explicit proofs of the same, which can be found in the arXiv version of this article.

Combining the results in [13], we summarize the results as follows.

Theorem 3.4. *Let X be a noetherian quasi-projective scheme as in Theorem 3.1 and $k \geq 0$ be a fixed integer. Consider the commutative diagram of functors of derived categories:*

$$\begin{tikzcd}
\mathcal{D}^b(C\mathbb{M}^{k+1}(X)) \arrow[r, "\zeta", "\sim"'] \arrow[d, "\alpha"] & \mathcal{D}^b(\mathbb{M}^{k+1}(X)) \arrow[r, "\iota", "\sim"'] \arrow[d, "\beta"] & \mathscr{D}^{k+1}(\mathbb{M}(X)) \arrow[d, "\gamma"] & \mathscr{D}^{k+1}(\mathscr{V}(X)) \arrow[l, "\iota'"', "\sim"] \arrow[d, "\eta"] \\
\mathcal{D}^b(C\mathbb{M}^{k}(X)) \arrow[r, "\sim", "\zeta"'] & \mathcal{D}^b(\mathbb{M}^{k}(X)) \arrow[r, "\sim", "\iota"'] & \mathscr{D}^{k}(\mathbb{M}(X)) & \mathscr{D}^{k}(\mathscr{V}(X)) \arrow[l, "\sim"', "\iota'"]
\end{tikzcd} \tag{6}$$

Then, all the horizontal functors are equivalences of derived categories and all the vertical functors are fully faithful.

Proof. The equivalences of the horizontal functors follow from Theorem 3.1 and the results in [13, Theorem 3.2]. It also follows from Theorem 3.2 that β is fully faithful. This completes the proof. $\square$

4. Implications in K-theory and others

In this section we discuss the implications of the equivalence Theorem 3.4. We will not repeat the prelude we provided in the introduction. First, we recall a notation and a lemma. For a noetherian scheme X, denote

$$X^{(k)} := \{Y \in X : co\dim(Y) = k\} \quad \text{and recall} \quad X_x := \operatorname{Spec}(\mathcal{O}_{X,x}).$$

We recall the following well known result that follows from [2,4].

Lemma 4.1. *Suppose X is a Cohen–Macaulay quasi-projective scheme over an affine scheme* Spec(A) *and $k \geq 0$ is an integer. Then, the sequence of derived categories*

$$\mathscr{D}^{k+1}(\mathscr{V}(X)) \longrightarrow \mathscr{D}^{k}(\mathscr{V}(X)) \longrightarrow \textstyle\coprod_{x\in X^{(k)}} \mathcal{D}^k(\mathscr{V}(X_x))$$

is exact up to factor. If X is regular, this sequence is exact.

4.1. K-theory

First, we consider the consequences in K-theory. Our standard reference for K-theory would be [18] and we freely use the definitions and notations from [18]. However, for an exact category $\mathscr{E}$ or a complicial exact category $\mathscr{E}$ with weak equivalences, $\mathbb{K}(\mathscr{E})$ will denote the $\mathbb{K}$-theory spectra of $\mathscr{E}$ and $\mathbb{K}_i(\mathscr{E})$ will denote the $\mathbb{K}$-groups. Likewise, $\mathbf{K}(\mathscr{E})$ would denote the $\mathbf{K}$-theory space of $\mathscr{E}$.

The following is the main application of Theorem 3.4 to $\mathbb{K}$-theory.

Theorem 4.2. *Suppose X is a noetherian quasi-projective scheme over an affine scheme $\operatorname{Spec}(A)$ and $k \geq 0$ is an integer. Consider the diagram of $\mathbb{K}$-theory spectra and maps:*

$$\begin{tikzcd}
\mathbb{K}\left(C\mathbb{M}^{k+1}(X)\right) \arrow[d, "\wr"'] & \mathbb{K}\left(C\mathbb{M}^{k}(X)\right) \arrow[d, "\wr"', "\Psi"] & \coprod_{x\in X^{(k)}} \mathbb{K}\left(C\mathbb{M}^{k}\left(X_x\right)\right) \arrow[d, "\wr"] \\
\mathbb{K}\left(\mathscr{C}h^{k+1}\left(\mathbb{M}(X)\right), \mathscr{Q}\right) \arrow[r] & \mathbb{K}\left(\mathscr{C}h^{k}\left(\mathbb{M}(X)\right), \mathscr{Q}\right) \arrow[r] & \coprod_{x\in X^{(k)}} \mathbb{K}\left(\mathscr{C}h^{k}\left(\mathbb{M}\left(X_x\right)\right), \mathscr{Q}\right) \\
\mathbb{K}\left(\mathscr{C}h^{k+1}\left(\mathscr{V}(X)\right), \mathscr{Q}\right) \arrow[r] \arrow[u, "\wr"] & \mathbb{K}\left(\mathscr{C}h^{k}\left(\mathscr{V}(X)\right), \mathscr{Q}\right) \arrow[r] \arrow[u, "\wr", "\Phi"'] & \coprod_{x\in X^{(k)}} \mathbb{K}\left(\mathscr{C}h^{k}\left(\mathscr{V}\left(X_x\right)\right), \mathscr{Q}\right) \arrow[u, "\wr"']
\end{tikzcd}$$

Then, the vertical maps are homotopy equivalences of $\mathbb{K}$-theory spectra. Further, if X is Cohen–Macaulay, then the second line and the third line are homotopy fibrations of $\mathbb{K}$-theory spectra.

Proof. Here the middle upward arrow $\Phi : \mathbb{K}\left(\mathscr{C}h^{k}\left(\mathscr{V}(X), \mathscr{Q}\right)\right) \to \mathbb{K}\left(\mathscr{C}h^{k}\left(\mathbb{M}(X), \mathscr{Q}\right)\right)$ is induced by the functor $\iota' : \left(\mathscr{C}h^{k}\left(\mathscr{V}(X), \mathscr{Q}\right)\right) \to \left(\mathscr{C}h^{k}\left(\mathbb{M}(X), \mathscr{Q}\right)\right)$ of complicial exact categories with weak equivalences. By Theorem 3.4, ι' induces an equivalence of the associated triangulated categories. Therefore, by [18, 3.2.29] Φ is a homotopy equivalence. Likewise, other two upward arrows are homotopy equivalences.

The middle downward arrow Ψ is a composition of three maps, as follows:

$$\begin{tikzcd}
\mathbb{K}\left(C\mathbb{M}^{k}(X)\right) \arrow[r, "\Psi'"] \arrow[d, "\Psi"'] & \mathbb{K}\left(Ch^{b}\left(C\mathbb{M}^{k}(X), \mathscr{Q}\right)\right) \arrow[d, "\zeta'"] \\
\mathbb{K}\left(\mathscr{C}h^{k}\left(\mathbb{M}(X), \mathscr{Q}\right)\right) & \mathbb{K}\left(Ch^{b}\left(\mathbb{M}^{k}(X), \mathscr{Q}\right)\right) \arrow[l, "\iota'"]
\end{tikzcd}$$

Now, ζ' and ι' are induced by the corresponding functors of complicial exact categories, with weak equivalences. Again, by [18, 3.2.29] in conjunction with Theorem 3.4, ζ' and ι' are homotopy equivalences. Now, Ψ' is a homotopy equivalence by the agreement theorem [18, 3.2.30]. Hence, so is Ψ.

It remains to show that, when X is Cohen–Macaulay, the third line is a homotopy fibrations of $\mathbb{K}$-theory spectra. To do this, consider sequence of complicial exact categories with weak equivalences (not necessarily exact):

$$\left(\mathscr{C}h^{k+1}\left(\mathscr{V}(X)\right), \mathscr{Q}\right) \longrightarrow \left(\mathscr{C}h^{k}\left(\mathscr{V}(X)\right), \mathscr{Q}\right) \longrightarrow \coprod_{x\in X^{(k)}} \left(\mathscr{C}h^{k}\left(\mathscr{V}\left(X_x\right)\right), \mathscr{Q}\right) \tag{7}$$

By Lemma 4.1, the corresponding sequence of the derived categories is exact up to factor. Therefore, by an application of the non-connective version of the Thomason–Waldhausen localization theorem (see [18, 3.2.27]) the third line in the statement of the theorem is a homotopy fibration of $\mathbb{K}$-theory spectra. The proof is complete. □

The following is an immediate consequence of Theorem 4.2.

Corollary 4.3. *Let X and k be as in Theorem 4.2. Assume X is Cohen–Macaulay. Then, for any integer n, there is an exact sequence of $\mathbb{K}$-groups,*

$$\cdots \longrightarrow \mathbb{K}_n\left(C\mathbb{M}^{k+1}(X)\right) \longrightarrow \mathbb{K}_n\left(C\mathbb{M}^{k}(X)\right) \longrightarrow \oplus_{x\in X^{(k)}}\mathbb{K}_n\left(\mathbb{M}^{k}(X_x)\right)$$

$$\longrightarrow \mathbb{K}_{n-1}\left(C\mathbb{M}^{k+1}(X)\right) \longrightarrow \cdots$$

Proof. Follows from Theorem 4.2. The proof is complete. □

Remark 4.4. If X is regular, statements exactly similar to Theorem 4.2 and Corollary 4.3, respectively, for **K**-theory spaces and groups, would be valid. Regularity is used to apply the connective version of Thomason–Waldhausen Localization theorem [18, 3.2.23], which requires that the corresponding sequence of derived categories is exact.

Remark 4.5. Let X be as in Theorem 4.2. Assume X is Cohen–Macaulay. The following are some remarks.

1. The usual diagram to compute the Gersten complex, reduces to

$$\begin{tikzcd}
\oplus_{x\in X^{(k-1)}}\mathbb{K}_{n+1}\left(C\mathbb{M}^{k-1}(X_x)\right) \arrow[d] \arrow[dr, dashed] & & \mathbb{K}_{n-1}\left(C\mathbb{M}^{k+2}(X)\right) \arrow[d] \\
\mathbb{K}_n\left(C\mathbb{M}^{k}(X)\right) \arrow[r] \arrow[d] & \oplus_{x\in X^{(k)}}\mathbb{K}_n\left(C\mathbb{M}^{k}(X_x)\right) \arrow[r] \arrow[dr, dashed] & \mathbb{K}_{n-1}\left(C\mathbb{M}^{k+1}(X)\right) \arrow[d] \\
\mathbb{K}_n\left(C\mathbb{M}^{k-1}(X)\right) & & \oplus_{x\in X^{(k+1)}}\mathbb{K}_{n-1}\left(C\mathbb{M}^{k+1}(X_x)\right)
\end{tikzcd}$$

   The dashed diagonal arrows form the Gersten complex. This provides further insight regarding the Gersten complexes in terms of the groups $\mathbb{K}_n\left(C\mathbb{M}^k(X)\right)$. This complex is analogous to the G-theoretic Gersten complex in [16, Proposition 5.8]. These complexes are clearly non-isomorphic.
2. The spectral sequence given in [3] takes the following form:

$$E_1^{p,q} = \bigoplus_{x\in X^{(p)}} \mathbb{K}_{-p-q}(C\mathbb{M}^p(X_x)) \Longrightarrow \mathbb{K}_{-n}(\mathscr{V}(X)) \qquad \text{along} \quad p+q=n.$$

4.2. Grothendieck–Witt theory

In this section, we develop a counter part of the results on K-theory (4.2), for Grothendieck–Witt theory. First, incorporating duality to the Theorem 3.4, we obtain the following.

Proposition 4.6. *Let X be a noetherian quasi-projective scheme, over an affine scheme* Spec (A)*, and $k \geq 0$ be an integer. Then, there is a duality preserving equivalence $\mathcal{D}^b\left(C\mathbb{M}^k(X)\right) \longrightarrow T^k\mathscr{D}^k\left((\mathscr{V}(X)\right)$ of the derived categories, where T denotes the shift, the duality on $\mathcal{D}^b\left(C\mathbb{M}^k(X)\right)$ is induced by $\mathcal{E}xt^k\left(-,\mathcal{O}_X\right)$ and that on $T^k\mathscr{D}^k\left((\mathscr{V}(X)\right)$ is $\# := T^k\mathcal{H}om\left(-,\mathcal{O}_X\right)$.*

Proof. It is a standard fact that there is a functor $\mathbb{M}(X) \longrightarrow \mathcal{D}^b\left(\mathscr{V}(X)\right)$, by resolution (e.g. see [11, 3.3]). The restriction to this functor to $C\mathbb{M}^k(X)$ extends to a functor $\mathcal{D}^b\left(C\mathbb{M}^k(X)\right) \to \mathscr{D}^k\left((\mathscr{V}(X)\right)$. It turns out that this functor represents the composite functor in Theorem 3.4. Hence the functor is an equivalence. Now, routine checking establishes that this functor preserves the duality, as required. The proof is complete. □

For clarity, we point out the technical differences in the literature among the basics of K-theory, derived Witt theory and Grothendieck Witt theory. Recall that K-theory is available for complicial exact categories with weak equivalences [18,22,23] and derived Witt theory was defined for triangulated categories with duality [1]. However, in [20], the Grothendieck Witt theory (GW) was developed for dg categories with weak equivalences and duality. Further, note that GW-theory of dg categories encompasses Witt theory of the same [20, Proposition 6.3]. As was pointed out in the introduction, K-theory is invariant of equivalences of the associated triangulated categories, and when 2 is invertible, so are GW-theory and Witt theory. The primary reason why we cannot directly imitate the methods in K-theory (§4.1), for Grothendieck Witt theory is that the dg category corresponding to $\mathscr{C}h^k(\mathbb{M}(X))$ does not have a natural duality structure. The remedy for this is obtained by embedding $\mathscr{C}h^k(\mathbb{M}(X))$ in the dg category of Perfect complexes. The following are some notations and background.

Notations 4.7. We establish some notations as follows. Let X denote a noetherian scheme.

1. We will denote the category of quasi-coherent $\mathcal{O}_X$-modules by $QCoh(X)$. Also, $Ch(QCoh(X))$ will denote the category of chain complexes of objects in $QCoh(X)$ and $\mathcal{D}(QCoh(X))$ will denote its derived category (see [18, A.3.2]).
2. Recall, that a complex $\mathcal{F}_\bullet \in Ch(QCoh(X))$ is called perfect complex, if for all $x \in X$, there is an affine open neighborhood U and a quasi-isomorphism $\mathcal{E}_\bullet \to (\mathcal{F}_\bullet)_{|U}$, for some $\mathcal{E}_\bullet \in Ch^b(\mathscr{V}(U))$. This is equivalent to saying $\mathcal{E}_\bullet$ is isomorphic to $(\mathcal{F}_\bullet)_{|U}$ in the derived category $\mathcal{D}(QCoh(U))$ (see [22, Lemma 2.2.9]).
3. Denote the category of perfect complexes of $\mathcal{O}_X$-modules by $Perf(X)$ and its derived category by $\mathcal{D}(Perf(X))$. In analogy to Notation 2.1(7), for integers $k \geq 0$,
$$\begin{cases} \mathscr{P}erf^k(X) := \left\{\mathcal{F}_\bullet \in Perf(X) : \forall\ i\ \mathcal{H}_i(\mathcal{F}_\bullet) \in Coh^k(X)\right\} \\ \mathscr{D}^k Perf(X) := \left\{\mathcal{F}_\bullet \in \mathcal{D}^b(Perf(X)) : \forall\ i\ \mathcal{H}_i(\mathcal{F}_\bullet) \in Coh^k(X)\right\} \end{cases}$$
would denote the full subcategory, of the respective categories, of such objects. Note, $\mathscr{D}^k Perf(X)$ is the derived category of $\mathscr{P}erf^k(X)$.
4. To avoid confusion, we will use prefix dg to denote the respective dg categories. So, $dgPerf(X)$ would denote the dg category whose objects are the same as those of $Perf(X)$. Likewise, $dg\mathscr{V}(X)$, $dg^k\mathscr{V}(X)$, $dgCM^k(X)$ will denote the dg categories whose objects are, respectively, the same as those of $Ch^b(\mathscr{V}(X))$, $\mathscr{C}h^k(\mathscr{V}(X))$, $Ch^b(C\mathbb{M}^k(X))$.
5. Throughout, we fix a minimal injective resolution $I_\bullet$ of $\mathcal{O}_X$, as follows:
$$0 \longrightarrow \mathcal{O}_X \longrightarrow I_0 \longrightarrow I_{-1} \longrightarrow I_{-2} \longrightarrow \cdots. \qquad \text{Clearly, } I_\bullet \in Perf(X).$$
For $\mathcal{F}_\bullet \in Perf(X)$, denote $\mathcal{F}^\vee := \mathcal{H}om(\mathcal{F}_\bullet, I_\bullet)$. For properties of such minimal resolutions and the nature of arguments, the readers are referred to [5] and [7].

The following addresses the duality aspect of $dgPerf(X)$.

Lemma 4.8. *Let X be a noetherian scheme. Let $I_\bullet$ be as in Notation 4.7(5). Then, the association $\mathcal{F}_\bullet \mapsto \mathcal{F}_\bullet^\vee$ endows $(dgPerf(X), \mathscr{Q})$ with a structure of a dg category with weak equivalences and duality, weak equivalences being the set of all quasi-isomorphism $\mathscr{Q}$.*

Proof. Consider $\mathcal{O}_X$ as a complex, concentrated at degree zero. Since $\mathcal{O}_X \to I_\bullet$ is a quasi-isomorphism, $I_\bullet \in Perf(X)$. Let $\mathcal{F}_\bullet \in Perf(X)$. So, there is an affine open subset U and a quasi-isomorphism $\mathcal{E}_\bullet \to (\mathcal{F}_\bullet)_{|U}$ for some $\mathcal{E}_\bullet \in Ch^b(\mathscr{V}(U))$. Then:

$$\text{In} \quad \mathcal{D}(QCoh(X)), \quad \mathcal{H}om(\mathcal{E}_\bullet, \mathcal{O}_U) \xrightarrow[\sim]{} \mathcal{H}om(\mathcal{E}_\bullet, (I_\bullet)_{|U}) \xleftarrow{\sim} \mathcal{H}om((\mathcal{F}_\bullet)_{|U}, (I_\bullet)_{|U})$$

are bijections. Since $\mathcal{H}om(\mathcal{E}_\bullet, \mathcal{O}_U) \in Ch^b(\mathscr{V}(U))$, it follows that $\mathcal{F}_\bullet^\vee \in Perf(X)$. Also, for $\mathcal{F}_\bullet \in Perf(X)$, we check, $\mathcal{F}_\bullet \to \mathcal{F}_\bullet^{\vee\vee}$ is a quasi-isomorphism. As above, let U be affine and $\mathcal{E}_\bullet \to (\mathcal{F}_\bullet)_{|U}$ be a quasi-isomorphism, where $\mathcal{E}_\bullet \in Ch^b(\mathscr{V}(X))$. Since, the question is local, we can assume that $X = U$. Consider the diagram

$$\begin{tikzcd} \mathcal{E}_\bullet \arrow[r] \arrow[d] & \mathcal{H}om\left(\mathcal{H}om\left(\mathcal{E}, \mathcal{O}_U\right), \mathcal{O}_U\right) \arrow[d] \\ \mathcal{F}_\bullet \arrow[r] & \mathcal{F}_\bullet^{\vee\vee} \end{tikzcd}$$

Since, two vertical arrows and the top horizontal arrow are quasi-isomorphisms, so is the bottom horizontal arrow. Now, we show if $f_\bullet : \mathcal{F}_\bullet \to \mathcal{G}_\bullet$ is a quasi-isomorphism in $Perf(X)$, so is $f_\bullet^\vee : \mathcal{G}_\bullet^\vee \to \mathcal{F}_\bullet^\vee$. Using similar arguments as above, we can assume that X is affine and f is a map in $Ch^b(\mathscr{V}(X))$, in which case the assertion is obvious. The proof is complete. □

The following proposition on derived equivalences is derived from results in [22].

Proposition 4.9. *Let X be noetherian separated scheme, with an ample family of line bundles and $k \geq 0$ be an integer. Let $I_\bullet$ be as in Notation 4.7(5). Then, $\mathscr{D}^k(\mathscr{V}(X)) \to \mathscr{D}^k Perf(X)$ is an equivalence of derived categories.*

Proof. We have $\mathcal{D}^b(\mathscr{V}(X)) \to \mathcal{D}(Perf(X))$ is an equivalence of derived categories ([22, Lemma 3.8], [18, Prop. 3.4.8]). Since $\mathscr{D}^k(\mathscr{V}(X)) \hookrightarrow \mathcal{D}^b(\mathscr{V}(X))$ and $\mathscr{D}^k Perf(X) \hookrightarrow \mathcal{D}(Perf(X))$ are full subcategories, the assertion follows. □

Of our particular interest would be the following equivalences of derived categories. Refer to [20] for definition of form functors.

Proposition 4.10. *Suppose X is a quasi-projective scheme over an affine scheme* Spec (A) *and $k \geq 0$ is an integer. Then:*

1. *The inclusion functor $dg^k\mathscr{V}(X) \hookrightarrow dg\mathcal{P}erf^k(X)$ is a duality preserving form functor* (see [20, 1.12, 1.7], for definition), *of pointed dg categories with weak equivalences and dualities, such that the associated functor of the triangulated categories $\mathscr{T}\left(dg^k\mathscr{V}(X)\right) \hookrightarrow \mathscr{T}\left(dg\mathcal{P}erf^k(X)\right)$ is an equivalence.*
2. *The inclusion functor $dgC\mathbb{M}^k(X) \hookrightarrow T^k\left(dgPerf^k(X)\right)$ is a duality preserving form functor, of pointed dg categories with weak equivalences and dualities, where T denotes the shift. Further, the associated functor of the triangulated categories $\mathscr{T}(dgC\mathbb{M}^k(X)) \to \mathscr{T}\left(T^k\left(dgPerf^k(X)\right)\right)$ is an equivalence.*

Proof. Since $\mathscr{T}\left(dg^k\mathscr{V}(X)\right) = \mathscr{D}^k(\mathscr{V}(X))$ and $\mathscr{T}\left(dg\mathcal{P}erf^k(X)\right) = \mathscr{D}^k(Perf(X))$, the latter part of (1) follows immediately from Proposition 4.9. The duality compatibility transformation is the obvious map $\mathcal{H}om(\mathcal{F}_\bullet, \mathcal{O}_X) \to \mathcal{H}om(\mathcal{F}_\bullet, I_\bullet)$, which is a weak equivalence. This establishes (1).

To prove (2), note that the duality on $dgC\mathbb{M}^k(X)$ is induced by the duality $\mathcal{E}xt^k(-, \mathcal{O}_X)$. For $\mathcal{F}_\bullet \in dg(C\mathbb{M}^k(X))$, let $\widehat{\mathcal{F}_\bullet}$ denote its dual. Note, $\mathscr{T}\left(dgC\mathbb{M}^k(X)\right) = \mathcal{D}^b\left(C\mathbb{M}^k(X)\right)$ and $\mathscr{T}\left(dg\mathcal{P}erf^k(X)\right) = T^k\mathscr{D}^k\left(Perf^k(X)\right)$. Now, it follows from (1) and Proposition 4.6, that $\mathscr{T}(dgC\mathbb{M}^k(X)) \to \mathscr{T}\left(T^k\left(dgPerf^k(X)\right)\right)$ is an equivalence. For an object $\mathcal{F} \in C\mathbb{M}^k(X)$, due to grade consideration it follows that $\forall\, j \leq k-1$, $\mathcal{H}om(\mathcal{F}, I_j) = 0$ (see [5, Prop. 3.2.9], [7, Theorem 1.15]). Therefore, for a complex $\mathcal{F}_\bullet \in dgC\mathbb{M}^k(X)$, we have a bounded double complex:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & \mathcal{E}xt^k(\mathcal{F}_{n+1}, \mathcal{O}_X) \arrow[r] \arrow[d] & \mathcal{E}xt^k(\mathcal{F}_n, \mathcal{O}_X) \arrow[r] \arrow[d] & \mathcal{E}xt^k(\mathcal{F}_{n-1}, \mathcal{O}_X) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathcal{H}om(\mathcal{F}_{n+1}, I_{-k}) \arrow[r] \arrow[d] & \mathcal{H}om(\mathcal{F}_n, I_{-k}) \arrow[r] \arrow[d] & \mathcal{H}om^k(\mathcal{F}_{n-1}, I_{-k}) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathcal{H}om\left(\mathcal{F}_{n+1}, I_{-(k+1)}\right) \arrow[r] \arrow[d] & \mathcal{H}om\left(\mathcal{F}_n, I_{-(k+1)}\right) \arrow[r] \arrow[d] & \mathcal{H}om^k\left(\mathcal{F}_{n-1}, I_{-(k+1)}\right) \arrow[r] \arrow[d] & \cdots \\
 & \cdots & \cdots & \cdots &
\end{tikzcd}$$

In this double complex, the vertical lines are exact. This gives a natural transformation $\varphi : \widehat{\mathcal{F}_\bullet} \to \mathcal{H}om(\mathcal{F}_\bullet, I_\bullet) =: \mathcal{F}_\bullet^\vee$, which is a weak equivalence. Now, it follows that φ defines a duality preserving transformation $dgC\mathbb{M}^k(X) \to T^k\left(dgPerf^k(X)\right)$. The proof is complete. $\square$

The following useful diagram is analogous to the diagram in the Equivalence Theorem 3.4, in the context of dg categories with weak equivalences and dualities.

Corollary 4.11. *Suppose X is a quasi-projective scheme over an affine scheme* Spec (A)*, and $k \geq 0, r$ are integers. Consider the diagram*

$$\begin{tikzcd}
T^{-1}dgC\mathbb{M}^{k+1}(X) \arrow[r] & T^k dg\mathcal{P}erf^{k+1}(X) \arrow[d] & T^k dg^{k+1}\mathscr{V}(X) \arrow[l] \arrow[d] \\
dgC\mathbb{M}^k(X) \arrow[r] & T^k dg\mathcal{P}erf^k(X) & T^k dg^k\mathscr{V}(X) \arrow[l]
\end{tikzcd} \tag{8}$$

In this diagram, all the arrows are form functors of dg categories with weak equivalence. Further, the horizontal arrows induce equivalences of the associated triangulated categories and the right hand square commutes. Note that there is no natural vertical functor on the left side.

Proof. Follows from Proposition 4.10. $\square$

Now we have the machinery to state our results on GW-theory. For background information regarding the definitions of $\mathbf{GW}$-spectrum and $\mathbb{G}W$-spectrum the readers are referred to Appendix A or [20].

Theorem 4.12. *Suppose X is a quasi-projective scheme over an affine scheme* Spec (A)*, with $1/2 \in A$ and $k \geq 0, r$ are integers. In the following, weak equivalences and dualities in the respective categories would be as in Proposition 4.10. Then, the maps in the following zig-zag sequences*

$$\mathbf{GW}^{[r]}\left(dgC\mathbb{M}^k(X)\right) \xrightarrow{\zeta} \mathbf{GW}^{[k+r]}\left(dgPerf^k(X)\right) \xleftarrow{\Phi} \mathbf{GW}^{[k+r]}\left(dg^k\mathscr{V}(X)\right) \qquad \text{in Sp}$$

$$\mathbb{G}W^{[r]}\left(dgC\mathbb{M}^k(X)\right) \xrightarrow[\zeta]{} \mathbb{G}W^{[k+r]}\left(dgPerf^k(X)\right) \xleftarrow[\Phi]{} \mathbb{G}W^{[k+r]}\left(dg^k(\mathscr{V}(X)\right) \qquad \text{in BiSp}$$

are stable homotopy equivalences in the respective categories.

Proof. Follows directly from Proposition 4.10 and [20, Theorem 6.5], [20, Theorem 8.9]. The proof is complete. □

Theorem 4.13. *Suppose X, k, r are as in Theorem 4.12. Assume further that X is a Cohen–Macaulay scheme. Consider the following diagram of $\mathbb{G}W$-spectra:*

$$\begin{tikzcd}
\mathbb{G}W^{[-1+r]}\left(dgC\mathbb{M}^{k+1}(X)\right) \arrow[d] & \mathbb{G}W^{[r]}\left(dgC\mathbb{M}^{k}(X)\right) \arrow[d] & \coprod_{x\in X^{(k)}} \mathbb{G}W^{[r]}\left(dgC\mathbb{M}^{k}\left(X_x\right)\right) \arrow[d] \\
\mathbb{G}W^{[k+r]}\left(dgPerf^{k+1}(X)\right) & \mathbb{G}W^{[k]}\left(dgPerf^{k}X\right) & \coprod_{x\in X^{(k)}} \mathbb{G}W^{[k+r]}\left(dgPerf(X_x)\right) \\
\mathbb{G}W^{[k+r]}\left(dg^{k+1}\mathscr{V}(X)\right) \arrow[u] \arrow[r] & \mathbb{G}W^{[k+r]}\left(dg^{k}\mathscr{V}(X)\right) \arrow[u] \arrow[r] & \coprod_{x\in X^{(k)}} \mathbb{G}W^{[k+r]}\left(dg^{k}\mathscr{V}(X_x)\right) \arrow[u]
\end{tikzcd}$$

In this diagram, all the vertical arrows are equivalences of homotopy Bispectra and the bottom sequence is a homotopy fibration of bispectra. Further, if X is regular then the corresponding statement for **GW***-spectra would be valid.*

Proof. It follows directly from Theorem 4.12 that the vertical rows are equivalences. It remains to show that the bottom row is a fibration. This follows from the localization theorem [20, Thm 8.10] and Lemma 4.1. When X is regular, use the localization theorem [20, Thm 6.6]. The proof is complete. □

Remark 4.14. The following are some remarks:

1. As in Corollary 4.3, each shift r, corresponding to the fiber sequence in Theorem 4.13, an exact sequence of $\mathbb{G}W$-groups would follow. Likewise, analogous to Remark 4.5, for each shift r, a spectral sequence of $\mathbb{G}W$-groups would follow.
2. For a scheme X and a rank one locally free sheaf $\mathcal{L}$, $\mathcal{H}om(-,\mathcal{L})$ induces a duality on $\mathscr{V}(X)$. All of the above would be valid, with dualities induced by $\mathcal{H}om(-,\mathcal{L})$, instead of $\mathcal{H}om(-,\mathcal{O}_X)$.
3. For an exact category $\mathscr{E}$ with duality, Grothendieck–Witt space $GW(\mathscr{E}) \in \mathrm{Top}_*$ and Grothendieck–Witt groups $\forall\, i \geq 0$ $GW_i(\mathscr{E}) := \pi_i\left(GW(\mathscr{E})\right)$ were defined by Schlichting [21]. With X as in Theorem 4.12 and integers $k \geq 0$, it follows from the Agreement theorem [21, Proposition 6], and [20, Proposition 5.6] that $GW(C\mathbb{M}^k(X))$ is naturally equivalent to the infinite loop space $\Omega^\infty \mathbf{GW}(dgC\mathbb{M}^k(X))$, of the spectra. In particular, $\forall\, i \geq 0$ $GW_i(C\mathbb{M}^k(X)) \cong \mathbf{GW}_i(dgC\mathbb{M}^k(X))$.

4.3. Derived Witt theory

In this subsection we comment on Witt theory. The following follows immediately.

Theorem 4.15. *Let X, A and k be as in Proposition 4.6. Assume $1/2 \in A$. Then, the maps of the shifted Witt groups $W^r\left(\mathcal{D}^b\left(C\mathbb{M}^k(X)\right)\right) \to W^{k+r}\left(\mathscr{D}^k\left(\left(\mathscr{V}(X)\right)\right)\right)$ are isomorphisms, for all $r \in \mathbb{Z}$. In particular, the maps*

$$\begin{cases} W\left(C\mathbb{M}^k(X)\right) \to W^{k+4r}\left(\mathscr{D}^k\left(\left(\mathscr{V}(X)\right)\right)\right) \\ W^-\left(C\mathbb{M}^k(X)\right) \to W^{k+2+4r}\left(\mathscr{D}^k\left(\left(\mathscr{V}(X)\right)\right)\right) \end{cases}$$

are isomorphisms, for all $r \in \mathbb{Z}$.

Proof. The first isomorphism follows from Proposition 4.6 and [1, Theorem 6.2], because the quotient category would be trivial. By [4, Theorem 1.4], $W\left(C\mathbb{M}^k(X)\right) \cong W^{4r}\left(\mathcal{D}^b\left(C\mathbb{M}^k(X)\right)\right)$ and $W^-\left(C\mathbb{M}^k(X)\right) \cong W^{2+4r}\left(\mathcal{D}^b\left(C\mathbb{M}^k(X)\right)\right)$. By combining these, with the first isomorphism, the latter two isomorphisms are established. □

Due to non-availability of Thomason–Waldhausen [18, 3.2.27] type of localization theorems in derived Witt theory, for the statement of the following theorem, we would assume that X is regular.

Theorem 4.16. *Let X be a quasi-projective regular scheme, over an affine scheme* $\mathrm{Spec}\,(A)$*, with $1/2 \in A$, and $k \geq 0$ be an integer. In this case, the sequence in Lemma 4.1 is exact. Now, the twelve term exact sequence in [1, Corollary 6.6], corresponding to the same exact sequence reduces to two five term exact sequences of Witt groups as follows:*

$$0 \longrightarrow W^{-1}\left(\mathcal{D}^b\left(C\mathbb{M}^{k+1}(X)\right)\right) \longrightarrow W\left(C\mathbb{M}^k(X)\right) \longrightarrow \oplus_{x\in X^{(k)}} W\left(\mathbb{M}^k(X_x)\right)$$

$$\longrightarrow W\left(C\mathbb{M}^{k+1}(X)\right) \longrightarrow W^1\left(\mathcal{D}^b\left(C\mathbb{M}^k(X)\right)\right) \longrightarrow 0$$

$$\longrightarrow W^1\left(C\mathbb{M}^{k+1}(X)\right) \longrightarrow W^-\left(CM^k(X)\right) \longrightarrow \oplus_{x\in X^{(k)}} W^-\left(\mathbb{M}^k(X_x)\right)$$

$$\longrightarrow W^-\left(C\mathbb{M}^{k+1}(X)\right) \longrightarrow W^3\left(\mathcal{D}^b\left(C\mathbb{M}^b(X)\right)\right) \longrightarrow 0$$

Proof. Write down the twelve term exact sequence [1, Corollary 6.6] of Witt groups, corresponding to exact sequence of triangulated categories above. The zero term in the second row corresponds to $\oplus_{x\in X^{(k)}} W^{k+1}(X_x) = 0$ by [4]. Likewise, the first and the last zero are established. The rest follows by identifying the other terms by Corollary 4.15. □

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Appendix A. Background on GW-spectrum and $\mathbb{G}W$-bispectrum

In this section, we include some background information on **GW**-spectrum and $\mathbb{G}W$-bispectrum. In fact, this is an overview from our primary source [20] on the same. Readers reluctant to deal with language of model categories may like to refer directly to formulas (9), (10).

Recall that for exact categories $\mathscr{E}$ and also for complicial exact categories $\mathscr{E}$ with weak equivalences the **K**-theory spaces $\mathbf{K}(\mathscr{E})$ were defined as pointed topological spaces. Further, the $\mathbb{K}$-theory spectra $\mathbb{K}(\mathscr{E})$ were defined as Ω-spectra of topological spaces, which is a sequence of pointed topological spaces, with bonding maps (see [18, §A.1.8]). Likewise, for a pointed dg category $\mathscr{A}$ with weak equivalences and duality two invariants are defined in GW-theory, in [20], as follows.

Definition A.1. For a dg category $\mathscr{A}$ with weak equivalences and duality, the Grothendieck–Witt spectrum $\mathbf{GW}(\mathscr{A})$ takes value in the category of symmetric spectra of pointed topological spaces. We only give some outlines of the definitions of symmetric spectrum (see [20, §B.1] for details) and of $\mathbf{GW}(\mathscr{A})$ as follows.

1. Let Σ denote the category, whose objects are $\underline{n} := \{1, 2, \ldots, n\}$, $\underline{0} = \phi$. The morphisms are: $\forall\ m \neq n$ $Mor(\underline{m}, \underline{n}) = \phi$ and $\forall\ n$ $Mor(\underline{n}, \underline{n}) =: \Sigma_n$ is the group of permutations elements in $\underline{n}$.
   The category of pointed topological spaces is denoted by Top_*. The smash product $\wedge$ provides the category Top_* a structure of a symmetric monoidal category. Also, denote $S^0 := \{0, 1\}$, and S^1 denotes the circle obtained by identifying $0 \sim 1$, in the unit interval $[0, 1]$. For $n \in \mathbb{N}$, denote by $S^n = S^1 \wedge \cdots \wedge S^1$, the n-fold smash product.
2. A symmetric spectrum is a functor $\Sigma \to Top_*$ form Σ to the category Top_*, together with base point preserving maps $\forall\ n, m \in \mathbb{N}$ $e_{n,m} : S^n \wedge X_m \to X_{n+m}$, to be called the bonding maps, with further compatibility conditions. Therefore, a symmetric spectrum is a sequence $X := \{X_0, X_1, X_2, \ldots\}$ of pointed topological spaces such that (1) $\forall\ n \in \mathbb{N}$, there is a continuous base point preserving left action of Σ_n on X_n, and (2) for $n, m \in \mathbb{N}$, there are pointed continuous $\Sigma_m \times \Sigma_m$-equivariant maps $e_{n,m} : S^n \wedge X_m \to X_{n+m}$ with natural compatibility conditions.
   The category of symmetric spectrum is denoted by Sp. The smash product $\wedge$ of pointed topological spaces extends to a smash product on Sp, denoted by the same notation $\wedge$. Further, four different model structures on Sp are discussed in [20, §B.2], namely, the (positive) projective level model structure and (positive) projective stable model structure. Two stable model structures on Sp have same weak equivalences and hence same homotopy category, to be called the stable homotopy category.
   For $X, Y \in \mathrm{Sp}$, let $[X, Y]$ denote the set of all morphisms $X \to Y$ in the stable homotopy category. Define the homotopy groups of X as
   $$\pi_n(X) := [S^n, X]$$
   Our interest, with respect to GW-theory, would remain limited to the case when the spectrum is a (positive) Ω-spectrum, meaning that the bonding maps $X_n \to \Omega X_{n+1}$ are weak equivalences of pointed topological spaces. In this case (see [20, §B.3]),
   $$\forall\ n \in \mathbb{Z} \qquad \pi_n(X) = \operatorname{co}\lim_k \pi_{n+k}(X_k),$$
   where $\pi_{n+k}(X_k)$ denote the usual homotopy groups.
3. For a pointed dg category $\mathscr{A}$, the Grothendieck–Witt spectrum $\mathbf{GW}(\mathscr{A})$ is defined as a symmetric spectrum [20, §4.4, Definition 5.4]
   $$\mathbf{GW}(\mathscr{A}) := \{\mathbf{GW}(\mathscr{A})_0, \mathbf{GW}(\mathscr{A})_1, \mathbf{GW}(\mathscr{A})_2, \ldots\}.$$
   Further, for $n \in \mathbb{Z}$ the n-shifted $\mathbf{GW}$-spectrum is defined to be
   $$\mathbf{GW}^{[n]}(\mathscr{A}) := \mathbf{GW}\left(\mathscr{A}^{[n]}\right) \quad \text{where } \mathscr{A}^{[n]} \text{ denotes the } n\text{-shifted dg category of } \mathscr{A}.$$
   In fact, $\mathbf{GW}(\mathscr{A})$, is a positive Ω-spectrum [20, Theorem 5.5]. For $n, i \in Z$, denote
   $$\mathbf{GW}_i^{[n]}(\mathscr{A}) := \pi_i\left(\mathbf{GW}^{[n]}(\mathscr{A})\right) = \operatorname{co}\lim_k \pi_{n+k}\left(\mathbf{GW}^{[n]}(\mathscr{A})_k\right) \tag{9}$$
   where the latter equality is a property of the positive Ω-spectra.

With respect to sequences $(\mathscr{A}_0, w) \longrightarrow (\mathscr{A}_1, w) \longrightarrow (\mathscr{A}_2, w)$ of pointed dg categories with weak equivalences and dualities, the $\mathbf{GW}$-spectra behave well (see [20, Theorem 6.6]), when the associated sequence of triangulated categories is exact (assuming 2 is invertible). However, while dealing with non-regular schemes, as noted in Lemma 4.1, the relevant sequence of associated triangulated categories is exact only up to factors. To remedy this situation, in analogy to $\mathbb{K}$-theory spectra, Karoubi–Grothendieck–Witt spectrum $\mathbb{G}W(\mathscr{A})$ of pointed dg categories $\mathscr{A}$ with weak equivalences and dualities are defined [20].

Definition A.2. For dg categories $\mathscr{A}$ with weak equivalences and dualities, the Karoubi–Grothendieck–Witt spectrum $\mathbb{G}W(\mathscr{A})$ takes value in the category of Bispectra. Again, we only outline the definition of Bispectra and $\mathbb{G}W(\mathscr{A})$ from [20].

1. Note that the category $(Top_*, \wedge, S^0)$ of pointed topological spaces, with smash product $\wedge$ is a symmetric monoidal category, where the unit is S^0. The process used to obtain the category Sp from $(Top_*, \wedge, S^0)$ is fairly formal and is described in [20, §B.9], where S^1 had a special role to play. The category of Bispectra is obtained, by iterating the same process on Sp, as follows.
   (a) Denote $S := \{S^0, S^1, \dots\} \in \mathrm{Sp}$. For $X, Y \in \mathrm{Sp}$ define a new smash product $X \wedge_S Y$ by push forward

$$\begin{tikzcd} X \wedge S \wedge Y \arrow[r, "1_X \wedge -"] \arrow[d, "- \wedge 1_Y"'] & X \wedge Y \arrow[d] \\ X \wedge Y \arrow[r] & X \wedge_S Y \end{tikzcd}$$

   Then, $(\mathrm{Sp}, \wedge_S, S)$ is cofibrantly generated closed symmetric monoidal model category, with the positive stable model structure on Sp. By abuse of notations, write $S^1 := S \wedge (S^1, pt, pt, \dots) = (S^1, S^1 \wedge S^1, S^2 \wedge S^1, \cdots)$. Also, let $\tilde{S}^1$ denote a cofibrant replacement of $S^1 \in \mathrm{Sp}$.
   (b) The process mentioned above [20, §B.9] is applied to this category $(\mathrm{Sp}, \wedge_S, S)$ with the special role played by $\tilde{S}^1 \in \mathrm{Sp}$ (see [20, §B.11]). The category thus obtained is called category $\tilde{S}^1$–S-bispectra, or simply the category of Bispectra, which is denoted by

$$\mathrm{BiSp} := \mathrm{Sp}(\mathrm{Sp}, \tilde{S}^1) := \mathrm{Sp}\left((\mathrm{Sp}, \wedge_S, S), \tilde{S}^1\right)$$

   We describe BiSp as follows (see [20, §B.11]):
      i. Let Sp^Σ denote the category of functors $\Sigma \to \mathrm{Sp}$. So, an object in Sp^Σ is a sequence $(\mathcal{X}_0, \mathcal{X}_1, \mathcal{X}_2, \dots,)$ where $\mathcal{X}_n \in \mathrm{Sp}$ are spectra, with a left action of the symmetric groups Σ_n.
      ii. The smash product $\wedge_S$ extends to a smash product $\wedge_S$ in Sp^Σ.
      iii. Write $\tilde{S} := (S^0, \tilde{S}^1, \tilde{S}^1 \wedge_S \tilde{S}^1, \tilde{S}^1 \wedge_S \tilde{S}^1 \wedge_S \tilde{S}^1, \dots)$.
      iv. The objects in BiSp are $\tilde{S}$-modules $M \in \mathrm{Sp}^\Sigma$. This means that there is a map (natural transformation) $\tilde{S} \wedge M \to M$, compatible with the action of Σ.
      v. The objects $\mathcal{X} \in \mathrm{BiSp}$ are also called $\tilde{S}^1$–S^1-bispectrum, or simply a bispectrum.
   (c) BiSp has a (positive) stable symmetric monoidal model structure (see [20, §B.9, B.11]), by results of Hovey [6]. For bispectra $\mathcal{X}, \mathcal{Y} \in \mathrm{BiSp}$, let $[\mathcal{X}, \mathcal{Y}]_{H(\mathrm{BiSp})}$ denote the set of all morphisms $\mathcal{X} \to \mathcal{Y}$ in this stable homotopy category. Define the homotopy groups of $\mathcal{X}$ as

$$\pi_n(\mathcal{X}) := [\tilde{S}^n, \mathcal{X}]_{H(Bi\mathrm{Sp})}$$

   If $\mathcal{X} = (\mathcal{X}_0, \mathcal{X}_1, \mathcal{X}_2, \dots) \in \mathrm{BiSp}$ is a level fibrant semistable $\tilde{S}^1$–S^1-bispectrum, then [20, Lemma B.16]

$$\pi_n(\mathcal{X}) := [\tilde{S}^n, \mathcal{X}]_{H(\mathrm{BiSp})} = \operatorname{co}\lim_k \left(\pi_{n+k}(\mathcal{X}_k)\right) \tag{10}$$

2. Suppose $\mathscr{A}$ is a pointed dg category with weak equivalence and duality. Let $\mathcal{S}^n\mathscr{A}$ denote the iterated n-fold suspension of $\mathscr{A}$ (see [20, §8.1]). Denote $\mathbb{G}W(\mathscr{A})_n := \mathbf{GW}(\mathcal{S}^n\mathscr{A}) \in \mathrm{Sp}$, the $\mathbf{GW}$-spectrum of

$\mathcal{S}^n\mathscr{A}$. Then, Σ_n has a left action on $\mathbb{G}W(\mathscr{A})_n$. The Karoubi–Grothendieck–Witt spectrum $\mathbb{G}W(\mathscr{A})$ of $\mathscr{A}$ is defined as the $\tilde{S}^1$–S^1-bispectrum (see [20, §8.2])

$$\mathbb{G}W(\mathscr{A}) := (\mathbb{G}W(\mathscr{A})_0, \mathbb{G}W(\mathscr{A})_1, \mathbb{G}W(\mathscr{A})_2, \ldots, \mathbb{G}W(\mathscr{A})_n, \ldots) \in \mathrm{BiSp}.$$

The Karoubi–Grothendieck–Witt spectrum is defined as a functor $\mathbb{G}W : dgCatWD_* \to \mathrm{BiSp}$ from the category of pointed small dg categories with weak equivalences and dualities to the category of Bispectra. For $n \in Z$, the Karoubi–Grothendieck–Witt groups are defined as, and are isomorphic to

$$\begin{aligned}\mathbb{G}W_n(\mathscr{A}) := \pi_n\left(\mathbb{G}W(\mathscr{A})\right) &\cong co\lim_k \pi_{n+k}\left(\mathbb{G}W(\mathscr{A})_k\right) \cong co\lim_k \pi_{n+k}\left(\mathbf{GW}(\mathcal{S}^k\mathscr{A})\right)\\ &\cong co\lim_k \left(co\lim_m \left(\pi_{n+k+m}\left(\mathbf{GW}(\mathcal{S}^k\mathscr{A})\right)_m\right)\right)\end{aligned}$$

The latter isomorphisms follow because the formulas (9), (10) would apply. For $n, i \in \mathbb{Z}$ define the n-shifted Karoubi–Grothendieck Witt spectrum and groups, respectively, as

$$\mathbb{G}W^{[n]}(\mathscr{A}) := \mathbb{G}W\left(\mathscr{A}^{[n]}\right) \quad \text{and} \quad \mathbb{G}W_i^{[n]}(\mathscr{A}) := \pi_i\left(\mathbb{G}W^{[n]}(\mathscr{A})\right).$$

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