Topology in Algebra

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Abstract

We will discuss how the theory of vector bundles in topology influenced developments in algebra, and the correspondences between the classical theory in topology and the newly developed theory in algebra.

Rings

- A ring A is a set with an addition $(+)$ and a multiplication. It is a commutative group under addition $+$, and the multiplication is distributive with respect to $+$.
- Any field is a ring. So, $\mathbb{R}, \mathbb{C}$ are rings.
- Let M be a topological space. Let $C(M)$ denote the set of all continuous real valued functions. Then $C(M)$ is a ring. This may be the most inspiring example of a ring.

Modules

- A module M over a ring A is what a vector space would be over a field.
- A free module F over a ring A is an $A-$module that has a basis. If F is a finitely generated free $A-$module, then $F \approx A^n$. In this case, $rank(F) := n$.

Projective Modules

- Suppose A is a commutative ring.
- An $A-$module P is said to be projective, if

$$P \oplus Q = Free$$

for some other $A-$module Q.

Projective Modules

Examples:

- A free $A-$module is a projective $A-$module.
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$$Let \quad A_n = \frac{\mathbb{R}[X_0, \ldots, X_n]}{(X_0^2 + \cdots + X_n^2 - 1)} = \mathbb{R}[x_0, x_1, \ldots, x_n]$$
be the algebraic coordinate ring of the real $n-$sphere. Let T_n be defined by the exact sequence
$$0 \longrightarrow T_n \longrightarrow A_n^{n+1} \xrightarrow{(x_0, \ldots, x_n)} A_n \longrightarrow 0.$$
Then T_n is a projective A_n-module. It "corresponds" to the tangent bundle.

Projective Modules

Examples:

- Now suppose, M is a smooth (compact) manifold. Let $\mathcal{T}$ be the tangent bundle on M. Let T be the set of all vector fields. Then T is a projective $C(M)-$module.
- It is known that the tangent bundles over even dimensional spheres $\mathbb{S}^n$ are not trivial. So, the projective $C(\mathbb{S}^n)-$module T is not free. (*The last slide proves it.*)
- Next, we define vector bundles.

Vector bundles

Suppose M is a topological space. A (real) **vector bundle** on M, is a continuous map $p : \mathcal{E} \to M$ such that

- Each fiber $\mathcal{E}_x = p^{-1}(x)$ has a vector space structure.
- M has an open cover $\{U_i\}$ and homeomorphisms (trivializations) φ_i such that the diagrams

$$\begin{array}{ccc} p^{-1}(U_i) & \xrightarrow[\sim]{\varphi_i} & U_i \times \mathbb{R}^r \\ & {\scriptstyle p}\searrow \quad \swarrow & \\ & U_i & \end{array} \qquad \textit{commute}.$$

- For each $x \in U_i$, the trivialization φ_i induces linear isomorphisms $\mathcal{E}_x \to \mathbb{R}^r$.

Vector bundles

- The rank of $\mathcal{E}$ is defined as $rank(\mathcal{E}) = r$.
- Example: $M \times \mathbb{R}^r \longrightarrow M$ *is the trivial bundle on* M, *to be denoted by* $\mathcal{R}^r$.
- Example: *The tangent bundle* $\mathcal{T}$ *over a manifold* M, *is a vector bundle.*

The Module of Sections

Let

$$\Gamma(\mathcal{E}) := \{s : M \to \mathcal{E} : ps = Id_M, \quad s \text{ is continuous}\}.$$

This means $s(x) \in \mathcal{E}_x \ \ \forall x \in M$.

1. Elements $s \in \Gamma(\mathcal{E})$ are called **sections** of $\mathcal{E}$.
2. Example: *vector fields are sections of the tangent bundle.*
3. $\Gamma(\mathcal{E})$ **has a natural** $C(M)-$**module structure.**

Correspondence

Theorem ([Swan 1962])

Suppose M is a (compact connected) Hausdorff topological space. Then the association

$$\mathcal{E} \longrightarrow \Gamma(\mathcal{E})$$

is an equivalence of catagories, from the category $\mathcal{V}(M)$ of vector bundles over M to the category $\mathcal{P}(C(M))$ of finitely generated projective $C(M)-$modules.

Correspondence

- Because of this correspondence, there is a lot in common between research on vector bundles in topology and that on projective modules in algebra.
- The ring $C(M)$ is too big. We work with the ring of algebraic functions.
- I will often talk about "noetherian commutative rings," because the ring of algebraic functions over a space M is "notherian and commutative".
- More often than not, research on vector bundles led the way for research on projective modules.

Never-Vanishing sections

- Let M be a real manifold with $\dim M = d$.
- Let $\mathcal{E}$ be a vector bundle of rank r.
- If $r > d$, then $\mathcal{E}$ has a never-vanishing section.
- Therefore,

$$\mathcal{E} \approx \mathcal{E}_0 \bigoplus \mathcal{R}^{r-d} \quad with \quad rank(\mathcal{E}_0) = d$$

where $\mathcal{R} = M \times \mathbb{R}$ is the trivial bundle of rank one.

Splitting

The above inspired the theorem of Serre ([Serre1957]):

- Let A be a noetherian commutative ring with $\dim A = d$.
- Let P be a projective $A-$module of rank r.
- If $r > d$, then P has a free direct summand.
- Therefore,

$$P \approx P_0 \oplus A^{r-d} \quad with \quad rank(P_0) = d.$$

Polynomial rings

- $\mathbb{R}^n$ is contractible. So, vector bundles over $\mathbb{R}^n$ are trivial.
- So, J.-P. Serre conjectured ([Serre1955]) the same for polynomial rings.
- Independently, Quillen and Suslin proved the conjecture:

Polynomial rings

Theorem ([Quillen1976], [Suslin1976])

Let $A = k[X_1, \ldots, X_n]$ be a polynomial ring over a field k. Then, finitely generated projective $A-$modules P are free.

Topological Obstructions

- In topology, there is a classical Obstruction theory (see [Steenrod1951]).
- Suppose M is a real smooth manifold with $\dim M = d \geq 2$ and $\mathcal{L}$ is a line bundle over M. Then, there are obstruction groups

$$\mathcal{H}^n(M, \mathcal{L}) \approx \mathcal{H}^n(M, \mathcal{L}^*) \qquad 0 \leq n \leq d.$$

- If $\mathcal{L}$ is trivial (the orientable case), these groups turn out to be the singular cohomology groups $H^n(M, \mathbb{Z})$. In the non-orientable case, they are the cohomology group $H^n(M, \mathcal{G}_{\mathcal{L}})$, with local coefficients in a bundle of groups.

Topological Obstructions

- For a vector bundle $\mathcal{E}$ on M with rank $r \leq d$, there is an invariant
$$w(\mathcal{E}) \in \mathcal{H}^r\left(M, \wedge^r \mathcal{E}\right).$$
- If $\mathcal{E}$ has a never-vanishing section, then $w(\mathcal{E}) = 0$.
- For rank $r = d$, conversely,
$$w(\mathcal{E}) = 0 \Longrightarrow \quad \mathcal{E} = \mathcal{F} \oplus \mathcal{R}.$$

Algebraic Obstructions

Obstruction theory in algebra is a more recent development, first outlined by M. V. Nori.

- Suppose A is a noetherian commutative ring with $\dim A = d \geq 2$ and L is a rank one projective $A-$module.
- Then, there is an obstruction group $E^d(A, L)$.
- ([BhatSri]) Given a projective $A-$module P of rank d, there is an obstruction class

$$e(P) \in E\left(A, \wedge^d P\right) \quad such \quad that$$

$$e(P) = 0 \Longleftrightarrow P = Q \oplus A.$$

Algebra and topology

Let $A = \frac{\mathbb{R}[X_1, X_2, \ldots, X_n]}{I} = \mathbb{R}[x_1, x_2, \ldots, x_n]$, where I is an ideal of the polynomial ring $\mathbb{R}[X_1, X_2, \ldots, X_n]$. Let M be the set of points $v \in \mathbb{R}^n$ such that $f(v) = 0$ for all $f \in I$.

- There are two types of maximal ideals m of A. If $A/m \approx \mathbb{C}$ then m is called a complex maximal ideal (or point).

Algebra and topology

- If $\mathbb{R} \xrightarrow{\sim} A/m$, then m is called a real maximal ideal (or point). In this case, $m = (x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n)$.

  $$m \longleftrightarrow (a_1, \ldots, a_n) \in M \quad \textit{is an } 1-1 \textit{ correspondence}$$

  between real maximal ideals of A and the points in M.
- **If A is smooth, then $M \subseteq \mathbb{R}^n$ is a smooth maifold.** Also $\dim M = \dim A$. (Implicit function theorem.)

Algebra and topology

Theorem (Mandal and Sheu): Let $A = \mathbb{R}[x_1, x_2, \ldots, x_n]$ be a smooth algebra over $\mathbb{R}$ and let $M \subseteq \mathbb{R}^n$ be the real manifold, as above. Let $\dim A = \dim M = d \geq 2$ and L be a rank one projective $A-$module and $\mathcal{L}$ be the corresponding line bundle over M.

- Then, there is a canonical homomorphism

$$\epsilon : E(A, L) \to \mathcal{H}^d\left(M, \mathcal{L}^*\right).$$

- For a projective $A-$module P of rank d, we have

$$\epsilon(e(P)) = w(\mathcal{E}^*) \quad \textit{where} \quad \mathcal{E} \quad \textit{is} \quad \textit{the vector bundle}$$

on M with the module of sections $= P \otimes C(M)$.

- The homomorphism ϵ, factors through an isomorphism

$$E(S^{-1}A, S^{-1}L) \xrightarrow{\sim} \mathcal{H}^d(M, \mathcal{L}^*) \quad \textit{where} \quad S \quad \textit{is}$$

the set of functions $f \in A$ never vanishing on M.

- **Remark:** In $S^{-1}A$, all the complex maximal ideals of A are killed. So, as sets $Max(S^{-1}A) = M$.

More Groups

1. (With Yong Yang) We were able to define
$$E^r(A, L) \quad for \quad 0 \leq r \leq d, \quad with \quad a$$
$$multiplicative\ structure\ on \quad \oplus_{r=0}^{d} E^r(A, A).$$
2. For a projective $A-$module P of rank r we were able to define an obstruction homomorphism:
$$w(P) : E^{d-r}(A, L) \to E^d(A, L \otimes (\wedge^r P)).$$
3. (Question) How to define a cannonical homomorphism
$$E^r(A, L) \to \mathcal{H}^r(M, \mathcal{L}^*)?$$

Definitions

- Let $A = \mathbb{R}[x_1, x_2, \ldots, x_n]$ be a smooth algebra and $\dim A = d \geq 2$. Let L be a rank one projective $A-$module.
- We will give a definition of the Euler class group $E^d(A, L)$.
- Let $\mathcal{G}(L)$ be the free abelian group generated by the set of pairs (m, ω), where m runs through all maximal ideals of A and $\omega : L/mL \xrightarrow{\sim} \wedge^d m/m^2$ is an isomorphism.

Definitions

- Let $I = m_1 \cap \cdots \cap m_r$ be an intersection of finitely many maximum ideals.
- An isomorphism $\omega_I : L/IL \xrightarrow{\sim} \wedge^d I/I^2$ is called a local $L-$orientation on I.
- Such a local orientation is called a Global $L-$orientation, if it is induced by a surjective homomorphism

$$\Omega : L \bigoplus A^{d-1} \twoheadrightarrow I.$$

Definitions

- As above $I = m_1 \cap \cdots \cap m_r, \quad \omega_I : L/IL \xrightarrow{\sim} \wedge^d I/I^2$. Then ω_I induces local orientations $\omega_i : L/m_i L \xrightarrow{\sim} \wedge^d m_i/m_i^2$.
- To such local orientations ω_I we associate

$$(I, \omega_I) := \sum (m_i, \omega_i) \in \mathcal{G}(L).$$

- Let $\mathcal{R}(L)$ be the subgroup of $\mathcal{G}(L)$ generated by (I, ω_I), such that ω_I is global.
- $$\textit{Define} \qquad E^d(A, L) = \frac{\mathcal{G}(L)}{\mathcal{R}(L)}.$$

Orientable case

- Let $A = \mathbb{R}[x_1, x_2, \ldots, x_n]$ be **oriented** smooth algebra over $\mathbb{R}$. Then the manifold M is orientable. We assume $\dim A = \dim M = d \geq 2$.
- Let $C_1, \ldots, C_r$ be the compact connected components of M. Then, the topological obstruction group $\mathcal{H}^d(M, \mathcal{R}) = H^d(M, \mathbb{Z}) = \oplus_{i=1}^r H^d(C_i) = \mathbb{Z}^r$.
- We will define a homomorphism $\epsilon_0 : \mathcal{G}(A) \longrightarrow H^d(M, \mathbb{Z})$ and check that it factors through $E^d(A, A) = \frac{\mathcal{G}(A)}{\mathcal{R}(A)}$.

Orientable case

- Suppose (m, ω) is a generator of $\mathcal{G}(A)$, where m is a maximal ideal of A and $\omega : A/m \xrightarrow{\sim} \wedge^d m/m^2$.
- If m is a complex maximal ideal, define $\epsilon_0(m, \omega) = 0$.
- Let m be a real maximal ideal and $v \in M$ be the corresponding real point. If
  $v \in M \setminus \cup C_i, \qquad \textit{define} \qquad \epsilon_0(m, \omega) = 0.$

Orientable case

- Suppose $v \in C_i$. Then $\omega : A/m \xrightarrow{\sim} \wedge^d m/m^2$ is given by a generator $\bar{f}_1 \wedge \bar{f}_2 \wedge \cdots \wedge \bar{f}_d$ of $\wedge^d m/m^2$, where $f_1, \ldots, f_d \in A$ and $(f_1, \ldots, f_d)$ has an isolated zero at v. Define $\epsilon_0(m, \omega) =$

$$index(f_1, \ldots, f_d) \in \mathbb{Z} = H^d\left(C_i, \mathbb{Z}\right) \subseteq H^d\left(M, \mathbb{Z}\right).$$

- **Remark:** The index is well defined because M is orientable.

Orientable case

- Above associations defines a homomorphism $\epsilon_0 : \mathcal{G}(A) \longrightarrow H^n(M, \mathbb{Z})$.
- Suppose $I = m_1 \cap \cdots \cap m_r$ is an intersection of (real) maximal ideals and ω_I is a global orientation. Then, $\epsilon_0(I, \omega_I)$ is the topological Euler class of the trivial bundle of rank d. So, $\epsilon_0(I, \omega_I) = 0$.
- So, ϵ_0 factors through a homomorphism

$$\epsilon : E^d(A, A) = \frac{\mathcal{G}(A)}{\mathcal{R}(A)} \longrightarrow H^d(M, \mathbb{Z}). \qquad \blacksquare$$

Non-Orientable case

The definition of the homomorphism is similar in the non-orientable case. The index is defined only "modulo 2".

On the real Sphere $\mathbb{S}^n$

- As before, let

$$A_n = \frac{\mathbb{R}[X_0, \ldots, X_n]}{(X_0^2 + \cdots + X_n^2 - 1)} = \mathbb{R}[x_0, x_1, \ldots, x_n]$$

be the algebraic coordinate ring of $\mathbb{S}^n$ with $n \geq 2$.
- All rank one projective A_n-modules are free. So, there is only one group $E(A_n, A_n)$.
- We have $E^n(A_n, A_n) = \mathbb{Z}$.

On the real Sphere $\mathbb{S}^n$

- As before let T_n be defined by the exact sequence

$$0 \longrightarrow T_n \longrightarrow A_n^{n+1} \xrightarrow{(x_0,\ldots,x_n)} A_n \longrightarrow 0.$$

- If n is odd, then $e(T_n) = 0$.
- If n is even, then $e(T_n) = \pm 2$. *This is a fully algebraic proof that T_n does not have a free direct summand. This result corresponds to the topological result that the tangent bundle on an even dimensional sphere, does not have a no-where vanishing section.*

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