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ON A QUESTION OF NORI : THE LOCAL CASE

S. MANDAL

UNIVERSITY OF KANSAS, LAWRENCE, KS 66045
E-mail address: mandal@math.ukans.edu

P.L.N. VARMA

MEHTA RESEARCH INSTITUTE, 10, KASTURBA GANDHI MARG, (OLD KUTCHERY ROAD) ALLAHABAD 211 002, INDIA
E-mail address: plnv@mri.ernet.in

ABSTRACT. In this paper we consider an algebraic problem which was motivated by a topological problem posed by Nori, about the homotopy of sections of projective modules. We give an affirmative answer in the case of some local rings, namely when the ring is a powerseries ring $k[[X_1, \cdots, X_n]]$ over a field k or when the ring is a regular k-spot over an infinite perfect field k.

INTRODUCTION

The following is the algebraic analogue of a problem of Nori :

Suppose $X = \text{Spec}A$ is a smooth affine variety of dim n. Let $\mathcal{P}$ be a projective A-module of rank r and $S : \mathcal{P} \to I$ be a surjective homomorphism from $\mathcal{P}$ onto an ideal I of A. Assume that the zero set of I, $V(I) = Y$ is a smooth affine subvariety of dim $n - r$. Also suppose that $Z = V(J)$ is a smooth closed subvariety of $X \times \mathbb{A}^1$ $= \text{Spec}(A[T])$; where T is a variable, such that Z intersects $X \times 0$ transversally in $Y \times 0$. Also suppose that $\varphi : \mathcal{P}[T] \to J/J^2$ (where $\mathcal{P}[T]$ denotes the tensor product $\mathcal{P} \otimes A[T]$) is a surjective map which is compatible with S. The question of Nori is whether there is a surjective map $\psi : \mathcal{P}[T] \to J$ such that $\psi|_{T=0} = S$ and $\psi|_Z = \varphi$? (See the appendix of [M]).

In [M], Mandal answers this affirmatively for the affine algebras, in the following two cases :

1. rank $\mathcal{P} \geq \dim A[T]/J + 2$ and J contains a monic polynomial.

2. $J = IA[T]$ is a local complete intersection ideal of height > 2 and I/I^2 is free.

In fact, Mandal proves the following two theorems :

Theorem A. *(See Theorem 2.1 of [M]).*

Let $R = A[T]$ be a polynomial ring over a commutative noetherian ring A and let I be an ideal of R that contains a monic polynomial. Suppose that $\mathcal{P}$ is a projective A-module of rank $\mathcal{P} = r \geq \dim R/I + 2$ and suppose that $S : \mathcal{P} \to I_0$ is a surjective map, where I_0 is the ideal $\{f(0) : f(T) \in I\}$. Also suppose that $\varphi : \mathcal{P}[T] \to I/I^2$ is a surjective map such that $\varphi(0) \equiv S \mod I_0^2$. Then there is a surjective map $\psi : \mathcal{P}[T] \to I$ such that ψ lifts φ and $\psi(0) = S$.

Theorem B. *(See Theorem 2.3 of [M]).*

Let $R = A[T]$ be a polynomial ring over an affine algebra A over a field k and let I_0 be a smooth and locally complete intersection ideal of height $r > 2$ in A with I_0/I_0^2 free. Write $I = I_0 R$ and suppose $\mathcal{P}$ is a projective A-module of rank r = height I_0. Let $S : \mathcal{P} \to I_0$ be a surjective map and let $\varphi : \mathcal{P}[T] \to I/I^2$ be a surjective map such that $\varphi(0) \equiv S \mod I_0^2$. Then there is a surjective map $\psi : \mathcal{P}[T] \to I$ such that $\psi(0) = S$ and ψ lifts φ.

Moreover, he asks if the condition on I that it contains a monic can be omitted? Our investigations started in this direction.

In view of Bhatwadekar's example (see [B]) of height 2 maximal ideal in $R[T]$, where R is a two dimensional normal ring, but not regular, which is not a complete intersection, we assume that R is a regular ring.

When R is a local ring the problem can be stated as follows :

Problem. *Suppose R is regular local ring, I is an ideal of $A = R[T]$. Assume that*

1. there exist $f_1, \cdots, f_r \in I$ such that $I = (f_1, \cdots, f_r) + I^2$

2. there exist $a_1, \cdots, a_r \in R$ such that $I(0) := \{f(0) : f(T) \in I\} = (a_1, \cdots, a_r)R$

3. $f_i(0) \equiv a_i \mod I(0)^2$, $i = 1, \cdots, r$

Then can we find polynomials $F_1, \cdots, F_r \in R[T]$ such that

4. $I = (F_1, \cdots, F_r)R[T]$

5. $F_i \equiv f_i \mod I^2$.

6. $F_i(0) = a_i$, $i = 1, \cdots, r$? □

In this paper, we give affirmative answer to this problem in the following two cases:

a) when R is a powerseries ring $k[[X_1, \cdots, X_n]]$ over a field k.

b) when R is a regular k-spot over an infinite perfect field k.

The Results

In this section we discuss our main results on the problem stated in the Introduction. Below we recall a definition and its easy consequence.

Definition : Let R be a ring and R_1 be a subring of R. Let $\alpha \neq 0$ be an element of R_1 such that α is not a zero divisor in R. We say $R_1 \subset R$ is an *analytic isomorphism along* α if $R_1/\alpha R_1 \simeq R/\alpha R$.

One can see easily that the above condition is equivalent to $R = R_1 + \alpha R$ and $\alpha R \cap R_1 = \alpha R_1$. Also observe that if $R_1 \subset R$ is an analytic isomorphism along α then it is also an analytic isomorphism along α^n for any positive integer n and $R_1[T] \subset R[T]$ is also an analytic isomorphism along α. Below we state a proposition whose proof is easy, for e.g. see [N].

Proposition 1. *Let R be a ring and R_1 be a subring of R. If I is an ideal in R, let $I_1 = I \cap R_1$. Suppose that $R_1 \subset R$ is an analytic isomorphism along α for some $\alpha \in I_1$. Then*
1. $R_1/I_1 \simeq R/I$.
2. $I = I_1 R$
3. *I_1/I_1^2 and I/I^2 are isomorphic as R_1/I_1 or R/I- modules.* □

Now we prove the result which deals with the ring of powerseries.

Theorem 2. *Let k be a field and $R = k[[X_1, \cdots, X_n]]$ be the powerseries ring over k in n variables. Let $A = R[T]$ and I be an ideal of height ≥ 2 in A and let r be an integer such that $r \geq \dim A/I + 2$. Suppose that there exist polynomials $f_1, \cdots, f_r$ in A such that $I = (f_1, \cdots, f_r) + I^2$ and $a_1, \cdots, a_r$ in R such that $I(0) = (a_1, \cdots, a_r)R$ and $f_i(0) \equiv a_i \mod I(0)^2$ for $i = 1, \cdots, r$. Then we can find polynomials $F_1, \cdots, F_r \in R[T]$ such that*
1. $I = (F_1, \cdots, F_r)$
2. $F_i \equiv f_i \mod I^2$
3. $F_i(0) = a_i$ *for* $i = 1, \cdots, r$.

Proof. Look at the ideal $I \cap R \subseteq R$. As ht $I \cap R \geq 1$, there exists a non-zero non-unit, say a_0, in $I \cap R$. We can write, after a change of variables, if necessary, $a_i = u_i \alpha_i$ $(i = 0, 1, \cdots, r)$ where u_i is a unit in R and α_i is a Weierstrass polynomial in X_n. Replacing f_i by $u_i^{-1} f_i$ and a_i by $u_i^{-1} a_i$, we can assume that a_i is a Weierstrass polynomial in X_n for all i and hence $a_0, a_1, \cdots, a_r \in k[[X_1, \cdots, X_{n-1}]][X_n]$. Also $f := a_0 a_1 \in I \cap R$ is a Weierstrass polynomial in X_n.

Let $R_1 = k[[X_1, \cdots, X_{n-1}]]$ and look at the ideal $I_1 = I \cap R_1[X_n, T]$. As $I_1 \cap R_1[X_n]$ contains a Weierstrass polynomial, namely, f, we get that $R_1[X_n] \hookrightarrow R$ is an analytic isomorphism along f (see [ZS]), and hence along f^2 also. Moreover $R_1[X_n, T] \hookrightarrow R[T]$ is an analytic isomorphism along f. Then, by Proposition 1 above, we have
1. $I = I_1 R$
2. $R_1[X_n, T]/I_1 \simeq R[T]/I$
3. $I_1/I_1^2 \simeq I/I^2$.

As I_1/I_1^2 and I/I^2 are isomorphic, we can get some $g_1, \cdots, g_r \in I_1$ such that g_i corresponds to f_i under this isomorphism. Thus $g_i - f_i \in I^2$. Hence $g_i(0) - f_i(0) \in I(0)^2$, i.e., $g_i(0) \equiv a_i \mod I(0)^2$.

Claim : $I_1(0) = (a_1, \cdots, a_r)R_1[X_n]$.

To see this, let $a \in I_1(0)$. Then $a = \sum_{i=1}^r \alpha_i a_i$ for some $\alpha_i \in R$. Writing α_i as $\alpha_i = r_i + q_i f$ with $r_i \in R_1[X_n]$, $q_i \in R$, we have $a = \sum_{i=1}^r r_i a_i + \sum_{i=1}^r q_i f a_i = \sum r_i a_i + f \sum q_i a_i$. This implies $a - \sum r_i a_i \in fR \cap R_1[X_n] = fR_1[X_n]$. Hence $a \in \sum a_i R_1[X_n]$, since $f \in (a_1)$.

Now look at the linear change of variables given by $X_n \mapsto X_n + T$ and $T \mapsto T$ on $R_1[X_n, T]$. Note that $f(X_n + T)$ is a monic in T in $R_1[X_n, T]$.

Now applying Mandal's theorem (Theorem 2.1, [M]) to $I_1 \subseteq R_1[X_n, T]$ we get $G_1, \cdots, G_r$ such that

$I_1 = (G_1, \cdots, G_r)$

$G_i \equiv g_i \mod I_1^2$, $i = 1, \cdots, r$

$G_i(0) = a_i$.

Hence $G_i \equiv f_i \mod I^2$ and $I = (G_1, \cdots, G_r)$. □

Theorem 3. *Let k be a field and $f(X_1) \in k[X_1]$ be a monic irreducible polynomial such that $D := k[X_1]_{f(X_1)}$ has infinite residue field. Let $R = k[X_1, \cdots, X_d]_M$ where M is the maximal ideal $(f(X_1), X_2, \cdots, X_d)$. Let I be an ideal of $A := R[T]$ of height ≥ 4 and let r be an integer such that $r \geq \dim A/I + 2$. Suppose that we are given $f_1, \cdots, f_r \in A$ such that $I = (f_1, \cdots, f_r) + I^2$ and $a_1, \cdots, a_r \in R$ such that $I(0) = (a_1, \cdots, a_r)R$. Moreover assume that $f_i(0) \equiv a_i \mod I(0)^2$ for $i = 1, \cdots, r$. Then we can find $F_1, \cdots, F_r \in I$ such that*

1. $I = (F_1, \cdots, F_r)$

2. $F_i \equiv f_i \mod I^2$

3. $F_i(0) = a_i$.

Proof. Let $I_1 = I \cap R$. First observe that after multiplying with units, if necessary, we can assume that $a_1, \cdots, a_r \in k[X_1, \cdots, X_d]$.

Write R as $D[X_2, \cdots, X_d]_{(\pi, X_2, \cdots, X_d)}$ where $D = k[X_1]_{(f(X_1))}$ is a d.v.r. with uniformising parameter $\pi = f(X_1)$. As ht $I_1 =$ ht $I \cap R \geq 3$, I_1 contains a form, say F, which represents a unit in D (see [N]). Let $B_1 = D_1[X_2]$ where $D_1 = D[X_3, \cdots, X_d]_{(\pi, X_3, \cdots, X_d)}$. Then there exists a monic polynomial, say, g in $B_1 \cap FR$ such that $B_1 \hookrightarrow R$ is an analytic isomorphism along g (see [L],[N]). Also note that $g \in I_1$.

Thus $B_1 = D_1[X_2] \hookrightarrow R = D[X_2, \cdots, X_d]_{(\pi, X_2, \cdots, X_d)}$ is an analytic isomorphism along g where g is a monic (in X_2) in $B_1 \cap FB_1$. As $g \in I_1$, $g \in I_1 \cap B_1$. After the linear change of variables $X_2 \mapsto X_2 + T$ and $T \mapsto T$ on $D_1[X_2, T]$, g, which is a monic in the variable X_2 in I_1, becomes a monic in the variable T and $g \in I' := I \cap B_1[T]$. Now as I/I^2 and I'/I'^2 are isomorphic we can construct $g_1, \cdots, g_r$ in $B_1[T] = D_1[X_2, T]$ such that f_i corresponds to g_i under the above isomorphism. Therefore $g_i - f_i \in I^2$ and hence $g_i(0) - f_i(0) \in I(0)^2$, i.e., $g_i(0) \equiv a_i \mod I(0)^2$. Recall that $I(0)$ is generated by $\{a_1, \cdots, a_r\}$ in R.

Claim : $I_1(0)$ is generated by $\{a_1, \cdots, a_r\}$ in B_1.

Proof of this claim is same as the proof of the claim in Theorem 2 and hence we omit the proof.

Thus we have an ideal I' in $B_1[T] = D_1[X_2, T]$ containing a monic polynomial, g, in T and $g_1, \cdots, g_r \in B_1[T]$ such that $I' = \sum_{i=1}^r B_1[T]g_i + I^2$, $I'(0) = (a_1, \cdots, a_r)$ in B_1 and $g_i(0) \equiv a_i \mod I(0)^2$. Applying Mandal's theorem (2.1 of [M]) in the above situation we get $G_1, \cdots, G_r$ in $B_1[T]$ such that $I' = (G_1, \cdots, G_r)$, $G_i \equiv g_i \mod I'^2$ and $G_i(0) = a_i$. Hence $G_i \equiv f_i \mod I^2$ and $I = (G_1, \cdots, G_r)$. □

Now we state a theorem which is a consequence of the above theorem. First we recall a definition.

By a *regular spot* over a field k, we mean a localization R_p of a finitely generated k-algebra R at a regular prime $p \in$ Spce R, i.e., R_p is a regular local ring.

Theorem 4. *Let A be a regular k-spot where k is an infinite perfect field. Let $I \subseteq A[T]$ be an ideal of height ≥ 4 and let r be an integer such that $r \geq \dim A/I + 2$. Assume that*
1. *$I = (f_1, \cdots, f_r) + I^2$ for some $f_1, \cdots, f_r \in A[T]$*
2. *$I(0) = (a_1, \cdots, a_r)$ for some $a_1, \cdots, a_r \in A$*
3. *$f(0) \equiv a_i \mod I(0)^2$ for $i = 1, \cdots, r$*
4. *Moreover, $I(0)$ is a complete intersection of height r or $I(0) = A$.*
   Then there exist $F_1, \cdots, F_r \in A[T]$ satisfying
5. *$I = (F_1, \cdots, F_r)$*
6. *$F_i \equiv f_i \mod I^2$ and*
7. *$F_i(0) = a_i$ for $i = 1, \cdots, r$.*

Proof. Since k is perfect, by ([BR], Proposition), there exists a field $K \supset k$ and a regular K-spot A_0 such that $A_0 = K[X_1, \cdots, X_d]_{(f(X_1), X_2, \cdots, X_d)}$ and $A_0 \hookrightarrow A$ is an analytic isomorphism along h_0 for some $h_0 \in I \cap A_0$. Put $h = h_0^2$. Let $I_1 = I \cap A_0[T]$. Then note that $h \in I_1(0)^2$. Moreover, we have
1. $I = I_1 A[T]$
2. $A_0[T]/I_1 \simeq A_1[T]/I$
3. $I_1/I_1^2 \simeq I/I^2$
4. $I(0) = I_1(0)A$
5. $A_0/I_1(0) \simeq A/I(0)$
6. $I(0) \cap A_0 = I_1(0)$
7. $I_1(0)/I_1(0)^2 \simeq I(0)/I(0)^2$
8. $I(0)^2 \cap A_0 = I_1(0)^2$.

Observe that (1) - (3) follow from the properties of analytic isomorphism and (4) follows from (1).

(5). To prove $A_0/I_1(0)$ and $A/I(0)$ are isomorphic, first observe that the natural map $A_0/I_1(0) \to A/I(0)$ is surjective. To prove injectivity, let $x \in A_0 \cap I(0)$. Then there exists $f \in I$ such that $f(0) = x$. Write $f = f(T) = x + a_1 T + \cdots + a_t T^t$, for some $a_i \in A$ for $i > 0$. Then a_i can be written as $a_i = b_i + \lambda_i h$ for some $b_i \in A_0, \lambda_i \in A$. Then $f(T) = (x + b_1 T + \cdots + b_t T^t) + h(x + \lambda_1 T + \cdots + \lambda_t T^t) = g(T) + h g_1(T)$ where $g(T) \in I_1$ and $g_1(T) \in A[T]$. This implies that $x = g(0) \in I_1(0)$.

(6) is clear.

(7). To prove that $I_1(0)/I_1(0)^2$ is isomorphic to $I(0)/I(0)^2$ first note that the natural map $I_1(0)/I_1(0)^2 \to I(0)/I(0)^2$ is surjective. To prove the injectivity, let $x \in I_1(0) \cap I(0)^2$. Then $x = \sum_{i=1}^m x_i y_i = \sum (x_i^0 + \lambda_i h)(y_i^0 + \mu_i h) = \sum x_i^0 y_i^0 + h\delta$ implies $x - \sum x_i^0 y_i^0 \in hA \cap A_0 = hA_0 \subseteq I_1(0)^2$ and hence $x \in I_1(0)^2$.

(8). To prove $I(0)^2 \cap A_0 = I_1(0)^2$.

Clearly we have $I_1(0)^2 \subseteq I(0)^2 \cap A_0$. Let $x \in I(0)^2 \cap A_0$. Writing $x = \sum_{i=1}^m x_i y_i = \sum (x_i' + \lambda_i h)(y_i' + \mu_i h) = \sum x_i' y_i' + h\lambda$ where $\lambda \in A$ and $\sum x_i' y_i' \in I_1(0)^2$ we get $x - \sum x_i' y_i' \in hA \cap A_0 = hA_0 \subseteq I(0)^2$ and hence $x \in I_1(0)^2$.

Let $b_i \in I_1(0)$ be such that $b_i \equiv a_i \mod I(0)^2$. Then there is a matrix, say α, in $M_r(A)$ such that $\begin{pmatrix} b_1 \\ \vdots \\ b_r \end{pmatrix} = \alpha \begin{pmatrix} a_1 \\ \vdots \\ a_r \end{pmatrix}$. Since $I(0)/I(0)^2$ is free of rank r, $\det\alpha$ is equivalent to a unit modulo $I(0)$ and hence $\det\alpha$ is a unit in A itself. So, $\alpha \in GL_r(A)$. Let $\begin{pmatrix} F_1 \\ \vdots \\ F_r \end{pmatrix} = \alpha \begin{pmatrix} f_1 \\ \vdots \\ f_r \end{pmatrix}$. Then $\begin{pmatrix} F_1(0) \\ \vdots \\ F_r(0) \end{pmatrix} = \alpha \begin{pmatrix} f_1(0) \\ \vdots \\ f_r(0) \end{pmatrix} \equiv \alpha \begin{pmatrix} a_1 \\ \vdots \\ a_r \end{pmatrix} \equiv \begin{pmatrix} b_1 \\ \vdots \\ b_r \end{pmatrix} \mod I(0)^2$. Now let $H_i \in I_1$ be such that $H_i - F_i \in I^2$. Then $H_i(0) - b_i = (H_i(0) - F_i(0)) + (F_i(0) - b_i) \in I(0)^2 \cap A_0 = I_1(0)^2$. Now observe that

1) $I = (H_1, \cdots, H_r) + I^2$
2) $I_1 = (H_1, \cdots, H_r) + I_1^2$
3) $I_1(0) = (b_1, \cdots, b_r)$
4) $H_i(0) \equiv b_i \mod I_1(0)^2$ by (8) above.

Now applying Theorem 3 above, to $I_1 \subseteq A_0[T]$ we get $G_1, \cdots, G_r \in I_1$ such that

5) $I_1 = (G_1, \cdots, G_r)A_0[T]$
6) $G_i \equiv H_i \mod I_1^2$
7) $G_i(0) = b_i$

Now write $\begin{pmatrix} G_1^* \\ \vdots \\ G_r^* \end{pmatrix} = \alpha^{-1} \begin{pmatrix} G_1 \\ \vdots \\ G_r \end{pmatrix}$. Then since $I = (G_1, \cdots, G_r)A[T]$, we have $I = (G_1^*, \cdots, G_r^*)A[T]$. Also $\begin{pmatrix} G_1^*(0) \\ \vdots \\ G_r^*(0) \end{pmatrix} = \alpha^{-1} \begin{pmatrix} G_1(0) \\ \vdots \\ G_r(0) \end{pmatrix} = \alpha^{-1} \begin{pmatrix} b_1 \\ \vdots \\ b_r \end{pmatrix} = \begin{pmatrix} a_1 \\ \vdots \\ a_r \end{pmatrix}$. Further $\begin{pmatrix} G_1^* \\ \vdots \\ G_r^* \end{pmatrix} = \alpha^{-1} \begin{pmatrix} G_1 \\ \vdots \\ G_r \end{pmatrix} \equiv \alpha^{-1} \begin{pmatrix} H_1 \\ \vdots \\ H_r \end{pmatrix} \equiv \alpha^{-1} \begin{pmatrix} F_1 \\ \vdots \\ F_r \end{pmatrix} = \begin{pmatrix} f_1 \\ \vdots \\ f_r \end{pmatrix}$. □

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