Projective modules over real smooth affine varieties

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1 Abstract

We consider the following question:

Let $X = Spec(A)$ be a smooth affine variety of dimension $n \geq 2$ over $\mathbb{R}$ (the field of real numbers) and P a projective $A-$module of rank n. Under what further restrictions, does

$$C_n(P) = 0 \Longrightarrow P \simeq A \oplus Q?$$

We answer this question completely. We show that, in some cases, additional topological obstruction does exist.

This is a joint work with S. M. Bhatwadekar and Mrinal Kanti Das.

2 Background

Swan did a lot of thinking on the the relationship between topological vector bundles and algebraic vector bundles. Let me start with the following theorem of Swan.

Theorem 2.1 (Swan) Let X be a compact connected Hausdorff space.

1. Let

   $$C(X) = \{f : f : X \to \mathbb{R} \; is \; continuous \; function\}$$

   be the ring of continuous functions on X.

2. Let $\pi : E \to X$ be a real vector bundle E on X. Let

   $$\Gamma(E) = \{s : s : X \to E \; is \; a \; section \; of \; E\}.$$

   Note $\Gamma(E)$ is a projective $C(X)-$module.

Then the association

$$E \to \Gamma(E)$$

is an equivalence of catagories. *The compactness and connectedness conditions can be relaxed.*

Theorem 2.2 Suppose $\widetilde{X} = Spec(A)$ is a real affine variety and $X = \widetilde{X}(\mathbb{R})$ is the topological space of real points in $\widetilde{X}$. Then

1. there is a natural ring homomorphism $A \to C(X)$
2. Let $\mathcal{P}(A)$ and $\mathcal{P}(C(X))$ be catagories of finitely generated projective modules over respective rings.
3. By tensor product there is a natural functor
$$\mathcal{P}(A) \to \mathcal{P}(C(X))$$
4. Let $K_0(A)$ and $K^{top}(X)$ denote the Grothendieck group of the catagory $\mathcal{P}(A)$ and $\mathcal{P}(C(X))$, respectively. The above functor induces a natural map
$$K_0(A) \to K^{top}(X)$$

The following is another theorem of Swan.

Theorem 2.3 (Swan) Let S^n denote real $n-$sphere and

$$A_n = \mathbb{R}[X_0, X_1, \ldots, X_n]/(X_0^2 + X_1^2 + \cdots + X_n^2 - 1)$$

be the affine coordinate ring of S^n. Then

$$K_0(A_n) \simeq K^{top}(S^n)$$

is an isomorphism.

3 Non-vanishing sections

Today, we are interested in the existance of nowhere vanishing sections and obstructions. First, let me state Serre's theorem.

Theorem 3.1 Suppose A is a commutative noetherian ring of dimension n and P is a projective $A-$module with $rank(P) > n$. Then

$$P \approx Q \oplus A,$$

for some projective $A-$module Q. In other words P has a nowhere vanishing section.

Similarly, if V is a vector bundle over a compact manifold X with $rank(V) > \dim(X)$, then V has a nowhere vanishing section.

Question. So, what happens when $rank(P) = n = \dim(A)$? Note that tangent bundle over real two sphere does not have nowhere vanishing section.

In topology, there is an obstruction theory ([MiS]) available to deal with similar quetions for vector bundles over smooth compact manifolds.

A search for such an Obstruction theory in Algebra began with the work of Mohan Kumar and Murthy ([MK2, MKM, Mu1]) on vector bundles over affine algebras over algebraically closed fields. The final theorem is the following.

Theorem 3.2 (Murthy) Suppose A is reduced affine algebra of dimension n over an algebraically closed field k. Suppose $F^n K_0(A)$ had no $(n-1)!-$torsion. Let P be a projective $A-$module of rank n. Then

$$P \approx Q \oplus A \Longleftrightarrow C_n(P) = 0$$

where $C_n(P)$ denotes the top Chern class of P.

But, for projective $A-$modules P with $rank(P) = \dim(A) = n$, vaninshing of the top Chern class $C_n(P) = 0$, is not a sufficient condition for for existance of nowhere vanishing section. **The tangent bundle over real two sphere is an example.**

At this point, M. V. Nori introduced the Euler Class Group program ([MS, BS1] around 1989). For an smooth affine variety $X = Spec(A)$ of dimension $n \geq 2$, he gave a definition of Euler class group $E(X)$ and he defined a Euler class $e(P) \in E(X)$ of vector bundles P of rank n over X with trivial determinant. He conjectured that

$$e(P) = 0 \Longleftrightarrow P = Q \oplus A.$$

I did some work on this program of Nori ([Ma1, MS, MV]) and S. M. Bhatwadekar and Raja Sridharan ([BS1]) settled the conjecture affirmatively.

4 Main Definitions

For a commutative noetherian ring A of dimension $n \geq 2$ and a line bundle L on $Spec(A)$ a more general definition of relative Euler class group $E(A, L)$ and relative weak Euler class group $E_0(A, L)$ was given by Bhatwadekar and Sridharan [BS2].

Definition 4.1 Let A be a noetherian commutative ring with $\dim A = n$ and L be line bundle. Write $F = L \oplus A^{n-1}$.

1. For an ideal I of height n, two surjective homomorphisms $\omega_1, \omega_2 : F/IF \twoheadrightarrow I/I^2$ are said to be equivalent if $\omega_1\sigma = \omega_2$ for some automorphism $\sigma \in SL(F/IF)$. An equivalence class of surjective homomorphisms $\omega : F/IF \to I/I^2$ will be called a **local $L-$orientation.**

$$\begin{tikzcd} F/IF \arrow[rr, "SL(F/IF)"] \arrow[dr, "\omega_2"'] & & F/IF \arrow[dl, "\omega_1"] \\ & I/I^2 & \end{tikzcd}$$

2. Let

$$G(A,L) = \mathbb{Z} < \{(N,\omega) : N \, primary, \ ht(N) = n\} >$$

be the free abelian group generated by the set of all pairs (N,ω) (resp. by the set of all ideals N) where N is a primary ideal of height n and ω is a local $L-$orientation of N. Similarly, let

$$G_0(A) = \mathbb{Z} < \{(N) : N \ primary, \ ht(N) = n\} > .$$

3. Let J be an ideal of height n and

$$\omega : F/IF \twoheadrightarrow J/J^2$$

be a local $L-$orientation of J and

$$J = N_1 \cap N_2 \cap \cdots \cap N_k$$

an irredundant primary decomposition of J. Then

$$(J,\omega) := \sum_{i=1}^{r} (N_i, \omega_i) \in G(A,L)$$

denotes the cycle determined by (J,ω). Also use

$$(J) := \sum_{i=1}^{r} (N_i) \in G_0(A).$$

4. A local $L-$orientation $\omega : F/IF \twoheadrightarrow I/I^2$ of an ideal I of height n is said to be a **global $L-$orientation,** if ω lifts to a surjection $\Theta : F \twoheadrightarrow I$.

$$\begin{tikzcd} F \arrow[r, two heads, "\Theta"] \arrow[d] & I \arrow[d] \\ F/IF \arrow[r, two heads, "\omega"] & I/I^2 \end{tikzcd}$$

5. Let

$$H(A, L) = Subgroup(\{(J, \omega) : it\, is\, GLOBAL\}) \subseteq G(A, L)$$

and

$$H_0(A, L) = Subgroup(\{(J) : it\, is\, GLOBAL\}) \subseteq G_0(L).$$

6. Define

$$E(A, L) := \frac{G(A, L)}{H(A, L)} \quad and \quad E_0(A, L) := \frac{G_0(A)}{H_0(A, L)}.$$

The group $E(A, L)$ is called the **Euler class group** of A (relative to L) and $E_0(A, L)$ is called the **weak Euler class group** of A (relative to L).

7. Now we assume that $\mathbb{Q} \subseteq A$. Given a projective $A-$module P with $rank(P) = n$ and an isomorphism (orientation) $\chi : L \xrightarrow{\sim} \wedge^n P$, we define **euler class** $e(P, \chi)$ as follows: Let

$$f : P \twoheadrightarrow I$$

be a surjective homomorphism, where I is an ideal of height n. Now suppose $\gamma : F/IF \to P/IP$ is an isomorphism such that $(\wedge^n \gamma) = \overline{\chi}$ where "overline" denotes "modulo I". Let $\omega = \overline{f}\gamma$.

$$\begin{tikzcd}
P \arrow[r, two heads, "f"] \arrow[d] & I \arrow[d] \\
P/IP \arrow[r, two heads, "\overline{f}"] & I/I^2 \\
F/IF \arrow[u, "\gamma", "\wr"'] \arrow[ur, "\omega"] &
\end{tikzcd}$$

Define the **Euler class of** (P, χ) as

$$e(P, \chi) = (I, \omega) \in E(A, L).$$

Also define **weak Euler class of** P as

$$e_0(P) = (I) \in E_0(A, L).$$

The final result on vanishing conjecture of Nori is the following, due to Bhatwadekar and Raja Sridharan ([BS2]).

Theorem 4.2 ([BS2]) Let A be a noetherian commutative ring of dimension $n \geq 2$, with $\mathbb{Q} \subseteq A$, and L be a line bundle on $Spec(A)$. Let P be an $A-$module of rank n and determinant L. Let $\chi : L \xrightarrow{\sim} \wedge^n P$ be an orientation. Then

$$e(P, \chi) = 0 \Longleftrightarrow P = Q \oplus A$$

for some $A-$module Q.

Bhatwadekar and Raja Sridharan ([BS2]) also proved:

Theorem 4.3 *([BS2])* Let A be a noetherian commutative ring of dimension $n \geq 2$, with $\mathbb{Q} \subseteq A$, and P be a projective $A-$module of rank n and $\det(P) = L$.

Suppose J is an ideal of height n and

$$\omega : (L \oplus A^{n-1})/J(L \oplus A^{n-1}) \twoheadrightarrow J/J^2$$

be a local $L-$orientation of J. Let $\chi : L \xrightarrow{\sim} \wedge^n P$ be an isomorphism and $e(P, \chi) = (J, \omega)$. Then there is a surjective map $\Theta : P \twoheadrightarrow J$, such that Θ and χ induces ω.

Following is also an useful theorem from [BS2].

Theorem 4.4 *([BS2])* Let $X = Spec(A)$ be a smooth affine variety of dimension $n \geq 2$ over $\mathbb{R}$. Then the canonical map

$$\Phi : E_0(A, L) \to CH_0(X)$$

is an isomorphism. In fact, we have the commutative diagram:

$$\begin{tikzcd} E_0(A,A) \arrow[rr, "\sim"] \arrow[dr, "\sim"'] & & E_0(A,L) \arrow[dl, "\sim"] \\ & CH_0(A) & \end{tikzcd} .$$

All maps here are well defined isomorphisms.

5 On Real Smooth affine Varieties

Recall, for tangent bundle T of the real two sphere S^2, **the top Chern class $C_2(T) = 0$, but T does not have a nowhere vanishing section.** Still, Bhatwadekar and Raja Sridharan posed and initiated an investigation the following question in [BS4].

Question. *Let $X = \text{Spec}(A)$ be a smooth affine variety of dimension $n \geq 2$ over $\mathbb{R}$ (the field of real numbers) and P a projective A-module of rank n.*

$$When \;\; does \;\; C_n(P) = 0 \Longrightarrow P \simeq A \oplus Q?$$

They proved ([BS4]), if n is odd and $K_A \simeq A \simeq \wedge^n(P)$ then the vanishing of the top Chern class $C_n(P)$ is sufficient to conclude that $P \simeq A \oplus Q$, where $K_A = \wedge^n(\Omega_{A/\mathbb{R}})$ denotes the canonical module of A.

This question was settled in complete generality ([BDM]) as follows.

Theorem 5.1 Let $X = \mathrm{Spec}(A)$ be a smooth affine variety of dimension $n \geq 2$ over the field $\mathbb{R}$ of real numbers. Let K denote the canonical module $\wedge^n(\Omega_{A/\mathbb{R}})$. Let P be a projective A-module of rank n and let $\wedge^n(P) = L$. Assume that $C_n(P) = 0$ in $CH_0(X)$. Then $P \simeq A \oplus Q$ in the following cases:

1. $X(\mathbb{R})$ has no compact connected component.
2. For every compact connected component C of $X(\mathbb{R})$, $L_C \not\simeq K_C$ where K_C and L_C denote restriction of (induced) line bundles on $X(\mathbb{R})$ to C.
3. n is odd.

Moreover, if n is even and L is a rank 1 projective A-module such that there exists a compact connected component C of $X(\mathbb{R})$ with the property that $L_C \simeq K_C$, then there exists a projective A-module P of rank n such that $P \oplus A \simeq L \oplus A^{n-1} \oplus A$ (hence $C_n(P) = 0$) but P does not have a free summand of rank 1.

Note that the last part says that apart from the possible nonvanishing of its top Chern class, further topologincal obstruction exists, for an algebraic vector bundle of top rank over X to split off a trivial subbundle of rank 1.

The main thrust of our proof in brief consists in showing that for a projective A-module P of rank n, $C_n(P) = 0$ implies that its Euler class $e(P, \chi)$ vanishes for some orientation χ.

To do this, we need the structure theorem for Euler class groups.

6 Structure Theorem

Theorem 6.1 Let $X = \mathrm{Spec}(A)$ be a smooth affine variety of dimension $n \geq 2$ over the field $\mathbb{R}$ of real numbers and let $K = \wedge^n(\Omega_{A/\mathbb{R}})$ be the canonical module of A. Let L be a projective A-module of rank 1. Let $C_1, \cdots, C_r, C_{r+1}, \cdots, C_t$ be the compact connected components of $X(\mathbb{R})$ in the Euclidean topology. Let K_{C_i} and L_{C_i} denote restriction of (induced) line bundles on $X(\mathbb{R})$ to C_i. Assume that

$$L_{C_i} \simeq K_{C_i} \qquad for \qquad 1 \leq i \leq r$$

and

$$L_{C_i} \not\simeq K_{C_i} \qquad for \qquad r+1 \leq i \leq t.$$

Then,

$$E(\mathbb{R}(X), L) = \mathbb{Z}^r \oplus (\mathbb{Z}/(2))^{t-r}.$$

Proof of the Main Theorem 5.1: Let $L = det(P)$. Fix an orientation $\chi : L \xrightarrow{\sim} \wedge^n P$. Let $e(P, \chi) \in E(A, L)$ be the euler class.

We have the following commutative diagram of exact sequences:

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & K_1 \arrow[r, "\sim"] \arrow[d] & K_2 \arrow[d] & \\
0 \arrow[r] & E^{\mathbb{C}}(L) \arrow[r] \arrow[d, "\varphi", "\wr"'] & E(A, L) \arrow[r] \arrow[d, two heads, "\Theta"] & E(\mathbb{R}(X), L) \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & CH(\mathbb{C}) \arrow[r] & CH_0(A) \arrow[r] & CH_0(\mathbb{R}(X)) \arrow[r] & 0
\end{tikzcd}
$$

From the sructure theorem above and the knowledge of the group $CH_0(\mathbb{R}(X))$, due to Colliot-Thélène and Schiderer ([CT-S]), it follows that

$$K_2 \simeq \mathbb{Z}^r.$$

Case 1 : $X(\mathbb{R})$ has no compact connected component: In this case $E(\mathbb{R}(X), L) = 0$ and so $\Theta : E(A, L) \to CH_0(A)$ is an isomorphism. Since $C_n(P) = 0$, we have $e(P, \chi) = 0$. By theorem 4.2, $P \approx Q \oplus A$.

Case 2 : For every compact connected component C of $X(\mathbb{R})$ we have $L_C \not\approx K_C$.:

So,$r = 0$ and $E(\mathbb{R}(X), L) = \mathbb{Z}/(2)^t$. By a THEOREM of Colliot-Thélène and Schiderer $\Theta : E(A, L) \to CH_0(A)$ is an isomorphism and theorem follows as above.

Case 3 : n odd: Let $\Delta : P \to P$ be multiplicatin by -1. Then $det(\Delta) = -1$. Let $\alpha : P \to I$ be a surjection where I is a (locally complete intersection) ideal of height n. Write $F = L \oplus A^{n-1}$. Using an isomorphism $\gamma : F/IF \xrightarrow{\sim} P/IP$, we get a $\omega : F/IF \to I/I^2$ so that

$$e(P, \chi) = (I, \omega) \in E(A, L).$$

Again $\alpha\Delta$ will induce $-\omega : F/IF \to I/I^2$ and hence

$$e(P, \chi) = (I, -\omega) \in E(A, L).$$

Therefore

$$2e(P, \chi) = (I, \omega) + (I, -\omega).$$

Also recall, $E_0(A,L) \approx CH_0(A)$. Since $C_n(P) = cycle(I) = 0$, we have class $[I] = 0 \in E_0(A,L)$. Therefore

$$2e(P,\chi) = (I,\omega) + (I,-\omega) = 0 \in E(A,L).$$

Also, $e(P,\chi) \in K_1$ the kernel of Θ. The serious part of the proof (4.29) is K_1 is torsion free. Therefore $e(P,\chi) = 0$ and theorem follows.

Case 4: n even and for some compact component $L_C \simeq K_C$: It is not a news that $\Psi_L : E_0(A, L) \simeq CH_0(X)$. Also $ker(\Theta)$ is free abelian group of rank at least one. So,

$$0 \to ker(\Theta) \to E(A, L) \to E_0(A, L) \to 0$$

is exact.

From definition of $E_0(A, L)$ it follows that $ker(\Theta)$ is generated by elements of the type (J, ω) where J is an ideal of height n and is image of $F = L \oplus A^{n-1}$.

Since $ker(\Theta) \neq 0$, we pick a generator $(I, \omega) \neq 0 \in ker(\Theta)$ such that F maps onto I.

Let $\alpha : F \to I$ be a surjective map and $\omega_0 : F/IF \to I/I^2$ be the induced orientation.

We can find f such that the diagram

$$\begin{tikzcd} F/IF \arrow[r, "\omega"] \arrow[d, "f"] & I/I^2 \arrow[d, equal] \\ F/IF \arrow[r, "\omega_0"] & I/I^2 \end{tikzcd}$$

commutes. Note that $det(f) = \bar{u}$ has to be a unit (check mod maximal ideals and use height). So, $\omega = \bar{u}\omega_0$ and

$\bar{u}\bar{v} = 1$. Let $M = ker(\alpha)$, and we have the exact sequence:

$$0 \to M \to F \xrightarrow{\alpha} I \to 0.$$

We can consider this exact sequence as an element of $z \in Ext(I, M)$. Then vz is given by the diagram

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, "v"] & F \arrow[r, "\alpha"] \arrow[d, "g"] & I \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & M \arrow[r] & P \arrow[r, "\beta"] & I \arrow[r] & 0 \end{tikzcd}$$

Therefore P is projective and $[P] = [F]$. Also

$$e(P, \chi) = (I, \omega) \neq 0$$

for some orientation $\chi : L \xrightarrow{\sim} \wedge^n P$.

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