EULER CYCLES

SATYA MANDAL

I will talk on the following three papers:

(1) A Riemann-Roch Theorem ([DM1]),
(2) Euler Class Construction ([DM2])
(3) Torsion Euler Cycle ([BDM])

In a series of papers that N. Mohan Kumar and M.P. Murthy ([MK2], [Mu1], [MKM]) wrote, the final theorem was the following.

Theorem 0.1. *Suppose A is reduced affine algebra of dimension n over an algebraically closed field k. Suppose $F^nK_0(A)$ had no $(n-1)!-$torsion. Let P be a projective $A-$module of rank n. Then $P \approx Q \oplus A$ if and only if the top Chern class $C_n(P) = 0$.*

This was the reason why Euler Class group was introduced by Nori as an obstruction group for projective modules to split off a free direct summand.

1. Main Definitions

First we define oriented projective module over noetherian affine schemes $X = spec(A)$.

Definition 1.1. *Let A be a noetherian commutative ring and L be a line bundle on $X = Spec(A)$. An $L-$oriented projective $A-$module of rank r is a pair (P, χ) where P is a projective $A-$module rank r and $\chi : L \xrightarrow{\sim} \wedge^r P$ is an isomorphism. Such an isomorphism χ will be called an $L-$orientation of P.*

The original definition of Euler class group for smooth affine algebras was given by Nori [MS], [BS1] (around 1989).

For a commutative noetherian ring A of dimension $n \geq 2$ and a line bundle L on $Spec(A)$ the definition of relative Euler class group $E(A, L)$ was given by Bhatwadekar and Sridharan [BS2].

Definition 1.2. *Let A be a noetherian commutative (Cohen-Macaulay) ring with $\dim A = n$ and L be line bundle. Write $F = L \oplus A^{n-1}$.*

(1) *For an ideal I of height n, two surjective homomorphisms $\omega_1, \omega_2 : F/IF \to I/I^2$ are said to be equivalent if $\omega_1 \sigma = \omega_2$ for some automorphism $\sigma \in SL(F/IF)$. An equivalence class of surjective homomorphisms $\omega : F/IF \to I/I^2$ will be called a* **local $L-$orientation.**

$$\begin{array}{ccc} F/IF & \xrightarrow{SL(F/IF)} & F/IF \\ & {\scriptstyle \omega_2} \searrow \quad \swarrow {\scriptstyle \omega_1} & \\ & I/I^2 & \end{array}$$

(2) *A local $L-$orientation $\omega : F/IF \to I/I^2$ of an ideal I of height n is said to be a* **global Euler $F-$orientation,** *if ω lifts to a surjection $\Theta : F \to I$.*

$$\begin{array}{ccc} F & \xrightarrow{\Theta} & I \\ \downarrow & & \downarrow \\ F/IF & \xrightarrow{\omega} & I/I^2 \end{array}$$

(3) *Let $G(A, L)$ (resp. $G_0(A)$) be the free abelian group generated by the set of all pairs (N, ω) (resp. by the set of all ideals N) where N is a primary ideal of height n and ω is a local*

$L-$orientation of N. Elements of $G(A, L)$ will be called Euler $L-$cycles.

(4) *Let J be an ideal of height n and $\omega : F/IF \to J/J^2$ be a local $L-$orientation of J. Let*

$$J = N_1 \cap N_2 \cap \cdots \cap N_k$$

be an irredundant primary decomposition of J. Then ω induces local $F-$orientations $\omega_i : F/N_iF \to N_i/N_i^2$ for $i = 1, \ldots, k$. We use the notation

$$(J, \omega) := \sum_{i=1}^{r} (N_i, \omega_i)$$

in $G(A, L)$. We say that (J, ω) is the Euler $F-$cycle determined by (J, ω).

Also use

$$(J) := \sum_{i=1}^{r} (N_i)$$

in $G_0(A)$.

(5) *Let $H(A, L)$ (resp. $H_0(A, L)$) be the subgroup of $G(A, L)$ (resp. of $G_0(A)$) generated by the set of all pairs (J, ω) (resp. by (J)) where ω is a global Euler $L-$orientation.*

(6) *Define*

$$E(A, L) := \frac{G(A, L)}{H(A, L)} \quad \textit{and} \quad E_0(A, L) := \frac{G_0(A)}{H_0(A, L)}.$$

The group $E(A, L)$ is called the **Euler class group** *of A (relative to L) and $E_0(A, L)$ is* *called the* **weak Euler class group** *of A (relative to L).*

(7) **Notation:** The image of an Euler $L-$cycle $(J,\omega) \in G(A,L)$ in $E(A,L)$ will be denoted by the same notation (J,ω) and also be called an Euler $L-$cycle. It will be clear from the context, whether we mean in $G(A,L)$ or in $E(A,L)$. Similar notations and terminologies will be use for elements in $G_0(A)$ or $E_0(A,L)$.

(8) *Let $\chi_0 : L \approx \wedge^n F$, be the obvious isomorphism. This isomorphism χ_0 will be called the* **standard orientation** *of F.*

(9) *Now we assume that* $\mathbb{Q} \subseteq A$. *Let* (P, χ) *be an* $L-$*oriented projective module over* A *with* $rank(P) = n$. So $det(P) = L$. *Let*

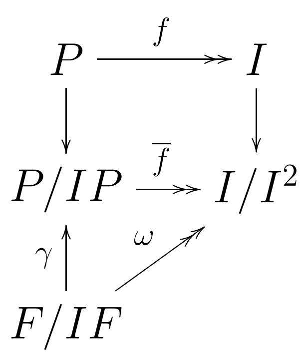
$$f : P \to I$$

be a surjective homomorphism, where I *is an ideal of height* n.

Define the **weak Euler class of** P *as*

$$e_0(P) = (I) \in E_0(A, L).$$

Now suppose $\gamma : F/IF \to P/IP$ *is an isomorphism such that* $(\wedge^n \gamma)\overline{\chi_0} = \overline{\chi}$ *where "overline" denotes "modulo* I*". Let* $\omega = \overline{f}\gamma$.



Define the **Euler class of** (P, χ) *as*

$$e(P, \chi) = (I, \omega) \in E(A, L).$$

2. Some Perspective

As I said, original definitions of Euler class groups and Euler classes were given by Nori. Nori told me about this program in his condo in Chicago in the summer of 1989. I am sure he told the same to Raja Sridharan and M. P. Murthy around the same time. Nobody clapped and there was no thumping of the desks. We worked diligently.

Euler class theory has matured. I am looking for some perspective. I have been asking how does this theory and the program compares with other existing programs that are able to draw a great deal of attention dew to the strength of publicity and money.

3. Some Background

Nori gave two conjectures. First one is called the homotopy conjecture and second one is called the vanishing conjecture. We will not state them. All these conjectures were eventually proved, at least when $\mathbb{Q}$ is in the ring.

In summer of 1989, before I left Chicago, I made the first break through and proved the following monic polynomial theorem on Homotopy conjecture.

Theorem 3.1. ([Ma1]) *Let $R = A[t]$ be a polynomial ring over a noetherian commutative ring A and let J be an ideal in R that contains a monic polynomial. Write $J_0 = \{f(0) : f \in J\}$.*

Suppose P is a projective A-module of rank $r \geq \dim R/J + 2$ and suppose

$$s : P \to J_0$$

is a surjective map. Now suppose that

$$\varphi : P[t] \to J/J^2$$

is a surjective map such that $\varphi(0) \equiv s$ modulo J_0^2.

Then there is a surjective map $\psi : P[t] \to J$ such that ψ lifts φ and $\psi(0) = s$.

With P.L.N. Varma, I proved a local case of the homotopy theorem ([MV]). I also wrote my joint paper with Raja Sridharan ([MS]). Final theorem on vanishing conjecture is due to Raja Sridharan and S. M. Bhatwadekar ([BS2]).

The final result on vanishing conjecture is the following ([BS2]).

Theorem 3.2. ([BS2]) *Let A be a noetherian commutative ring of dimension $n \geq 2$, with $\mathbb{Q} \subseteq A$, and L be a line bundle on Spec(A). Let (P, χ) be an $L-$oriented projective $A-$module. Then $e(P, \chi) = 0 \in E(A, L)$ if and only if P has a unimodular element.*

Bhatwadekar and Raja Sridharan ([BS2]) also proved the following theorem.

Theorem 3.3. *(*[BS2]*) Lat A be a noetherian commutative ring of dimension $n \geq 2$, with $\mathbb{Q} \subseteq A$, and L be a rank one projective $A-$module. Let J be an ideal of height n. Write $F = L \oplus A^{n-1}$. Suppose $\omega : F/JF \to J/J^2$ be a local $L-$orientation of J.*

Assume that $(J, \omega) = 0 \in E(A, L)$. Then ω lifts to a surjective map $\Theta : F \to J$.

4. Problems

The above theorem inspired us to pose the following problem.

Problem 4.1. *Let A be a Cohen-Macaulay ring of dimension $n \geq 2$ and J be local complete intersection ideal of height n. Suppose (J, ω) is a torsion element in $E(A, A)$ for some local $A-$orientation ω of J.*

Is J set theoretic complete intersection?

We can ask similar questions for cycles (J) in the weak Euler class group $E_0(A, A)$ and for zero cycle$(J) \in CH^n(A)$ in the Chow group.

These questions will be answered affimatively.

Another open problem we want to consider in this talk is the following.

Problem 4.2. *(*[BS1, Mu2]*) Let A be a smooth affine algebra over a field k of dimension $n \geq 2$. Let $CH^n(A)$ denote the Chow group of zero cycles. Is the natural map $E_0(A, A^n) \to CH^n(A)$ an isomorphism?*

Because of the result of Murthy [Mu1], this problem has an affirmative answer when the ground field k is algebraically closed.

The problem has an affirmative answer for smooth affine varieties over $\mathbb{R}$ ([BS2]).

We will be able to answer these questions affimatively, upto torsion.

5. Boratynski's Construction

Motivation also came from the following construction of Boratynski ([B]).

Theorem 5.1. ([B]) *Let R be any commutative ring. Let I be an ideal in R and*

$$I = (f_1, \ldots, f_{n-1}, f_n) + I^2.$$

Write

$$J = (f_1, \ldots, f_{n-1}) + I^{(n-1)!}.$$

Then J is image of a projective $R-$module P with $rank(P) = n$.

This theorem of Boratynski served as a central motivation for some of the developments in this theory and of some techniques. We introduce the following notation.

Notation 5.1. *Let I be an ideal of a ring A such that I/I^2 is generated by k elements. Suppose $I = (f_1, \cdots, f_k) + I^2$. For integers $r \geq 2$, let*

$$I^{(r)} := (f_1, \ldots, f_{k-1}) + I^r.$$

Note that $I^{(r)}$ depends on first $k-1$ generators of I/I^2 but not the last generator f_k.

We quote the following from [Ma3].

Theorem 5.2. (**[Ma3]**) *Let A and I be as above. Further, assume that A is Cohen-Macaulay.*

1. *Then I is a local complete intersection ideal of height k if and only if so is $I^{(r)}$ for any integer $r \geq 2$.*
2. *If I is local complete intersection of height n then*
$$[A/I_r] = r[A/I]$$
in $K_0(A)$.

We investigated whether we can prove similar results for cycles in Euler class groups $E(A, A)$ or weak Euler class groups $E_0(A, A)$. We had good luck with cycles in $E_0(A, A)$.

6. Euler Cycle Calculus

With Mrinal Das ([DM1]) we prove the following.

Lemma 6.1. ([DM1]) *Let A be a noetherian commutative ring of dimension n and L be line bundle on $Spec(A)$. Let J be a local complete intersection ideal of height n. Let $J = (f_1, \ldots, f_n) + J^2$ and $J^{(r)} = (f_1, \ldots, f_{n-1}) + J^r$. If $f_1, \ldots, f_n$ is a regular sequence, then the class*

$$(J^{(r)}) = r(J)$$

in $E_0(A, L)$.

Proof. I am including this proof to give a flavor of arguments involved. The proof is given by usual moving techniques.

We will write $f_n = g_1$. We can find a local complete intersection ideal K_1 of height n such that

$$J \cap K_1 = (f_1, \ldots, f_{n-1}, g_1)$$

and $J + K_1 = A$.

By induction, we can find, for $i = 1, \ldots, r$, elements $g_i \in J$ and local complete intersection ideals K_i of height n such that

(1) $J = (f_1, f_2, \ldots, f_{n-1}, g_i) + J^2$.
(2) $J \cap K_i = (f_1, f_2, \ldots, f_{n-1}, g_i)$.
(3) $J + K_i = A$ and $K_i + K_j = A$, for $i \neq j$.

It follows that

$$J^{(r)} \cap K_1 \ldots \cap K_r = (f_1, f_2, \ldots, f_{r-1}, g)$$

where

$$g = g_1 g_2 \cdots g_r.$$

Therefore,

$$(J^{(r)}) = -\sum (K_i) = r(J)$$

in $E_0(A, A)$.

Since the natural map

$$\eta_L : E_0(A, A) \to E_0(A, L)$$

is an isomorphism, the theorem follows.

We were not as lucky with Euler cycles, because lifting generators of I/I^2 and K_i/K_i^2 could not be done in a compatible manner.

But, with Bhatwadekar and Mrinal Das we can prove the following.

Proposition 6.1. ([BDM]) *Suppose A is a noetherian commutative ring of dimension n. Let I be a local complete intersection ideal of height n. Assume I/I^2 has a square generator, namely, $I = (f_1, \ldots, f_{n-1}, f_n^2) + I^2$ for some $f_1, \ldots, f_{n-1}, f_n \in A$. Let ω_I be any local $A-$orientation of I. For positive integers r, define*

$$I^{(r)} = (f_1, \ldots, f_{n-1}) + I^r.$$

Then,

$$r(I, \omega_I) = (I^{(r)}, \omega_r)$$

in $E(A, A)$ *for some* $A-$*orientation* ω_r *on* I_r*. Also the right hand side is independent of* ω_r*.*

Having a square generator f_n^2 helps. For $I^{(2)}$ we can prove the following.

Proposition 6.2. ([BDM]) *Suppose* A *is a noetherian commutative ring of dimension* n*. Let* I *be a local complete intersection ideal of height* n *and* $I = (f_1, \ldots, f_{n-1}, f_n) + I^2$ *for some* $f_1, \ldots, f_n$ *in* A*. Let* ω_I *denote the local* $A-$*orientation of* I *defined by* $f_1, \ldots, f_{n-1}, f_n$*. Define*

$$I^{(2)} = (f_1, \ldots, f_{n-1}) + I^2.$$

Then, for any local $A-$*orientation* ω_0 *on* $I^{(2)}$

$$(I^{(2)}, \omega_0) = (I, \omega) + (I, -\omega)$$

in $E(A, A)$*.*

7. Equivalence of cycles

All these follows from the following equivalance theoem ([BDM]). Mrinal was of great help.

Proposition 7.1. ([DM1]) *Suppose A is a commutative ring of dimension n and L is a line bundle on $Spec(A)$. Let J be a local complete intersection ideal of height n and $J = (f_1, \ldots, f_{n-1}, f_n^2) + J^2$. Then, for any two local $L-$orientations $\omega_1, \omega_2 : F/JF \to J/J^2$ we have*

$$(J, \omega_1) = (J, \omega_2) \in E(A, L).$$

This one works because an unimodular row (a, b, c^2) is completeable to an invertible matrix.

8. Unimodular Element Theorem

Following improves the corresponding results of Mohan Kumar ([MK2]) and Mandal ([Ma3]).

Theorem 8.1. ([DM1]) *Suppose A is a noetherian commutative ring of dimension n with $\mathbb{Q} \subseteq A$. Let $f_1, \ldots, f_{n-1}, f_n$ be a regular sequence Suppose P is a projective $A-$module of rank n and there is a surjective homomorphism*

$$\varphi : P \to (f_1, \ldots, f_{n-1}, f_n^2).$$

Then $P \approx Q \oplus A$ for some projective $A-$module Q.

Proof. Note that $(I, \omega) = 0$ in $E(A, L)$ for any local $L-$oriantation ω. So, Euler class $e(P, \chi) = 0$.

9. Time for Riemann-Roch Theorem

Further inspiration came from the following version of Boratynski's theorem, due to Murthy ([Mu1]).

Theorem 9.1. (**[Mu1]**) *Let A be a noetherian commutative ring and $I \subset A$ be a local complete intersection ideal of height r. Suppose $I = (f_1, \cdots, f_r) + I^2$ and $J = (f_1, \cdots, f_{r-1}) + I^{(r-1)!}$. Assume $f_1, \ldots, f_r$ is a regular sequence. Then there is a projective A-module P of rank r and a surjective homomorphism $P \to J$, such that* $[P] - [A^r] = -[A/I] \in K_0(A)$.

Notation 9.1. Let A be a noetherian commutative ring of dimension n and $X = Spec(A)$.

(1) $F^1K_0(A)$ will denote the kernel of the rank map $\epsilon : K_0(A) \to \mathbb{Z}$.

(2) Define
$$F^2K_0(A) = \{x \in F^1K_0(A) : det(x) = A\}.$$

(3) Define
$$F^nK_0(A) = \{[A/I] \in K_0(A) : I\, is\, a\, LCI\, ideal\, of\, height\, n\}.$$

It was established in ([Ma3]) that $F^nK_0(A)$ is a subgroup of $K_0(A)$.

9.1. **Two Homomorphisms.** With Mrinal Das we define two maps/homomorphisms.

Definition 9.1. ([DM1]) Let A be a ring of dimension n with $\mathbb{Q} \subseteq A$ Let L be a line bundle on $Spec(A)$. Write $F = L \oplus A^{n-1}$.

Define a map $\Phi_L : F^2K_0(A) \to E_0(A, L)$ as

$$\Phi_L(x) = e_0(P)$$

the weakEuler class of P, where $x \in F^2K_0(A)$ is written as $x = [P] - [F]$ for some projective $A-$module P with $rank(P) = n$ and $det(P) = L$.

Φ_L is well defined because the weak Euler class respects stable isomorphism.

We will be more concerned with the restriction map

$$\varphi_L : F^nK_0(A) \to E_0(A, L)$$

of Φ_L to $F^nK_0(A)$. Both the maps Φ_L and φ_L will be called the **weak Euler class map.** φ_L is a group homomorphism, while Φ_L is not a group homomorphism.

We also define a map in the opposite direction.

Definition 9.2. ([DM1]) *Let A be a Cohen-Macaulay ring of dimension n. Define*

$$\psi_L : E_0(A, L) \to F^nK_0(A)$$

as the natural map that sends the class (J) of an ideal J to the class $[A/J]$. Note that J is local complete intersection ideal and $[A/J] \in F^nK_0(A)$.

10. Riemann-Roch Theorem

First, we prove the following commutativity theorem ([DM1]).

Theorem 10.1. ([DM1]) *Let A be a commutative noetherian ring of dimension n and L be a line bundle on $Spec(A)$. Assume that A contains the field of rationals $\mathbb{Q}$. Then the diagram*

$$\begin{array}{ccc} F^nK_0(A) & \xrightarrow{\varphi_A} & E_0(A, A^n) \\ & \searrow^{\varphi_L} & \downarrow \eta_L \\ & & E_0(A, F) \end{array}$$

commutes where

$$\eta_L : E_0(A, A) \to E_0(A, L)$$

is the natural isomorphism.

With Mrinal Das, we prove the following theorem ([DM1]), in analogy to the Riemann-Roch theorem, without denominator, for the Chern class map.

Theorem 10.2. ([DM1]) *Let A be a Cohen-Macaulay ring of dimension n with $\mathbb{Q} \subseteq A$. Then,*

$$\varphi_L \psi_L = -(n-1)! Id_{E_0(A,L)}$$

and

$$\psi_L \varphi_L = -(n-1)! Id_{F^n K_0(A)}.$$

11. Weak Euler class group and the Chow Group

For a noetherian commutative ring A of dimension n, $CH^n(A)$ will denote the Chow group of zero cycles. There is a natural map

$$\Theta : E_0(A, A) \to CH^n(A)$$

that sends the weak Euler cycles (J) to the Chow cycle of J.

Following theorem follows from the Riemann-Roch theorems.

Theorem 11.1. ([DM1]) *Let A be a regular ring with $\mathbb{Q} \subseteq A$ and $\dim A = n$. Let $\Theta : E_0(A, A) \to CH^n(A)$ be the natural homomorphism. Then*

$$\mathbb{Q} \otimes E_0(A, A) \approx \mathbb{Q} \otimes CH^n(A).$$

This theorem answers Problem 4.2,, affirmatively, upto torsion

Proof. The proof follows from the following commutative diagram:

$$\begin{array}{ccc} F^n K_0(A) & \xrightarrow{\varphi_A} & E_0(A, A^n) \\ & \searrow^{C} & \downarrow{\scriptstyle (-1)^n \Theta} \\ & & CH^n(A) \end{array}$$

where C the top Chern class map.

12. Torsion Cycles

With Bhatwadekar and Das ([BDM]) we answer the first problem 4.1 as follows.

Theorem 12.1. *Let A be a Cohen-Macaulay ring of dimension $n \geq 2$ with $\mathbb{Q} \subseteq A$. Let J be an ideal of height n such that $\mu(J/J^2) = n$.*

Then J is set theoretically generated by n elements in all the following situations:

(1) *$[A/J] \in K_0(A)$ is torsion then J,*
(2) *the weak Euler cycle $(J) \in E_0(A, A)$ is torsion,*
(3) *A is a regular ring and the Chow cycle $[J] \in CH^n(A)$ is torsion.*

Proof. Because of the above diagram, it is enough to prove (2). Write $J = (f_1, \ldots, f_{n-1}, f_n) + J^2$ where $f_1, \ldots, f_{n-1}, f_n$ is a regualr sequence. As before, write

$$J^{(r)} = (f_1, \ldots, f_{n-1}) + J^r.$$

Note J and $J^{(r)}$ have same radical and

$$r(J) = (J^{(r)}).$$

Replacing (J) by $J^{(r)}$ we can assume $(J) = 0$.

Look at the map $\varphi : E(A, A) \to E_0(A, A)$. Let $\omega : (A/J)^n \to J/J^2$ be any local orientation. Since $\varphi(J, \omega) = 0$, by ([BS2]) we have

$$(J^{(2)}, \omega^*) = (J, \omega) + (J, -\omega) = 0.$$

Therefore $J^{(2)}$ is complete intersection.

13. Euler Class Construction

The heading is the title of the second paper [DM2] with Mirnal Das. Our main theorem in ths paper is the following construction.

Theorem 13.1. *Let A be a commutative noetherian ring of dimension n. Let L be a line bundle on Spec(A) and $F = A^{n-1} \oplus L$. Let J be a local complete intersection ideal of height n with J/J^2 free and $J = (f_1, \ldots, f_{n-1}, f_n) + J^2$. Let*

$$I = (f_1, \ldots, f_{n-1}) + J^{(n-1)!}.$$

Let (I, ω) be any $L-$cycle in $E(A, L)$.

Then, there is an $L-$oriented projective $A-$module (P, χ) of rank n such that

(1) $[P] - [F] = -[A/J]$ *in* $K_0(A)$,

(2) P *maps onto* I,

(3) *the Euler class* $e(P, \chi) = (I, \omega) \in E(A, F)$, *if* $\mathbb{Q} \subseteq A$.

As simple construction of projective modules, without (3), with a given determinant L, it improves Murthy's theorem.

Recall the following diagram for a comparison:

$$\begin{array}{ccc} F^n K_0(A) & \xrightarrow{\varphi_A} & E_0(A, A^n) \\ & \searrow^{\varphi_L} & \downarrow \eta_L \\ & & E_0(A, F) \end{array}$$

Euler classes do not respect stable isomorphism. So, there is no maps to work with.

References

[AM] M. F. Atiyah and I. G. Macdonald, *Introduction to commutative algebra,* Addison-Wesley, Reading, Mass. (1969), ix+128 pp.

[B] M. Boratynski, *A note on set-theoretic complete intersection ideals,* J. Algebra **54**(1978).

[BS1] S. M. Bhatwadekar and Raja Sridharan *Projective generation of curves in polynomial extensions of an affine domain and a question of Nori* Invent. math. 133, 161-192 (1998).

[BS2] S. M. Bhatwadekar and Raja Sridharan, *The Euler Class Group of a Noetherian Ring,* Compositio Mathematica, **122:** 183-222,2000.

[BS3] S. M. Bhatwadekar and Raja Sridharan, *On Euler classes and stably free projective modules,* Algebra, arithmetic and geometry, Part I, II (Mumbai, 2000), 139–158, TIFR Stud. Math.,**16**, TIFR., Bombay, 2002..

[BK] S. M. Bhatwadekar and Manoj Kumar Keshari, *A question of Nori: Projective generation of Ideals,* K-Theory **28** (2003), no. 4, 329–351.

[BDM] S. M. Bhatwadekar, Mrinal Kanti Das and Satya Mandal, *Torsion Euler Cycles,* Preprint

[D1] Mrinal Kanti Das and Raja Sridharan, *The Euler class group of a polynomial algebra,* J. Algebra **264** (2003), no. 2, 582–612.

[D2] Mrinal Kanti Das and Raja Sridharan, *The Euler class groups of polynomial rings and unimodular elements in projective modules,* JPAA **185** (2003), no. 1-3, 73–86.

[DM1] Mrinal Kanti Das and Satya Mandal, *A Riemann-Roch Theorem,* Preprint.

[DM2] Mrinal Kanti Das and Satya Mandal, *Euler Class Construction,* to appear in JPAA.

[F] W. Fulton, *Intersection Theory,* Springer-Verlag, 1984.

[Ma1] Satya Mandal, *Homotopy of sections of projective modules,* J. Algebraic Geom. 1 (1992), no. 4, 639–646.

[Ma2] Satya Mandal, *Decomposition of projective modules,* K-Theory 22 (2001), no. 4, 393–400.

[Ma3] Satya Mandal, *Complete Intersection and K-Theory and Chern Classes,* Math. Zeit. 227, 423-454 (1998).

[Ma4] Satya Mandal, *Projective Modules and Complete Intersections,* LNM 1672, Springer(1997),1-113.

[Ma5] Satya Mandal, *On efficient generation of ideals,* Invent. Math. **75** (1984), 59-67.

[MM] Satya Mandal and M.P.Murthy, *Ideals as Sections of Projective Modules,* J. Ramanujan Math. Soc. **13** No.1 (1998) pp.51-62.

[MP] Satya Mandal and Bangere Purnaprajna, *Cancellation of projective modules and complete intersections,* J. Ramanujan Math. Soc. **18** (2003), no. 1, 77–85.

[MS] Satya Mandal and Raja Sridharan, *Euler Classes and Complete Intersections,* J. of Math. Kyoto University, 36-3 (1996) 453-470.

[MV] Satya Mandal and PLN Varma, *On a Question of Nori : the Local Case,* Comm. in Algebra, 25(2), 451-457(1997).

[MK1] N. Mohan Kumar, *Stably Free Modules,* Amer. J. of Math. **107** (1985), no. 6, 1439–1444.

[MK2] N. Mohan Kumar, *Some theorems on generation of ideals in affine algebras,* Comment. Math. Helv. 59 (1984), no. 2, 243–252.

[Mu1] M. P. Murthy, *Zero cycles and projective modules,* Ann. of math. **140** (1994), 405-434

[Mu2] M. P. Murthy *A survey of obstruction theory for projective modules of top rank,* Algebra, K-theory, groups, and education (New York, 1997), 153–174, Contemp. Math., **243**, AMS, Providence, 1999.

[MKM] N. Mohan Kumar and M. P. Murthy, *Algebraic cycles and vector bundles over affine three-folds,* Annals of MAth., **116** (1982), 579–591.

[Q] D. Quillen, *Projective modules over polynomial rings,* Invent. Math. **36** (1976), 167-171.

[S] A. A. Suslin, *A cancellation theorem for projective modules over affine algebras,* Soviet Math. Dokl. **17** (1976), 1160-1164.