The Homotopy Program of Nori

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We start with the main question we deal with:

Question 0.1. Suppose A is a noetherian commutative ring, with $\dim A = d$ and P is projective A-module, with $rank(P) = n$. Suppose

$$J \quad \text{is an ideal and} \quad \omega : P \twoheadrightarrow \frac{J}{J^2} \quad \text{is a surjective map.}$$

The question is, whether or when can we lift ω to a surjective map $f : P \twoheadrightarrow J$?

Admittedly, things work out better if A is regular.

Such pairs (J, ω) would be referred to as P-local orientations. Set of all such P-local orientations will be denoted by $\mathcal{L}O(P)$.

Prelude: This question was first considered in [MMu]. When A is a smooth affine algebra over an algebraically closed field k, and $rank(P) = \dim A$, an obstruction class $\zeta(P, I) \in CH^d(X)$ was written down [MMu]. It was established that there is a surjective map $P \twoheadrightarrow J$ if and only if $\zeta(P, J) = 0$ and there is a surjective map $P \twoheadrightarrow \frac{J}{J^2}$.

1 Introduction to the Homotopy Program

Subsequent to that [MMu], based on some Homotopy Relations, Madhav V. Nori (around 1990) laid out a set of ideas to deal with the questions of such obstructions in broader contexts, like when X is a regular or a noetherian affine scheme. These were communicated verbally to some in a very informal and open ended manner. Because of the nature of these communications, not everyone heard the same thing and these ideas took the form of some folklores. As a result, versions of this set of ideas available (or not) in the literature (e. g. [M3, MV, MS, BS1, BS2, BS3, BK]) have been up to the interpretations and adaptations by the recipients of these communications, much to their credit, and the stated hypotheses may differ. Because of the openendedness and broadness of these ideas, they appeared to be more like a research Program (*the Homotopy Program*) to this author, which is how we would refer to the same. Sometimes it may even be difficult to say whether certain part of the program was actually explicitly articulated by Nori or were part of the adaptations by others. There is no systematic exposition of this program available in the literature and certain aspects failed to receive deserved traction. Nori never classified these as conjectures or otherwise. However, some results followed too quickly [M3], to treat them as anything less than conjectures.

Analogy to the Obstruction Theory for vector bundles ([St]) was the main backdrop behind this program and central to this Program was the Homotopy conjecture of Nori. The following is the statement of the Homotopy Conjecture from [M3], which would most likely be an adaptation by the respective author.

Conjecture 1.1 (Homotopy Conjecture). Suppose A is a commutative noetherian ring, with $\dim A = d$ and P is a projective A-module. Let $R = A[T]$ be a polynomial ring and $I \subseteq A[T]$ be an ideal. Write $P[T] = P \otimes A[T]$. Write $I(0) = \{f(0) : f \in I\}$. Let

$$\varphi : \frac{P[T]}{IP[T]} \twoheadrightarrow \frac{I}{I^2} \quad \text{and} \qquad f_0 : P \twoheadrightarrow I(0) \qquad \text{be surjective homomorphisms.}$$

1. Substituting $T = 0$, we obtain $\varphi(0) : \frac{P}{I(0)P} \twoheadrightarrow \frac{I(0)}{I(0)^2}$,
2. Also, $f_0 \otimes \frac{A}{I(0)} : \frac{P}{I(0)P} \twoheadrightarrow \frac{I(0)}{I(0)^2}$.

We assume $\varphi(0) = f_0 \otimes \frac{A}{I(0)}$. Now, the question is: Whether there is surjective map

$$f : P[T] \twoheadrightarrow I \quad \ni \quad f(0) = f_0 \quad \text{and} \quad f \otimes \frac{A[T]}{I} = \varphi.$$

Remark: I stated it in an open-ended manner. It would be safe to assume A is regular or smooth. However, it fails even when A is regular [BS1, Example 3.15]. Often, it is assumed that $I(0)$, I are locally complete intervention ideals (*the Transversality Condition*). Existing results (see [M3, BS1, BK]) indicate that with suitable hypotheses the regularity and/or transversality hypotheses may be spared. The best result, up to date, is due to Bhatwadekar and Keshari [BK]. All the main results assume the following condition:

$$2rank(P) \geq d + 3$$

Nori informed me that there is very little topological obstructions, meaning the topological analogue is true, without such bounds.

Why Homotopy Conjecture?

1. Recall, our main object of study are $(J, \omega) \in \mathcal{L}O(P)$, where I is an ideal of A and $\omega : P \twoheadrightarrow \frac{J}{J^2}$ is a surjective map. Question is whether ω lifts to a surjective map $P \twoheadrightarrow J$.

2. In the statement of the Homotopy Conjecture $(I(0), \varphi(0)), (I(1), \varphi(1)) \in \mathcal{L}O(P)$.

   (a) Hypothesis of the of the conjecture says $(I(0), \varphi(0))$ is "GOOD".

   (b) Conclusion of the conjecture would imply $I(1), \varphi(1))$ is "GOOD".

3. The hypothesis also suggest a relation, as follows: for $(J_0, \omega_0), (J_1, \omega_1) \in \mathcal{L}O(P)$ define (J_0, ω_0) is equivalent to (J_1, ω_1), if there is a surjective map $\varphi : \frac{P[T]}{IP[T]} \twoheadrightarrow \frac{I}{I^2}$ such that

   $$(I(0), \varphi(0)) = (J_0, \omega_0) \quad \text{and} \quad (I(1), \varphi(1)) = (J_1, \omega_1).$$

   This need not be an equivalence relation. However, it defines a chain Equivalence relation.

   Let $\pi_0 (\mathcal{L}O(P))$ denote the set of all equivalence classes.

4. Also, notice $(I, \varphi) \in \mathcal{L}O(P[T])$. So, $\pi_0 (\mathcal{L}O(P))$ fits in to

the following push forward diagram:

$$\begin{array}{ccc} \mathcal{L}O(P[T]) & \xrightarrow{T=0} & \mathcal{L}O(P) \\ {\scriptstyle T=1}\big\downarrow & & \big\downarrow \\ \mathcal{L}O(P) & \longrightarrow & \pi_0\left(\mathcal{L}O(P)\right) \end{array} \qquad \text{in} \qquad \underline{Sets}. \tag{1}$$

5. Try to think, what all these means when $P = A^n$ is free.

6. TheHomotopy Obstruction set $\pi_0\left(\mathcal{L}O(P)\right)$ has other descriptions, which we proceed to discuss. Fir.st, we have the following lemma that follows from Nakayama's lemma

Lemma 1.2. Suppose, A is a commutative noetherian ring and P is a projective A-module. Let $(J, \omega) \in \mathcal{L}O(P)$. By priorities of projective modules, ω lifts as follows:

$$\begin{array}{ccc} P & \xrightarrow{f} & J \\ & {\scriptstyle \omega}\searrow & \big\downarrow \\ & & \frac{J}{J^2} \end{array} \qquad f \quad \text{need not be surjective.}$$

So, $J = f(P) + I^2$. By Nakayama's Lemma

$\exists\ s \in I \ \ni \ (1-s)I \subseteq f(P)$. So $\exists\ p \in P \ \ni \ f(p) = s(1-s)$.

In particular, $I = (f(P), s)$ and

$$\text{with} \quad J = (f(P), 1-s) \quad f(P) = I \cap J, \qquad I + J = A.$$

2 Homotopy Obstructions

Now, we proceed to give other descriptions of $\pi_0(\mathcal{LO}(P))$. Alway, keep track of what all these means when $P = A^n$ is free.

Definition 2.1. Let A be a noetherian commutative ring, $X = \mathrm{Spec}\,(A)$ and P be a projective A-module. By a local P-orientation, we mean a pair (I, ω) where I is an ideal of A and $\omega : P \twoheadrightarrow \frac{I}{I^2}$ is a surjective homomorphism, which is identified with surjective homomorphism $\frac{P}{IP} \twoheadrightarrow \frac{I}{I^2}$, induced by ω. Denote

$$\begin{cases} \mathcal{LO}(P) = \{(I, \omega) : (I, \omega) \text{ is a local } P \text{ orientation}\} \\ \mathcal{Q}(P) = \{(f, s) \in P^* \oplus A : s(1-s) \in f(P)\} \\ \widetilde{\mathcal{Q}}(P) = \{(f, p, s) \in P^* \oplus P \oplus A : f(p) + s(s-1) = 0\} \\ \widetilde{\mathcal{Q}}'(P) = \{(f, p, z) \in P^* \oplus P \oplus A : f(p) + z^2 = 1\} \end{cases}$$

In addition to the pushout diagram (1), we have three more pushout diagrams in <u>*Sets*</u>:

$$\begin{array}{ccc} \mathcal{Q}(P[T]) & \xrightarrow{T=0} & \mathcal{Q}(P) \\ {\scriptstyle T=1}\downarrow & & \downarrow \\ \mathcal{Q}(P) & \longrightarrow & \pi_0\left(\mathcal{Q}(P)\right) \end{array} \quad \begin{array}{ccc} \widetilde{\mathcal{Q}}(P[T]) & \xrightarrow{T=0} & \widetilde{\mathcal{Q}}(P) \\ {\scriptstyle T=1}\downarrow & & \downarrow \\ \widetilde{\mathcal{Q}}(P) & \longrightarrow & \pi_0\left(\widetilde{\mathcal{Q}}(P)\right) \end{array} \quad \begin{array}{ccc} \widetilde{\mathcal{Q}}'(P[T]) & \xrightarrow{T=0} & \widetilde{\mathcal{Q}}'(P) \\ {\scriptstyle T=1}\downarrow & & \downarrow \\ \widetilde{\mathcal{Q}}'(P) & \longrightarrow & \pi_0\left(\widetilde{\mathcal{Q}}'(P)\right) \end{array} \tag{2}$$

By completing the square $s(s-1) = \left(s - \frac{1}{2}\right)^2 - \frac{1}{4}$, we obtain a bijection

$$\kappa : \widetilde{\mathcal{Q}}(P) \xrightarrow{\sim} \widetilde{\mathcal{Q}}'(P) \quad \text{sending} \quad (f, p, s) \mapsto (2f, 2p, 2s-1) \tag{3}$$

So, for all practical purposes, these two are same.

There is a commutative diagram of set theoretic maps, as denoted:

$$\begin{array}{ccc} \widetilde{\mathcal{Q}}(P) & \xrightarrow{\nu} & \mathcal{Q}(P) \\ {\scriptstyle \eta}\downarrow & \swarrow{\scriptstyle \eta'} & \\ \mathcal{LO}(P) & & \end{array} \quad \text{where, for } (f, p, s) \in \widetilde{\mathcal{Q}}(P), \quad \nu(f, p, s) = (f, s) \tag{4}$$

and $\eta'(f,s) = \eta(f,p,s) = (I,\omega)$, where $I = f(P) + As$ and $\omega : P \twoheadrightarrow \frac{I}{I^2}$ is the homomorphism is induced by f. These maps η, η', ν are surjective.

We have four descriptions of the same.

Lemma 2.2. *Use the notations, as above (2.1). The bijections κ, ν, η, η', induces a bijections*

$$\begin{array}{ccccc} \pi_0\left(\widetilde{\mathcal{Q}}'(P)\right) & \xleftarrow[\sim]{\overline{\kappa}} & \pi_0\left(\widetilde{\mathcal{Q}}(P)\right) & \xrightarrow[\sim]{\overline{\nu}} & \pi_0\left(\mathcal{Q}(P)\right) \\ & & \overline{\eta}\downarrow\wr & \swarrow_{\overline{\eta}'}^{\sim} & \\ & & \pi_0\left(\mathcal{L}O(P)\right) & & \end{array}$$

Corollary 2.3. *Interpret, all these when $P = A^n$ is free.*

Before we proceed, we introduce the following notions.

Notations 2.4. Suppose A is a commutative noetherian ring, with $\dim A = d$ and P is a projective A-module, with $rank(P) = n$. Denote $\zeta = \overline{\nu}^{-1}\chi : \mathcal{L}O(P) \longrightarrow \pi_0\left(\widetilde{Q}(P)\right)$ and $\zeta_0 : \widetilde{Q}(P) \longrightarrow \pi_0\left(\widetilde{Q}(P)\right)$. So, we have a commutative diagram:

$$\begin{array}{ccc} \widetilde{Q}(P) & & \\ \eta\downarrow & \searrow^{\zeta_0} & \\ \mathcal{L}O(P) & \xrightarrow[\zeta]{} & \pi_0\left(\widetilde{Q}(P)\right) \end{array}$$

2.1 Off the Topic Comment

Four pushout diagrams (1, 2.1) is special case of a much general concept, as follows. Recall, a pre-sheaf is a contravariant functor.

Definition 2.5. Given a pre-sheaf $\mathcal{F} : \underline{Sch}_A \to \underline{Sets}$, and a scheme $X \in \underline{Sch}_A$, define $\pi_0(\mathcal{F})(X)$ by the pushout

$$\begin{array}{ccc} \mathcal{F}(X \times \mathbb{A}^1) & \xrightarrow{T=0} & \mathcal{F}(X) \\ {\scriptstyle T=1}\downarrow & & \downarrow \\ \mathcal{F}(X) & \longrightarrow & \pi_0(\mathcal{F})(X) \end{array} \quad \text{in } \underline{Sets} \tag{5}$$

if $X = \mathrm{Spec}\,(A)$, the diagram would looklike

$$\begin{array}{ccc} \mathcal{F}(A[T]) & \xrightarrow{T=0} & \mathcal{F}(A) \\ {\scriptstyle T=1}\downarrow & & \downarrow \\ \mathcal{F}(A) & \longrightarrow & \pi_0(\mathcal{F})(A) \end{array} \quad \text{in } \underline{Sets} \tag{6}$$

For example, let S be a fixed scheme (think of the sphere). Then

$$\mathcal{F}_S(X) = Hom_{\underline{Sch}}(X, S) \quad \text{is a pre} - \text{sheaf.}$$

In fact, one can include more variables and define higher homotopy sheaves $\pi_n\,(\mathcal{F}(X))$.

2.2 Homotopy Triviality

In this subsection, we establish that, for $(I, \omega_I) \in \mathcal{LO}(P)$, under some additional conditions, that the triviality of $\zeta(I, \omega_I)$ implies that ω_I lifts to a surjective map $P \twoheadrightarrow I$. First, we fix a base point, as follows.

Definition 2.6. Suppose A is a commutative noetherian ring, with $\dim A = d$ and P is a projective A-module, with $rank(P) = n$.

1. If $P = P_0 \oplus A$, let $f_0 \in P^*$ be the projection map $P_0 \oplus A \twoheadrightarrow A$.
2. If P does not split, let $f_0 \in P^*$ be such that, $height(f_0(P)) = n$. Recall, existence of such a map f_0 is assured by basic element theory.

We fix $\mathbf{v}_0 := (f_0, 0, 0) \in \widetilde{Q}(P)$ and treat is it as the **base point** of $\widetilde{Q}(P)$. In fact, $\zeta_0(\mathbf{v}_0) = \zeta_0(0,0,0) \in \pi_0\left(\widetilde{Q}(P)\right)$ (see my paper).

The following is, in deed, reinterpretation of [BK, Theorem 4.13].

Theorem 2.7. *Suppose A is an essentially smooth ring over an infinite perfect field k, with $1/2 \in k$ and $\dim A = d$. Let P be a projective A-module with $rank(P) = n$, with $2n \geq d+3$. Let $\mathbf{v}_0 = (f_0, 0, 0) \in \widetilde{Q}(P)$ be a base point, as in (2.6). Suppose $(I, \omega_I) \in \mathcal{LO}(P)$, with $height(I) \geq n$. Then, ω_I lifts to a surjective map $P \twoheadrightarrow I$ if and only if $\zeta(I, \omega_I) = \zeta_0(\mathbf{v}_0)$.*

Proof. Extra hypotheses on A and k are needed, because we use [BK, Theorem 4.13]. ∎

2.3 Homotopy and the Equivalence Relation

The following Corollary would be of some use for our future discussions.

Theorem 2.8. *Let A be a regular ring over a field k, with $1/2 \in k$. Let P be a projective A-module, with $rank(P) = n \geq 2$, and $(\mathbb{Q}(P), q) = \mathbb{H}(P) \perp A$. Let $\mathbf{u}, \mathbf{v} \in \widetilde{Q}'(P)$ such that $[\mathbf{u}] = [\mathbf{v}] \in \pi_0\left(\widetilde{Q}'(P)\right)$. Then, there is a homotopy $H(T) \in \widetilde{Q}'(P[T])$ such that $H(0) = \mathbf{u}$ and $H(1) = \mathbf{v}$. Equivalently, for $\mathbf{u}, \mathbf{v} \in \widetilde{Q}'(P)$ if $\zeta_0(\mathbf{u}) = \zeta_0(\mathbf{v}) \in \pi_0\left(\widetilde{Q}(P)\right)$, then there is a homotopy $H(T) \in \widetilde{Q}(A[T])$ such that $H(0) = \mathbf{u}$ and $H(1) = \mathbf{v}$.*

In other words, the homotopy relation on $\widetilde{Q}(P)$ is actually an equivalence relation.

3 The Involution and the Monoid Structure

Proposition 3.1. *Suppose A is a commutative ring and P is a projective A-module, with rank$(P) = n$.*

$$\text{For} \quad (f, p, s) \in \widetilde{Q}(P), \quad \text{define} \quad \Gamma(f, p, s) = (f, p, 1 - s)$$

Then,

$$\Gamma : \widetilde{Q}(P) \xrightarrow{\sim} \widetilde{Q}(P), \quad \text{is a bijection, such that} \quad \Gamma^2 = 1.$$

We say Γ is an involution on on $\widetilde{Q}(P)$.

We record the following obvious lemma.

Lemma 3.2. *Suppose A is a commutative ring and P is a projective A-module, with rank$(P) = n$ and $\Gamma : \widetilde{Q}(P) \xrightarrow{\sim} \widetilde{Q}(P)$ is the involution. Let $\mathbf{v} = (f, p, s) \in \widetilde{Q}(P)$ and denote*

$$\eta(\mathbf{v}) = (I, \omega_I), \qquad \eta(\Gamma(\mathbf{v})) = (J, \omega_J) \quad \text{where} \quad I = (f(P), s), \ J = (f(P), 1-s)$$

Then,

1. *$I \cap J = f(P)$.*
2. *For $H(T) \in \widetilde{Q}(P[T])$, we have $\Gamma(H(T))_{T=t} = \Gamma(H(t))$.*
3. *Therefore, $\forall\, \mathbf{v}, \mathbf{w} \in Q_{2n}(S) \qquad \zeta_0(\mathbf{v}) = \zeta_0(\mathbf{w}) \Longleftrightarrow \zeta_0(\Gamma(\mathbf{v})) = \zeta_0(\Gamma(\mathbf{w}))$.*

In deed, Γ factors through an involution on $\pi_0(Q_{2n})(A)$, as follows.

Corollary 3.3. *Suppose A is a commutative ring and P is a projective A-module, with rank$(P) = n$. Then, the involution $\Gamma : \widetilde{Q}(P) \xrightarrow{\sim} \widetilde{Q}(P)$ induces a bijective map $\widetilde{\Gamma} : \pi_0\left(\widetilde{Q}(P)\right) \xrightarrow{\sim} \pi_0\left(\widetilde{Q}(P)\right)$, such that $\widetilde{\Gamma}^2 = 1$ and $\zeta_0\Gamma = \widetilde{\Gamma}\zeta_0$. We say $\widetilde{\Gamma}$ is an involution.* (The notation $\widetilde{\Gamma}$ will also be among our standard notations throughout this article.)

Proof. First, consider the map $\zeta_0\Gamma : \widetilde{Q}(P) \longrightarrow \pi_0\left(\widetilde{Q}(P)\right)$. For, $H(T) \in \widetilde{Q}(P[T])$, we have $\zeta_0\Gamma(H(0)) = \zeta_0\Gamma(H(1))$. Therefore, $\zeta_0\Gamma$ is homotopy invariant. Hence, it induces the a well defined map $\widetilde{\Gamma} : \pi_0\left(\widetilde{Q}(P)\right) \xrightarrow{\sim} \pi_0\left(\widetilde{Q}(P)\right)$. Clearly, $\widetilde{\Gamma}^2 = 1$ and $\widetilde{\Gamma}$ is a bijection. The proof is complete. ■

3.1 The Monoid Structure on $\pi_0\left(\widetilde{Q}(P)\right)$

The following would be a natural way to define addition, when conditions are met.

Definition 3.4. *Let A be a commutative noetherian ring and P be a projective A-module, with $rank(P) = n \geq 2$. Let $(I, \omega_I), (J, \omega_J) \in \mathcal{LO}(P)$ be such that $I + J = A$. Let $\omega := \omega_I \star \omega_J : P \twoheadrightarrow \frac{IJ}{(IJ)^2}$ be the unique surjective map induced by ω_I, ω_J. We define a pseudo-sum*

$$(I, \omega_I)\hat{+}(J, \omega_J) := \zeta(IJ, \omega) \in \pi_0\left(\widetilde{Q}(P)\right) = \pi_0\left(\mathcal{LO}(P)\right).$$

Also, for $\mathbf{u}, \mathbf{v} \in \widetilde{Q}(P)$ with $\eta(\mathbf{u}) = (I, \omega_I)$ and $\eta(\mathbf{v}) = (J, \omega_J)$, if $I + J = A$, define pseudo-sum

$$\mathbf{u}\hat{+}\mathbf{v} := \eta(\mathbf{u})\hat{+}\eta(\mathbf{v}) \in \pi_0\left(\widetilde{Q}(P)\right).$$

This, in deed, extends to an addition on $\pi_0\left(\widetilde{Q}(P)\right)$, when A is a regular ring over a field k, with $1/2 \in k$.

Now we define a pseudo-difference in the spirit of (3.4).

Definition 3.5. *Let A be a commutative noetherian ring and P be a projective A-module, with rank$(P) = n \geq 2$. Suppose $(K, \omega_K), (I, \omega_I) \in \mathcal{LO}(P)$. Assume*

$$\exists\, \mathbf{u} = (f, p, s) \in \widetilde{Q}(P) \ni \eta(\mathbf{u}) = (I, \omega_I),\ \eta(\Gamma(\mathbf{u})) = (J, \omega_J),$$

and $J + K = A$. So, $\quad f(P) = I \cap J \quad I + J = K + J = A.$

Define the pseudo-difference

$$(K, \omega_K)\hat{-}(I, \omega_I) := (K, \omega_K)\hat{+}(J, \omega_J) \in \pi_0\left(\widetilde{Q}(P)\right).$$

We remark: (1) Under additional conditions, we prove that the pseudo-difference does not depend on the choice of J. (2) By Moving Lemma (Basic Element Theory), such choices $\mathbf{u} = (f, p, s) \in \widetilde{Q}(P)$ would be available if $2n \geq \dim A + 1$ and $height(K) \geq n$.

Extend pseudo-difference to $\pi_0\left(\widetilde{\mathcal{Q}}(P)\right) \times \pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$:

Theorem 3.6. Suppose A is a regular ring over a field k, with $1/2 \in k$ and $\dim A = d$. Let P be a projective A-module with $rank(P) = n$. Assume $2n \geq d+2$. Then, there is a well defined set theoretic map

$$\Theta : \pi_0\left(\widetilde{\mathcal{Q}}(P)\right) \times \pi_0\left(\widetilde{\mathcal{Q}}(P)\right) \longrightarrow \pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$$

such that, for $(K, \omega_K) \in \mathcal{LO}(P)$, with $height(K) \geq n$, and $(I, \omega_I) \in \mathcal{LO}(P)$,

$$\Theta\left(\zeta(K, \omega_K), \zeta(I, \omega_I)\right) = (K, \omega_K) \hat{-} (I, \omega_I). \qquad (7)$$

Remark: For $x \in \pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$, by Moving Lemma, we can write $x = \zeta(K, \omega_K)$ such that $height(K) \geq n$.

Finally, define the binary structure on $\pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$.

Definition 3.7. *Suppose A is a regular ring over a field k, with $1/2 \in k$ and $\dim A = d$. Let P be a projective A-module with rank$(P) = n$. Assume $2n \geq d+2$. Then, for $x, y \in \pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$, define*

$$x + y := \Theta\left(x, \widetilde{\Gamma}(y)\right)$$

This operation is well defined because so are Θ and $\widetilde{\Gamma}$.

The following is a final statement on the binary structure on $\pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$, which is not necessarily a group.

Theorem 3.8. *Suppose A is a regular ring over a field k, with $1/2 \in k$ and $\dim A = d$. Let P be a projective A-module with rank$(P) = n$. Assume $2n \geq d+2$. Then, the addition operation on $\pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$, defined in (3.7) has the following properties. Let $\mathbf{e}_0 = \zeta_0(0,0,0)$ and $\mathbf{e}_1 = \zeta_0(0,0,1)$.*

1. *The addition in $\pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$ is commutative and associative. Further, $\mathbf{e}_1$ acts as the additive identity in $\pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$. In other words, $\pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$ has a structure of an abelian monoid.*
2. *For any $x \in \pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$, $x + \widetilde{\Gamma}(x) = \mathbf{e}_0$.*
3. *If $\mathbf{e}_0 = \mathbf{e}_1$, then $\pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$ is an abelian groups under this addition.* (In particular, if $P = Q \oplus A$, then $\pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$ is an abelian groups.)

4 The Euler Class Groups

Suppose A is a noetherian commutative ring with $\dim A = d$ and P is a projective A-module, with $rank(P) = n$. In this section, in analogy to the definition of the Euler class groups $E^n(A)$ in [BS2, MY], we define a group $E(P)$, which would also be called the Euler class group of P.

(*In the sequel, for a set S, the free abelian group generated by S will be denoted by $\mathbb{Z}(S)$*).

Definition 4.1. Suppose A is a noetherian commutative ring, with $\dim A = d$ and P is a projective A-module, with $rank(P) = n \geq 0$. Denote,

$$\begin{cases} \mathcal{LO}^n(P) = \{(I, \omega_I) \in \mathcal{LO}(P) : height(I) \geq n\}, \\ \mathcal{LO}^n_c(P) = \{(I, \omega_I) \in \mathcal{LO}(P) : V(I) \text{ is connected and } height(I) \geq n\}. \end{cases}$$

Let $(I, \omega_I) \in \mathcal{LO}^n(P)$.

1. We can write $I = \cap_{i=1}^m I_i$, where $V(I_i) \subseteq \operatorname{Spec}(A)$ are connected (that means $\frac{A}{I_i}$ has not nontrivial idempotent).

2. Now, ω_I induce a $\omega_{I_i} : \frac{P}{I_i P} \twoheadrightarrow \frac{I_i}{I_i^2}$. Hence $(I_i, \omega_{I_i}) \in \mathcal{LO}^n_c(P)$, for $i = 1, \ldots, m$.

3. Denote
$$\varepsilon(I, \omega_I) = \sum_{i=1}^m (I_i, \omega_{I_i}) \in \mathbb{Z}\left(\mathcal{LO}^n_c(P)\right)$$
A local orientation $(I, \omega_I) \in \mathcal{LO}(P)$ would be called global, if ω_I lifts to a surjective map $P \twoheadrightarrow I$.

4. Let $\mathscr{R}(P)$ denote the subgroup of $\mathbb{Z}\left(\mathcal{LO}^c(P)\right)$, generated by the set
$$\{\varepsilon(I, \omega_I) : (I, \omega_I) \in \mathcal{LO}^n(P) \text{ and, is global}\}.$$
(*Note $\eta(0, 0, 1)$ is global, only if $P = Q \oplus A$.*)

Define

$$E(P) = \frac{\mathbb{Z}\left(\mathcal{L}O^n_c(P)\right)}{\mathscr{R}(P)} \quad \text{to be called the Euler class group of } \; P.$$

Images of $\varepsilon(I, \omega_I)$ in $E(P)$ with be denoted by $\overline{\varepsilon}(I, \omega_I)$. So, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{L}O^n_c(P) & & \\ \big\downarrow & \searrow & \\ \mathcal{L}O^n(P) & \xrightarrow[\overline{\varepsilon}]{} & \pi_0\left(\widetilde{\mathcal{Q}}(P)\right) \end{array} \tag{8}$$

Subsequently, we assume $P = Q \oplus A$. We define a homomorphism $\rho : E(P) \longrightarrow \pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$, as follows.

Definition 4.2. Suppose A is a regular ring over a field k, with $1/2 \in k$ and $\dim A = d$, and $P = Q \oplus A$ is a projective A-module with $rank(P) = n$ and $2n \geq d + 2$. Since $\pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$ has a group structure, the diagonal map in diagram (8) induce a homomorphism

$$\mathbb{Z}\left(\mathcal{L}O^n_c(P)\right) \longrightarrow \pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$$

Now suppose $(I, \omega_I) \in \mathcal{L}O^n(P)$ be global. Then, $\varepsilon(I, \omega) = \mathbf{e}_0 = \mathbf{e}_1$, the identity element. Therefore, ρ_0 factors through a group homomorphism

$$\rho : E(P) \twoheadrightarrow \pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$$

In fact, ρ is surjective.

We comment on the question of injectivity of ρ.

Theorem 4.3. *Suppose k is an infinite perfect field, with $1/2 \in k$ and A is an essentially smooth ring over k, with $\dim A = d$. Suppose P is a projective A-module with $rank(P) = n$ and $2n \geq d+3$. Assume $P \cong Q \oplus A$. Then, the homomorphism $\rho : E(P) \longrightarrow \pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$ is an isomorphism.*

Proof. The extra hypothesis on A and k, is needed because we are using [BK, Theorem 4.13]. ■

4.1 The Vanishing of Euler cycles

I borrowed the word "cycle", from the jargons of Chow groups. An element $x \in E(P)$ of may be referred to as an Euler cycle. If $x = \overline{\varepsilon}(I, \omega)$, the x may be called the cycle of (I, ω).

In [BS2], the Euler class groups $E^n(A)$ was defined. In fact, we got rid of some superfluous aspects in the definition in [BS2]. Then, $E^n(A)$ coincides with $E(A^n)$, as defined above. In this section, we extend the main theorem [BS2, Theorem 4.2], for $E(P)$, as follows. We will follow the arguments in the proof of [BS2, Theorem 4.2], which mainly depends on the availability of Subtraction and Addition Principles.

The following is the version of Corollary 4.1.

Theorem 4.4. *Suppose A is a commutative noetherian ring with $\dim A = d$ and P is a projective A-module, with $rank(P) = n$. Assume $2n \geq d+3$ and $P \cong Q \oplus A$. If $A = R[X]$ is a polynomial ring over a regular ring R, over an infinite field k, assume $2n \geq \dim R[T] + 2$. Let $(J, \omega_J) \in \mathcal{L}O^n(P)$ and $\overline{\varepsilon}\,(J, \omega_J) = 0 \in E(P)$. Then, ω_J lifts to a surjective map $P \twoheadrightarrow J$.*

A The Motivic Approach

We assume A is a commutative ring containing a field k, with $1/2 \in k$. In some literature (e. g. [F, AF]), perhaps known as Motivic Approach, driven by the desire to view the data in $\mathcal{LO}(A^n) = \mathcal{LO}(A,n)$ in a functorial manner, one writes

$$\begin{cases} Q_{2n}(A) = \{(s; f_1, \ldots, f_n; g_1, \ldots, g_n) \in A^{2n+1} : \sum_{i=1}^n f_i g_i + s(s-1) = 0\} \\ Q'_{2n}(A) = \{(s; f_1, \ldots, f_n; g_1, \ldots, g_n) \in A^{2n+1} : \sum_{i=1}^n f_i g_i + s^2 = 1\} \end{cases}$$

and the homotopy sets $\pi_0(Q_{2n})(A)$, $\pi_0(Q'_{2n})(A)$ were defined. It was pointed out thatt there is a bijection $Q_{2n}(A) \xrightarrow{\sim} Q'_{2n}(A)$, which induces a bijection $\pi_0(Q_{2n})(A) \xrightarrow{\sim} \pi_0(Q'_{2n})(A)$. So, we would comment only on $Q'_{2n}(A)$. Recall, the pref-sheaf (functoriality) structure of $A \mapsto Q'_{2n}(A)$ was obtained from bijection $Q'_{2n}(A) \cong Hom(\mathrm{Spec}(A), \mathrm{Spec}(\mathscr{B}_{2n+1}))$, where $\mathscr{B}_{2n+1} = \frac{k[X_1,\ldots,X_n;Y_1,\ldots,Y_n,Z]}{\left(\sum_{i=1}^n X_iY_i + Z^2 - 1\right)}$.

By analogy, for our purpose, for a projective A-module P, we considered

$$\begin{cases} \widetilde{Q}(P) = \{(f,p,s) \in P^* \oplus P \oplus A : f(p) + s(s-1) = 0\}, \\ \widetilde{Q}'(P) = \{(f,p,s) \in P^* \oplus P \oplus A : f(p) + s^2 = 1\}. \end{cases}$$

and define $\pi_0\left(\widetilde{Q}(P)\right)$, $\pi_0\left(\widetilde{Q}'(P)\right)$. As usual, a bijection $\widetilde{Q}(P) \xrightarrow{\sim} \widetilde{Q}'(P)$, is obtained by completing the square $s(s-1) = \left(s - \frac{1}{2}\right)^2 - \frac{1}{4}$. It was established (see Theorem **??**) that there is a bijection $\pi_0(\mathcal{LO}(P)) \xrightarrow{\sim} \pi_0\left(\widetilde{Q}(P)\right) \xrightarrow{\sim} \pi_0\left(\widetilde{Q}'(P)\right)$. We clarify the pre-sheaf structure (functoriality) on $\widetilde{Q}'(P)$ as follows.

Suppose Q is a projective A-module and $S(Q^*) = \bigoplus_{i \geq 0} S_i(Q^*)$ denote the symmetric algebra of Q^*. Let $quad(Q) = \{\varphi \in Hom(Q, Q^*) : \varphi^* = \varphi\}$ denote the A-module of all the quadratic forms on Q. Given $\varphi \in quad(Q)$, let $B(\varphi) :\in Hom(Q \otimes Q, A) \cong Q^* \times Q^*$ be the corresponding bilinear map. In fact, this association $\varphi \mapsto B(\varphi)$ induces a bijection $quad(Q) \xrightarrow{\sim} S_2(Q^*)$ (see [Sw, § 2]). Consider the commutative diagram of bijections

$$\begin{array}{ccc} Q & \xrightarrow[\sim]{ev} & Hom(Q^*, A) \\ & {\searrow}^{\sim}_{\lambda} & \downarrow \wr \\ & & Hom(S(Q^*), A) \end{array}$$

Fix $x \in Q$. For $f, g \in Q^*$, $\lambda(x)(f) = f(x)$ and $\lambda(x)(fg) = f(x)g(x)$. Let "overline" denote the images of elements in $Q^* \otimes Q^* \cong Hom(Q \otimes Q, A)$ in $S_2(Q^*)$ Given bilinear map $\beta : Q \otimes Q \longrightarrow A$, $\beta = \sum f_i \otimes g_i$ for some $f_i, g_i \in Q^*$. So, $\lambda(x)(\overline{\beta}) = \sum f_i(x)g_i(x) = \beta(x,x)$.

Fix quadratic form $\varphi : Q \longrightarrow Q^*$ and $B(\varphi) : Q \otimes Q \longrightarrow A$ be the corresponding bilinear map. More precisely, $B(\varphi)(x,y) = \varphi(x)(y)$. As usual, define $q : Q \longrightarrow A$ by $q(x) = B(x,x)$. Then,

$$\text{for } x \in Q \quad \lambda(x)(\overline{B(\varphi)}) = B(\varphi)(x,x) = q(x).$$

For our purpose, we summarize the above, as follows.

Proposition A.1. *Suppose A is a commutative noetherian ring, containing a field k, with $1/2 \in k$. Now, let (Q, φ) be a quadratic space, over A. Define*

$$\mathbb{S}(Q,\varphi) = \{x \in Q : q(x) = 1\}, \qquad \mathscr{B}(Q,\varphi) = \frac{S(Q^*)}{(\overline{B(\varphi)} - 1)}$$

Then, there are bijections, as follows

$$\mathbb{S}(Q,\varphi) \xrightarrow{\sim} Hom\left(\mathscr{B}(Q,\varphi), A\right) = Hom\left(\mathrm{Spec}\,(A)\,, \mathrm{Spec}\left(\mathscr{B}(Q,\varphi)\right)\right).$$

Proof. Follows from above discussions. ■

Remark A.2. Suppose A_0 is a commutative noetherian ring, containing a field k, with $1/2 \in k$ and (Q_0, φ_0) is a quadratic space over A_0. There is pre-sheaf

$$\underline{Sch}_{A_0} \longrightarrow \underline{Sets} \quad \text{sending} \quad \mathrm{Spec}\,(A) \mapsto \mathbb{S}\left((Q_0,\varphi_0) \otimes A\right)$$

In fact, there are bijections, as follows

$$\mathbb{S}\left((Q_0,\varphi_0) \otimes A\right) \xrightarrow{\sim} Hom\left(\mathscr{B}\left((Q_0,\varphi_0) \otimes A\right), A\right) \xrightarrow{\sim} Hom\left(\mathscr{B}(Q_0,\varphi_0), A\right).$$

Corollary A.3. *Suppose A is a commutative noetherian ring, containing a field k, with $1/2 \in k$. Let P be a projective A-module, $\mathbb{H}(P) = P^* \oplus P$ be the hyperbolic space and $(Q, \varphi) = \mathbb{H}(P) \perp A$, and $\mathbb{H}(P) = P^* \oplus P$. Let $B : Q \otimes Q \longrightarrow A$ be the bilinear form of (Q, φ). Let $\mathscr{A}(P) = \frac{S(P \oplus P^* \oplus A)}{(B-1)}$. Then, there are bijections, as follows*

$$\widetilde{Q}(P) \xrightarrow{\sim} Hom\left(\mathscr{A}(P), A\right) = Hom\left(\operatorname{Spec}(A), \operatorname{Spec}(\mathscr{A}(P))\right).$$

Proof. Follows from (A.1). This completes the proof. ■.

B From Intro Section

In this article, we mainly investigate various aspects of the Homotopy Program that concerns the structure of the obstruction set $\pi_0(\mathcal{LO}(P))$. Under some additional hypotheses, we prove that $\pi_0(\mathcal{LO}(P))$ has a natural structure of Monoid, which is a groups structure, when $P \cong Q \oplus A$. We give a definition of the Euler Class groups $E(P)$, which coincides with the same in [BS2] when $P = A^n$ and, likewise that in [MY]. Then, we compare $\pi_0(\mathcal{LO}(P))$ with the Euler class group $E(P)$.

Now on, we assume that A contains a field k, with $1/2 \in k$. The results in this article are extension of the results in [MM, AF], where the case of free modules $P = A^n$ was dealt with. When $P = A^n$ is a free A-module, $\mathcal{LO}(P)$ was denoted by $\mathcal{LO}(A, n)$ in [MM]. In some literature [F, AF], driven by the desire to view the same in a functorial manner, one writes $Q_{2n}(A) = \{(s; f_1, \ldots, f_n; g_1, \ldots, g_n) \in A^{2n+1} : \sum_{i=1}^n f_i g_i + s(s-1) = 0\}$ and the homotopy obstruction set $\pi_0(Q_{2n})(A)$ was defined. It was established in [MM] that $\pi_0(\mathcal{LO}(A, n)) \cong \pi_0(Q_{2n})(A)$. By analogy or due to the same desire, for a projective A-module P, we consider

$$\begin{cases} \widetilde{Q}(P) = \{(f, p, s) \in P^* \oplus P \oplus A : f(p) + s(s-1) = 0\}, \\ \widetilde{Q}'(P) = \{(f, p, s) \in P^* \oplus P \oplus A : f(p) + s^2 = 1\}. \end{cases}$$

and define homotopy obstruction sets $\pi_0\left(\widetilde{Q}(P)\right)$, $\pi_0\left(\widetilde{Q}'(P)\right)$. Note that there is a bijection $\widetilde{Q}(P) \xrightarrow{\sim} \widetilde{Q}'(P)$. We establish (see Theorem **??**) that there is a bijection $\pi_0(\mathcal{L}O(P)) \xrightarrow{\sim} \pi_0\left(\widetilde{Q}(P)\right)$. Recall, there is a bijection $Q_{2n}(A) \cong Hom\,(\mathrm{Spec}\,(A)\,, \mathrm{Spec}\,(\mathscr{A}_{2n+1}))$, where $\mathscr{A}_{2n+1} = \frac{k[X_1,\ldots,X_n;Y_1,\ldots,Y_n,Z]}{\left(\sum_{i=1}^n X_iY_i+Z(Z-1)\right)}$. By analogy, define $\mathscr{A}(P) = \frac{S(P\oplus P^*\oplus A)}{(B-1)}$, where $S(P \oplus P^* \oplus A)$ denotes the symmetric algebra and $B \in S_2(P \oplus P^* \oplus A)$ is the bilinear form of the quadratic space $P^* \oplus P \oplus A$. In Section A, an interpretation of $\widetilde{Q}'(P)$, analogous to that of $Q'_{2n}(A)$, given by establishing a bijection $\tilde{Q}'(P) \xrightarrow{\sim} Hom(\mathscr{A}(P), A)$.

Among the main results in this article, is the following reinterpretation (see (2.7)) of [BK, Theorem 4.13] on the Homotopy Conjecture 1.1, in terms of vanishing of obstruction classes.

Theorem B.1. *Suppose A is an essentially smooth ring over an infinite filed k, with $1/2 \in k$ and $\dim A = d$. Let P be a projective A-module with $rank(P) = n$, with $2n \geq d+3$. Let $\mathbf{v}_0 = (f_0, 0, 0) \in \widetilde{Q}(P)$ be a base point, as in (2.6). Suppose $(I, \omega_I) \in \mathcal{L}O(P)$, with $height(I) \geq n$. Then, ω_I lifts to a surjective map $P \twoheadrightarrow I$ if and only if $\zeta(I, \omega_I) = \zeta_0(\mathbf{v}_0)$.*

In Section 2.3, we develop some machinery regarding Homotopy. In particular, we establish the homotopy relation on $\widetilde{Q}(P)$ is, in deed, an equivalence relation (see (2.8), (**??**)), when A is a regular ring over a filed k, with $1/2 \in k$.

In Section 3, we consider the involution $\widetilde{Q}(P) \longrightarrow \widetilde{Q}(P)$ sending $(f, p, s) \mapsto (f, p, 1-s)$. For any commutative noetherian rings A, this involution induces an involution on $\pi_0\left(\widetilde{Q}(P)\right)$, which serves as a key tool, for the subsequent definition of the Monoid structure on $\pi_0(\mathcal{L}O(P))) = \pi_0\left(\widetilde{Q}(P)\right))$. In Section 3.1, we establish that, when A is a regular ring over a filed k, with $1/2 \in k$, with $2rank(P) \geq d+2$, $\pi_0\left(\widetilde{Q}(P)\right))$ has the structure of a natural Monoid (see Theorem refabelianGroup). There are two distinguished elements in $\widetilde{Q}(P)$, namely $\mathbf{e}_0 = \zeta_0(0,0,0), \mathbf{e}_1 = \zeta_0(0,0,1) \in \pi_0\left(\widetilde{Q}(P)\right)$. It turns out that, if $\mathbf{e}_0 = \mathbf{e}_1$ then the Monoid structure on $\pi_0\left(\widetilde{Q}(P)\right)$ becomes an abelian group, where $\mathbf{e}_0 = \mathbf{e}_1$ would be the identity. It is easy to see that, if $P = Q \oplus A$ for some projective A-module Q, then $\mathbf{e}_0 = \mathbf{e}_1$.

It had always remained conjectural , as a part of the Homotopy Program, that the Homotopy obstruction sets $\pi_0\left(\widetilde{Q}(P)\right)$ would have a group structure, when $rank(P)$ is hight enough. However, when $P = A^n$ is free, an obstruction group $E^n(P)$ was explicitly given by Nori, when $n = \dim A$. Subsequently, this definition was extended in [BS2]. This was further extended in [MY], when $P = L \oplus A^{n-1}$, where L is rank one projective A-module. These groups came to be known as Euler class groups. In Section 4, we extend these and give a natural definition of an Euler class groups $E(P)$, for any projective A-module P over a commutative noetherian ring A. It is helpful that certain superfluous aspect of the definition in [BS2] was pointed out in [MM]. in analogy the the results in [BS2, MY], we prove the following:

Theorem B.2. *Suppose A is a commutative noetherian ring with $\dim A = d$ and P is a projective A-module, with $rank(P) = n$. Assume $2n \geq d + 3$ and $P \cong Q \oplus A$. If $A = R[X]$ is a polynomial ring over a regular ring R, over an infinite field k, assume $2n \geq \dim R[T] + 2$. Let $(J, \omega_J) \in \mathcal{LO}^n(P)$ and $\overline{\varepsilon}(J, \omega_J) = 0 \in E(P)$. Then, ω_J lifts to a surjective map $P \twoheadrightarrow J$.*

When $\pi_0\left(\widetilde{Q}(P)\right)$ is a group, one can naturally define a group homomorphism $E(P) \longrightarrow \pi_0\left(\widetilde{Q}(P)\right)$. Consequently, we prove the following.

Theorem B.3. *Suppose k is a field, with $1/2 \in k$ and $\dim A = d$. Suppose $P = Q \oplus A$ is a projective A-module with $rank(P) = n$ and $2n \geq d + 2$. Then, there is a surjective group homomorphism $\rho : E(P) \twoheadrightarrow \pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$.*

In particular, if A is an essentially smooth ring over an infinite perfect field k and $2n \geq d + 3$ then, the homomorphism $\rho : E(P) \longrightarrow \pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$ is an isomorphism. (We make further comments when $\pi_0\left(\widetilde{\mathcal{Q}}(P)\right)$ fails to be a group or when P fails to have a unimodular element.)

C From Homo Obs Section

Before we proceed, we introduce some notations.

Notations C.1. Throughout, k will denote a field, with $1/2 \in k$. Also A will denote a commutative ring with $\dim A = d$. For A-modules M, N, we denote $M[T] := M \otimes A[T]$ and $M^* = Hom(M, A)$. For $f \in Hom(M, N)$, denote $f[T] := f \otimes 1 \in Hom(M[T], N[T])$. Homomorphisms $f : M \longrightarrow \frac{I}{I^2}$ would be identified with the induced maps $\frac{M}{IM} \longrightarrow \frac{I}{I^2}$.

For surjective homomorphisms $\omega_1 : M \twoheadrightarrow \frac{I_1}{I_1^2}$, $\omega_2 : M \twoheadrightarrow \frac{I_2}{I_2^2}$, where I_1, I_2 be two ideals, with $I_1 + I_2 = A$, $\omega_1 \star \omega_2 : M \twoheadrightarrow \frac{I_1 I_2}{(I_1 I_2)^2}$ will denote the unique surjective map induced by ω_1, ω_2.

For a projective A-module P, $\mathbb{Q}(P) = (\mathbb{Q}(P), q)$ will denote the quadratic space $\mathbb{H}(P) \perp A$, where $\mathbb{H}(P) = P^* \oplus P$ is the hyperbolic space. So, $P^* \oplus P \oplus A$ id the underlying projective module of $\mathbb{Q}(P)$ and, for $(f, p, s) \in P^* \oplus P \oplus A$, $q(f, p, s) = f(p) + s^2$.

The category of schemes over $\mathrm{Spec}\,(A)$ will be donated by $\underline{Sch}_A$. The category of sets will be denoted by $\underline{Sets}$. Given a pre-sheaf $\mathcal{F} : \underline{Sch}_A \to \underline{Sets}$, and a scheme $X \in \underline{Sch}_A$, define $\pi_0(\mathcal{F})(X)$ by the pushout

$$\begin{array}{ccc} \mathcal{F}(X \times \mathbb{A}^1) & \xrightarrow{T=0} & \mathcal{F}(X) \\ {\scriptstyle T=1}\downarrow & & \downarrow \\ \mathcal{F}(X) & \longrightarrow & \pi_0(\mathcal{F})(X) \end{array} \quad \text{in } \underline{Sets} \qquad (9)$$

For an affine scheme $X = \mathrm{Spec}\,(B) \in \underline{Sch}_A$ and a pre-sheaf $\mathcal{F}$, as above, we write $\mathcal{F}(A) := \mathcal{F}(\mathrm{Spec}\,(B))$ and $\pi_0(\mathcal{F})(B) := \pi_0(\mathcal{F})(\mathrm{Spec}\,(B))$. (*For our purpose, we could restrict ourselves to affine schemes* $\mathrm{Spec}\,(A)$ *and its polynomial extensions* $\mathrm{Spec}\,(A[T])$.)

D Bibekananda Lemma

We also record the following obvious observation.

Lemma D.1. *Suppose A is a commutative noetherian ring with* $\dim A = d$ *and $P = P_0 \oplus A$ is a projecttive A-module. Denote $p_0 = (0,1) \in P_0 \oplus A$ and $f_0 : P \twoheadrightarrow A$ be the projection map $P_0 \oplus A \longrightarrow A$. Let $\mathbf{u}_0 = (f_0, 0, 0) \in \widetilde{Q}(P)$ be the base point and $\mathbf{u_1} = (0, 0, 1)$. Then, $\zeta_0(\mathbf{u}_0) = \zeta_0(\mathbf{u}_1) \in \pi_0\left(\widetilde{Q}(P)\right)$*

Proof. Write $H(T) = ((1-T)f_0, Tp_0, T)$. Then, $(1-T)f_0(Tp_0) = T(1-T)$. So, $H(T) \in \widetilde{Q}(P[T])$. We have $H(0) = \mathbf{u}_0$ and $H(1) = \mathbf{u}_1$. The proof is complete. ■

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